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This handbook is intended for those interested in international developments and future directions in educational research, particularly mathematics education research. The book was conceived in response to a number of major global catalysts for change, including the impact of national and international mathematics comparative assessment studies; the social, cultural, economic, and political influences on mathematics education and research; the influence of the increased sophistication and availability of technology; and the increased globalization of mathematics education and research.

From these catalysts for change have emerged a number of priority themes and issues for mathematics education research for the new millennium. Three key themes have been identified for inclusion in this handbook: lifelong democratic access to powerful mathematical ideas, advances in research methodologies, and influences of advanced technologies. Each theme is examined in terms of learners, teachers, and learning contexts, with theory development being an important component of all these aspects.

The book comprises four sections. The first, Priorities in International Mathematics Education Research, provides important background information on the key themes of the book and introduces new and emerging research trends in the field. Following my introductory chapter, Carol Malloy (chapter 2) explores democratic access to mathematics through democratic education, and Richard Lesh (chapter 3) looks at research design in mathematics education with a focus on design experiments. In chapter 4, James Kaput, Richard Noss, and Celia Hoyles examine the evolution of representational infrastructures and related artifacts and technologies; they also show how changes in representational infrastructure are closely linked to “learnability and the democratization of intellectual power” (p. 73).

Section 2 of the handbook, Lifelong Democratic Access to Powerful Mathematical Ideas, is divided into two parts: Learning and Teaching and Learning Contexts and Policy Issues. With respect to learning and teaching, the authors consider students’ learning during the preschool and beginning school years (Bob Perry and Sue Dockett, chapter 5), in the elementary and middle school years (Graham Jones, Cynthia Langrall, and Carol Thornton, chapter 6), in the secondary school (Teresa Rojano, chapter 7), and, finally, at the advanced levels of mathematics education (Joanna Mamona-Downs and Martin Downs, chapter 8). Issues pertaining to representation in mathematical learning and problem solving (Gerald Goldin, chapter 9), teacher education (Dina Tirosh and Ruhama Even, chapter 10), and theoretical aspects of school mathematics (Boero, Douek, and Ferrari, chapter 11) are also included in this first part of section 2.

The second part of section 2, Learning Contexts and Policy Issues, covers a range of globally significant topics such as access and opportunity within the political and social context of mathematics education (William Tate and Celia Rousseau, chapter 12), democratic access to mathematical learning in developing countries (Luis Moreno
and David Block, chapter 13), and mathematical learning in out-of-school contexts (Guida de Abreu, chapter 14). Research and mathematics education reform are also addressed in this section (Miriam Amit and Michael Fried, chapter 15), together with Ole Skovsmose and Paolo Valero’s analysis of democratic access to powerful mathematical ideas. The professional community of mathematics teachers (James Middleton, Daiyo Sawada, Eugene Judson, and Jeff Turley, chapter 17) completes this section.

In section 3, Advances in Research Methodologies, among the many avenues explored are past, current, and possible future trends in conceptual frameworks and paradigms used in mathematics education research (Alan Schoenfeld, chapter 18), ways of making more productive contributions to policy and practice (Frank Lester and Dylan Wiliam, chapter 19), mathematics learning and levels of analysis and application (Eric Hamilton, Eamonn Kelly, and Finbar Sloane, chapter 20), and some methodological problems of innovative research paradigms (Ferdinando Arzarello and Maria Bartolini Bussi, chapter 21). The importance of linking research with practice is also emphasized in this section, in particular in the chapters by Nicolina Malara and Rosetta Zan (chapter 22), Kenneth Ruthven (chapter 23), and Douglas Clements (chapter 24). In chapter 25, Fulvia Furinghetti and Luis Radford discuss how the pedagogical use of the history of mathematics can serve as a means to transform teaching.

In the final section of the book, Influences of Advanced Technologies on Mathematical Learning and Teaching, the chapters include Rina Herskowitz and her colleagues’ analysis of the complexities of CompuMath, a large-scale curriculum development, implementation, and research project for the junior high school level. A focus on curricular change with graphing technology and its impact on students’ learning and teachers’ knowledge base is presented by Daniel Chazan and Michal Yerushalmy (chapter 28), and Giampaolo Chiappini and Rosa Maria Bottino (chapter 29) analyze the teaching and learning processes occurring within technologically and culturally rich social environments.

In the final chapter of the handbook, we consider future issues and directions in international research in mathematics education. These include ways in which research can support more equitable curriculum and learning access to powerful mathematical ideas; how we might assess the extent to which students have gained such access and whether they can make effective use of these ideas; how research can inform such assessment; how we might best develop and evaluate research methodologies in mathematics education; and how research can illuminate issues pertaining to mathematics education and society.

To each of the authors of this handbook, I convey my heartfelt thanks and appreciation. Indeed, the book would not have been possible without their valuable contributions. In addition, the associate editors, Graham Jones, Maria Bartolini Bussi, Richard Lesh, and Dina Tirosh, have provided me with wonderful guidance and support throughout the book’s development. In the final stages of its production, Graham Jones and I were fortunate to work together during his sabbatical in Brisbane. I express my sincere appreciation of his expert contribution here.

I also wish to thank the team of international reviewers of the handbook chapters. Their attention to detail and their insightful comments and suggestions were invaluable to the authors in improving their chapters. A list of the reviewers appears on the following page. Last, but by no means least, I express my heartfelt thanks to the editorial team at Lawrence Erlbaum. In particular, Naomi Silverman and Lori Hawver have been of immense support to me—always positive, always encouraging, and always giving freely of their time and expertise. And Larry Erlbaum, as ever, has been the ideal publisher.

Lyn D. English
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SECTION I

Priorities in International Mathematics Education Research
CHAPTER 1

Priority Themes and Issues in International Research in Mathematics Education

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...research is similar to other forms of learning in the sense that an important goal of research is to look beyond the immediate and the obvious, and to focus on what could be in addition to what is. Consequently, some of the most important contributions that researchers make to practice often involve finding new ways to think about problems and potential solutions rather than simply providing answers to specific questions.

—Lesh and Lovitts (2000, p. 53)

This handbook was initiated in response to a number of recent global catalysts that have had an impact on mathematics education and mathematics education research. In proposing the book, I made two fundamental claims. First, I noted—not surprisingly—that many nations are experiencing a considerable challenge in their quest to improve mathematics education for their students, the future leaders of society. Second, I claimed that mathematics education research was static for much of the 1990s and currently is not providing the much-needed direction for our future growth. In connection with the latter point, I argued that the most important questions are not being answered. Other researchers expressed similar sentiments in the late 1990s. Bauersfeld (1997), for example, likened development in mathematics education research to “a mere change of recipes” (p. 615), claiming that “too often, the choice of a research agenda follows actual models, easily available methods, and local preferences rather than an engagement in hot problems that may require unpleasant, arduous, and time-intensive investigations” (p. 621). More recently, Lesh and Lovitts (2000) commented that the mathematics education research community is often perceived to be
“driven by whims and curiosities of researchers rather than by an attempt to address real problems” (p. 52).

In an effort to redress these concerns, I asked the handbook authors to be proactive rather than reactive in examining the emerging and anticipated problems in our field. The problems we face today are quite different from those of 10, or even 5, years ago, and we are witnessing many more powerful catalysts for change at all levels of mathematics learning. In advancing our discipline we need early detection of these catalysts, a careful analysis of their likely impact, and well-planned strategies for dealing with change.

CATALYSTS FOR CHANGE

National and International Mathematics Testing

The findings from recent international mathematics testing, such as the Third International Mathematics and Science Study (e.g., National Research Council, 1996), have led many nations to question the substance of their school mathematics curricula. Indeed in some nations, such as the United States, this testing has led to significant divisions among states as to what mathematics should be taught and how it should be taught (e.g., Jacob, 1999). The development of mathematical standards in several nations, including those developed by government advisory bodies (e.g., The Japanese Curriculum Council; Hashimoto, 2000) and by professional organizations (e.g., National Council of Teachers of Mathematics [NCTM], 2000) has added fuel to the debate on what is the “best” mathematics curricula for the new millennium.

Indeed, as Amit and Fried point out in chapter 15 of this volume, performance on standard tests has been a motive for change almost from the conception of a standard examination. The authors refer to the 1845 citywide examination in Boston used to determine the achievements of the city’s school system, where the original intention was to prove to the Massachusetts Board of Education that Boston schools deserved increased funding. However, the poor outcomes on the mathematical and other components of the examination prompted a review of instructional practices and school organization, which led to the use of such tests as a tool for improving education.

Influences From Social, Cultural, Economic, and Political Factors

These factors are having an unprecedented impact on mathematics education and its research endeavours, with many of the current educational problems being fuelled by the opposing values that policy makers, program developers, professional groups, and community organizations hold (Silver, 2001; Skovsmose & Valero, this volume; Sowder, 2000; Tate & Rousseau, this volume).

Throughout the current controversies, a core goal of mathematics education remains, that of meeting the needs of all students. As Tate and Rousseau (chapter 12) stress, the lack of access to a quality education—in particular, a quality mathematics education—is likely to limit human potential and individual economic opportunity. Given the importance of mathematics in the ever-changing global market, there will be increased demand for workers to possess more advanced and future-oriented mathematical and technological skills. Together with the rapid changes in the workplace and in daily living, the global market has alerted us to rethink the mathematical experiences we provide for our students in terms of content, approaches to learning, ways of assessing learning, and ways of increasing access to quality learning.

Unfortunately, many nations have been faced with political and economic challenges as they attempt to address the above concerns, and when politics intrude in
educational issues the intended messages often get lost in the rhetoric (Roitman, 1999). A well-known example of such political intrusion is the Californian “Math Wars,” where the more-or-less independent development of the state mathematics framework and the National Council of Teachers of Mathematics Standards document (NCTM, 2000) has resulted in intense political debates and factions. A worrying side effect of this political situation was the California State Board of Education’s request for a summary of all relevant research in mathematics education. By relevant research, the board meant that only studies using experimental design would be considered. Judith Sowder lamented this situation when she observed, “It seems that only experimental studies can speak authoritatively to many policy makers” (NCTM 2000 Presession Address).

Severe cuts in education budgets in the past decade also have had a heavy impact on mathematics education and research (Niss, 1999). While some nations now appear to be increasing their budgets to target specific areas of mathematics learning such as numeracy (e.g., Brown, Denvir, Rhodes, Askew, Wiliam, & Ranson), there remain unacceptable funding shortages in several developed and underdeveloped countries. As financial assistance from government agencies continues to decline in many countries, researchers must increasingly seek financial support from independent bodies. This raises the question of whether research in mathematics education will be increasingly shaped by issues that agencies deem important and decreasingly influenced by issues identified by mathematics educators as in need of attention.

Increased Sophistication and Availability of Technology

New technologies are giving rise to major transformations of mathematics education and research (Niss, 1999; Roschelle, Kaput, & Stroup, 2000). Numerous opportunities are now available for both students and teachers to engage in mathematical experiences that were scarcely contemplated a decade ago. For example, international learning communities that are linked via videoconferencing and other computer-networking facilities are taking mathematics education and research into higher planes of development. As many researchers have emphasized, however (e.g., Maurer, 2000; Niss, 1999), the effective use of new technologies does not happen automatically and will not replace mathematics itself. Nor will technology lead to improvements in mathematical learning without improvements being made to the curriculum itself. The words of Kaput and Roschelle (1999) are apt here: “Technological revolutions in transportation and communications would be meaningless or impossible if core societal institutions and infrastructures remained unchanged in their wake. Today’s overnight shipments and telecommuting workers would be a shock to our forebears 100 years ago, but our curriculum would be recognized as quite familiar” (p. 167).

As students and teachers become more adept at capitalizing on technological opportunities, the more they need to understand, reflect on, and critically analyze their actions; and the more researchers need to address the impact of these technologies on students’ and teachers’ mathematical development (Niss, 1999).

A discussion of the impact of technology on mathematics learning cannot be separated from a consideration of globalization as the process responsible for establishing the “world village” (Skovsmose & Valero, this volume).

Increased Globalization of Mathematics Education and Research

Numerous interpretations of globalization appear in the literature, but for our purposes here, we adopt the ideas of Skovsmose and Valero (this volume): Globalization refers to the fact that events in one part of the world may be caused by, and at the same time influence, events in other parts. Our environment—described in political,
sociological, economic, or ecological terms—is continuously reconstructed in a process that receives inputs from all corners of the world (p. 384).

Major technological advances have increased the globalization of mathematics education in numerous aspects, including curriculum development and the nature of research. Interestingly, as Atweh and Clarkson (2001) noted, increased globalization has led to similarities rather than differences in the nature of mathematics curriculum documents. Furthermore, the similarities have proven to be rather stable over the years, with changes in curriculum in one country often being reflected in other countries within a few years. This globalization of curriculum development has led to an abundance of international comparative research studies of mathematics achievement, such as the Third International Mathematics and Science Study.

The impact of globalization is also evident in the conferences and publications of international associations for mathematics education. The International Group for the Psychology of Mathematics Education, founded in 1976, has played a powerful role in reducing geographic barriers among mathematics educators, thereby enabling rich exchanges of social, cultural, and mathematical heritages. Indeed, the goals of the International Group include promoting international contacts and sharing of scientific information in the psychology of mathematics education, fostering and stimulating interdisciplinary research in this field, and furthering a deeper understanding of the psychological aspects of mathematics teaching and learning. The achievement of these aims can be seen in the annual conference proceedings published by the group.

The International Commission on Mathematical Instruction (ICMI), which began its first set of studies in 1984, has also been a major force in the globalization of our discipline. The 1994 ICMI study, for example, addressed the question, What is research in mathematics education and what are its results? The study was designed to bring together representatives of different groups of researchers so that they could “confront one another’s views and approaches, and seek a better mutual understanding of what we might be talking about when we speak of research in mathematics education” (Sierpinska & Kilpatrick, 1998, p. 4).

Emerging from these catalysts for change are a number of key themes and issues for mathematics education research in the new century. The next section addresses these themes and issues, which formed the basis of the framework for this handbook.

**PRIORITY THEMES AND ISSUES FOR MATHEMATICS EDUCATION RESEARCH IN THE 21ST CENTURY**

In her editorial of the first millennium issue of the *Journal for Research in Mathematics Education*, Judith Sowder (2000, pp. 2–4) listed some of the research questions considered in need of attention in the coming decade. These questions include the following:

1. How should mathematics education researchers be prepared?
2. Could mathematics educators profit from professional development addressing the many new research methodologies being used in our field?
3. What counts as evidence in mathematics education research?
4. How can we get more support for research in mathematics education?
5. How can we communicate mathematics education research beyond our own research community to reach broader audiences?

Issues pertaining to research methodologies and paradigms are inherent in the first three of Sowder’s questions. We have seen a multiplicity of theoretical approaches and research designs evolve around the world in the past decade, including paradigms developed by particular nations such as the teacher–researcher approach in the Italian
model for innovation research (see chapter 22 of this volume by Malara and Zan). As Schoenfeld highlights in chapter 18 (this volume), this proliferation of research designs necessitates a closer scrutiny of the trustworthiness, generality, and importance of the claims made in mathematics education research. This point is revisited in a later discussion.

Sowder’s remaining two questions are concerned with how we might best convey our research to important stakeholders including funding agencies, classroom teachers, mathematicians, and policy developers. This long-standing issue has been highlighted in recent plenary addresses. A plea for reducing the ever-widening gap between researchers and practitioners in mathematics education was made by Mogens Niss (2000) in his plenary lecture at the 9th International Congress of Mathematics Educators. He stated (in the plenary and in personal communication, 8.11.00) that mathematics education as a domain of research is rapidly expanding, with new researchers “joining the conveyer belt” to address research that has been set by the research community at large. Niss argued that such research does not necessarily investigate issues of significance to the classroom, where teachers usually desire specific assistance to inform their practice. As researchers become more cautious about issuing advice that is not based on substantial research, they become less able to provide the concrete assistance concerning teaching and other forms of practice that teachers want. In summary, Niss claimed that researchers are not addressing issues that focus on shaping practice; rather their issues focus on practice as an object of research.

Although teachers might request specific practical assistance, they also need to draw on research that is conceptually relevant to them. In her National Council of Teachers of Mathematics 2000 presession speech, Judith Sowder addressed ways in which education might make a greater difference to practice. Drawing on the work of Kennedy (1997), Sowder stressed that mathematics education research must be conceptually relevant and accessible to teachers. That is, “The problem of accessibility is not merely one of placing research knowledge within physical reach of teachers, but rather one of placing research knowledge within the conceptual reach of teachers, for if research encouraged teachers to reconsider their prior assumptions, it might ultimately pave the way for change” (Kennedy, 1997, p. 7).

These important plenary addresses highlight the fact that research in mathematics education must aim for a greater conceptual and practical contribution to the field. This was one of the cornerstones of my proposal for the handbook. Drawing upon the work of Lesh and Lovitts (2000, p. 61), I asked authors to address research that makes a difference to both theory and practice. I defined such research as that which

1. anticipates problems and needed knowledge before they become impediments to progress;
2. translates future-oriented problems into researchable issues;
3. translates the implications of research and theory development into forms that are useful to practitioners and policy makers; and
4. facilitates the development of research communities to focus on neglected priorities or strategic opportunities.

To assist authors in addressing research that makes a difference to both theory and practice, I provided a framework for targeting the critical issues in mathematics education in the new millennium. Table 1.1 displays the matrix that formed the basis of the framework.

The columns in Table 1.1 represent the priority themes that authors were to address, namely, lifelong democratic access to powerful mathematical ideas, advances in research methodologies, and influences of advanced technologies. Each theme was to be explored in terms of learners, teachers, and learning contexts. Importantly, the
TABLE 1.1
Matrix of Priorities in Mathematics Education Research

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<tr>
<th>ADVANCES IN THEORIES</th>
<th>Life-long Democratic Access to Powerful Ideas</th>
<th>Advances in Research Methodologies</th>
<th>Influences of Advanced Technologies</th>
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<td>Learners</td>
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<td>Learning Contexts</td>
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Heading “Advances in Theories” serves to indicate that theory development is an important feature of each theme, including ways in which other disciplines might contribute to this theory development.

Expanding on the matrix to provide the overall framework for the book, I presented authors with a number of issues that might be examined within each cell. These issues, as presented to the authors, are outlined in the remainder of this chapter.

Priority Theme 1: Lifelong Democratic Access to Powerful Mathematical Ideas

Students are facing a world shaped by increasingly complex, dynamic, and powerful systems of information and ideas. As future members of the workforce, students will need to be able to interpret and explain structurally complex systems, to reason in mathematically diverse ways, and to use sophisticated equipment and resources. Mathematics education systems cannot afford to remain with the “powerful mathematical ideas” that were in vogue for a good part of the 20th century. These ideas were associated largely with computational skills that were considered necessary for effective citizenship and continued mathematical development beyond the elementary school (see Jones, Langrall, Thornton, and Nisbet, chapter 6 of this volume, for a discussion on this point). Today’s mathematics curricula must broaden their goals to include key concepts and processes that will maximize students’ opportunities for success in the 21st century. These include, among others statistical reasoning, probability, algebraic thinking, mathematical modeling, visualizing, problem solving and posing, number sense, and dealing with technological change. Opportunities for all students to access mathematics of this nature should be a primary goal of all mathematics education programs (Amit, 1999; Er-sheng, 1999).

In achieving this goal, we should examine the notion of “powerful mathematical ideas.” Skovsmose and Valero (chapter 16, this volume) provide a comprehensive analysis of the notion of powerful mathematical ideas from four perspectives: (a) a logical perspective (where “power refers to the characteristic of some key ideas that enable us to establish new links among theories and provide new meaning to previously defined concepts,” p. 390); (b) a psychological perspective (as associated with one’s experience in learning mathematics, that is, power is defined in relation to learning potentialities); (c) a cultural perspective (as related to the opportunities for students to “participate in the practices of a smaller community or of the society at large”; that is, mathematical ideas can become powerful to students “in as much as they provide opportunities to envision a desirable range of future possibilities,” pp. 393–394); and (d) a sociological perspective (in relation to the extent to which powerful mathematical ideas can be used as a resource for operating in society).
The notion of “democratic access” is also complex and multifaceted, as both Tate and Rousseau (chapter 12) and Skovsmose and Valero (chapter 16) point out in this volume. They take up the challenge of articulating the language and meaning of democratic access as it is used in the theoretical perspectives of our discipline. Tate and Rousseau argue that the field of mathematics education “is in need of ‘democratic access’ hermeneutics, or theory of interpretation” (p. 274) and provide the beginnings of such a theory in their chapter. Skovsmose and Valero remind us that the provision of “mathematics for all” needs to be considered from the many arenas where mathematics education practices take place, including the classroom, the overall school organization, the workplace, the local community, and the global society.

In presenting to the handbook authors the challenge of addressing lifelong democratic access to powerful mathematical ideas, I asked them to consider issues related to learners, to teachers, and to learning contexts. Specifically, I offered the following issues for consideration.

**Issues Related to Learners.**

- What key mathematical understandings, skills, and reasoning processes will students need to develop for success in the 21st century? What developments need to take place at each level of learning, from preschool through the adult level?
- To what extent are students currently developing these understandings, skills, and processes?
- How will the nature and form of students’ learning change as they develop these new skills and processes?
- How can we facilitate all students’ learning of powerful mathematical ideas?
- What theoretical models are emerging or need to be developed with respect to each of the above issues?

**Issues Related to Teachers.** Teachers need to be aware of and understand their students’ mathematical thinking and learning (as emphasized by Tirosh and Even, chapter 10 of this volume). Likewise, teachers must be cognizant of the rapidly changing nature of the “basics in mathematics” and should be willing to implement mathematical learning experiences that will enable all students to succeed mathematically, both within and beyond the classroom. As noted in reactions to the results of the Third International Mathematics and Science Study (e.g., National Research Council, 1996), teachers need to shift their attention from covering lots of topics superficially to addressing a few key topics in depth. At the same time, capitalizing on technological innovations requires special attention, as addressed later.

Among the issues that the handbook authors were invited to consider are the following:

- What is the nature of the key topics that should be addressed in depth? What conceptual models, technologies, principles, and reasoning processes exist or are needed to deal effectively with these topics?
- What understandings and strategies do teachers need to develop with respect to the above?
- What understandings and strategies do teachers need to acquire with respect to the nature and form of students’ learning and development in mastering these new topics?
- To what extent are teachers presently developing the above understandings and strategies? What research is needed here?
- What are the implications for preservice teacher education and the professional development of teachers?
Another important issue pertaining to teachers’ knowledge base is addressed by Ruthven (chapter 23) in section 3 of this text. Ruthven sees the knowledge base for teaching as drawing on both scholarly knowledge constructed through the practice of researching and on craft knowledge created within the practice of teaching. He tackles the difficult methodological question of how “greater synergy can be fostered between these distinctive practices, their characteristic forms of knowledge, and the associated processes of knowledge creation” (p. 581).

**Issues Related to Learning Contexts.** The learning context cells of the matrix (see Table 1.1) incorporate mathematical tools and representations, instructional programs, and learning environments. Attending to each of these is essential in democratizing access to powerful mathematics. In their work with SimCalc, for example, Kaput and Roschelle (1999) have demonstrated how the use of computational media can provide young students with access to mathematical ideas and forms of reasoning that traditionally are considered beyond their level. However, as Roschelle et al. (2000) pointed out, democratic access to important mathematical ideas is not just a matter of choosing the right media, but of creating the appropriate learning conditions where students develop their capability to solve and understand increasingly challenging problems. This requires the careful reformulation of curricula.

With respect to learning contexts in lifelong access to powerful mathematics, the authors were invited to consider the following issues:

- What kinds of mathematical tools and representations, including computational media, are needed to promote all students’ access? How should these tools and representations be implemented within the students’ learning environment? What research is taking place, and what research is needed?
- With respect to program development, what innovative instructional programs have been designed to promote lifelong access to powerful mathematical ideas? What are the special features of these programs? Are these new programs a significant advance on previous ones? For example, do recent programs offer more long-term coherence and more time for the development of the major strands of mathematical ideas? Are both the development and implementation of new programs couched within a sound theoretical framework? What research into program development is needed?

With respect to learning environments, the following issues were raised for consideration:

- What kinds of environments are needed to promote this democratic access, that is, to encourage students to develop, test, extend, or refine their own increasingly powerful understandings?
- What might recent research in learning environments in other disciplines offer mathematics education?
- How might we draw upon other disciplines, such as philosophical inquiry, in addressing research needed for opening our learning environments to extend all children?
- What theoretical models are being developed or need to be developed with respect to issues related to learning contexts?

**Advances in Research Methodologies**

Research in mathematics education has undergone a number of major paradigm shifts, both in its theoretical perspectives (e.g., from behaviorism to cognitive psychology) and in its research methodologies (e.g., from a focus on quantitative experimental
methodologies where controlled laboratory studies were the norm, to qualitative ap-
proaches where analyses of mathematical thinking and learning within complex social
environments have been possible; see Schoenfeld, chapter 18 of this volume). As a
consequence, we have seen over the years a multitude of thought-provoking inter-
pretations of mathematical thinking, learning, and problem solving. As Schoenfeld
highlights in his chapter, the phenomenal growth of research methodologies over
the past couple of decades has been largely chaotic, making it imperative that we
critically analyze our foundational assumptions, our methods of investigating var-
ious empirical phenomena, and our ways of providing warrants for the claims we
make:

As is absolutely characteristic of young fields experiencing rapid growth, much of the
early work has been revealed to be seriously flawed . . . unarticulated theoretical biases
or unrecognized methodological difficulties undermined the trustworthiness of a good
deal of work that seemed perfectly reasonable at the time it was done. This should
not cause hand wringing—such is the nature of the enterprise—but it should serve as
a stimulus for devoting seriously increased attention to issues of theory and method.
As the field matures, it should develop and impose the highest standards for its own
conduct. (Schoenfeld, chapter 18, p. 484).

In addressing ways of advancing methodologies for increasing our knowledge of
learners, teachers, and learning contexts, I recommended that the handbook authors
reflect on the following issues:

- What is the current status regarding our methodologies, and what changes are
  needed?
- What are the new and emerging methodologies that have import for mathematics
  education research?
- How can mathematics education develop more of its own research paradigms
  rather than “borrow” from other disciplines? What might be the nature of these
  research paradigms?
- What developments are taking place in complex research systems, such as those
  involving the interactions among the development of students, teachers, curricu-
lum materials, and instructional programs?
- What developments are needed in the criteria that we use to optimize and judge
  the quality of our research results?
- What developments are needed in research designs that increase, rather than
decrease, important links between researchers and practitioners?
- What are some examples of research projects with innovative designs that are
  having a significant impact on the development of students, teachers, curriculum
  materials, or instructional programs?

One of the many challenges we face in the advancement of our research methodolo-
gies is how best to capture the impact of technology on mathematics education. This
challenge is increased as we witness the growth in technology outstrip developments
in school mathematics programs.

Influences of Advanced Technologies

The third priority theme presents many challenges for mathematics educators as
they try to keep abreast of, and capitalize on, the rapid advances in technology. Of
fundamental concern is the question of how we can make maximum use of these
 technological developments in teaching and learning, as well as in the management
of communication among the various stakeholders. We need to be more innovative in
the ways we use technology in the teaching of mathematics. As Roschelle et al. (2000)
emphasized, “Routine applications of technology will not meet the order of magnitude of challenges we face in bringing much more mathematics learning to many more students of diverse backgrounds” (p. 72). There is much research that needs to be done on technological advances in mathematics education, including the design and implementation of appropriate learning experiences and how they impact on the development of both students and teachers.

**Issues Related to Learners.** “...the mere availability of powerful, globally connected computers is not sufficient to insure that students will learn, particularly in areas that pose considerable conceptual difficulties such as in science and mathematics” (Jacobson & Kozma, 2000, p. xiii).

Appropriately designed technological tools, incorporating reconstructed curriculum content, provide students with opportunities to both enhance their mathematical understandings and to “creatively construct, authentically experience, and socially develop and represent their understanding” (Jacobson, Angulo, & Kozma, 2000, p. 2). In our efforts to make optimal use of advanced technologies, we face significant issues pertaining to learners’ interactions with technological tools. Among the issues presented to the authors for consideration are the following:

- What is the impact of technological advances on the ways in which students learn mathematics? the content of what students learn? the mathematical content that becomes accessible to students? the types of learning situations students are able to handle?
- How are technological advances changing students’ thinking processes? visualization skills? communication skills? representational skills? abilities to research and solve problems? mathematical understandings? mathematical achievements? mathematical self-awareness (e.g., their perceptions of mathematics, attitudes towards the subject, self-confidence)? abilities to interact productively with one another, with teachers, and with data?
- What theoretical models are emerging or need to be developed with respect to the above issues?

**Issues Related to Teachers.** Technological advances afford teachers many opportunities to appropriate, apply, and implement new technological learning experiences within their classroom. These opportunities are not being seized in many mathematics classrooms, however, where teachers remain apprehensive about using technological tools to foster innovative, inquiry-based approaches to learning (Blumenfeld, Fishman, Krajcik, Marx, & Soloway, 2000; Mariotti, chapter 28 of this volume). This perhaps is not surprising when we think about the various factors that need to be addressed in implementing technological advances within the mathematics curriculum. As Lesh and Lovitts (2000) noted,

One reason for this lack of impact is that intended technological innovations have tended to be superimposed on existing sets of practices which were taken as given. Yet, it is clear that realizing the full potential of new technologies must be systemic. It will require deep changes in curriculum, pedagogy, assessment, teacher preparation and credentialling, and even the relationships among school, work and home. (p. 70)
Technological innovations clearly make a number of demands on teachers and the learning environments in which they work. Research agendas need to consider how we might best meet the professional needs of teachers at both the undergraduate and graduate levels. The chapters in section 4 address many issues that warrant substantial research, including the following:

- How are teachers capitalizing on opportunities provided by the latest technological tools to promote active learning for all students?
- How are teacher education programs and professional development programs capitalizing on the opportunities provided by technology?
- How are teachers using technology in their assessment practices?
- What is the impact of technological advances on teachers’ perceptions of mathematics? the ways in which teachers structure mathematical learning experiences for their students? teachers’ personal and professional growth? teachers’ local, national, and international collaborations?
- What theoretical models are developing or need to be developed with respect to teachers working with technology?

**Issues Related to Learning Contexts.** Information technology must be embedded carefully and thoughtfully within the overall design and implementation of a particular learning environment. As the handbook authors and others have emphasized, it is insufficient to simply take existing curricula and add on some technology. Subject matter and pedagogical reconstruction must accompany technological innovation (Lesh & Lovitts, 2000; Roschelle et al., 2000; Bottino & Chiappini, chapter 29 of this volume). The words of Bottino and Chiappini are apt here: “It is pointless from a pedagogical point of view to make computers available at school if the educational strategies and activities the students engage in are not suitably revised” (p. 759). Bottino and Chiappini make the important point that the introduction of information and communication technologies in education has often been linked to a view of learning as an individual process, whereby knowledge develops from the interaction between the student and the computer. This is not surprising, given that the literature frequently refers to educational software applications as “learning environments,” thus emphasizing the fact that it is the software itself, through interaction with the student, that forms the environment where learning is to take place.

This point was highlighted in the 1980s by Pea (1987), who demonstrated that the value of a software tool for mathematics learning does not depend solely on its inherent features but also on the context in which the activity takes place. Conversely, the learning context can have a powerful impact on the technology itself. Many of the latest developments in technology have yet to be realized fully in the educational arena; the realization of their potential is governed in large part by the learning contexts in which they might be used.

Issues pertaining to technology and learning contexts include the following:

- How are new technological tools provoking or initiating different learning environments?
- What theoretical models are emerging or need to be developed that address technology and learning environments?
- How are technological developments changing perceptions of what mathematics should be taught and learned?
- How are technological developments changing the nature of mathematics as a discipline and as an applied domain?
• What developments are taking place in mathematics curricula that exploit new technological tools?

CONCLUDING POINTS

This introductory chapter has provided the background and context for the chapters that follow. In closing, it is worth returning to the overall goal of this handbook, namely, to advance the discipline of mathematics education in both theory and practice. In achieving this goal, it is imperative that we rethink the nature of the mathematics and the mathematical experiences we are providing our young generation of learners, the future of our nations. We cannot simply transport the mathematics of the last century into today’s curricula and assume that we are equipping learners with the mathematical power needed for their success in the 21st century. We must give careful thought to what aspects of 20th century mathematics should be discarded, what should be retained, what should be modified, and what new ideas and experiences should be incorporated. Importantly, these curriculum decisions need to be informed by sound research.

In particular, we cannot afford to miss the opportunities offered by advancements in technology. As Kaput et al. demonstrate in chapter 4, “Computational media have provided a next step in the evolution of powerful, expressive systems for mathematics” (p. 73). As a consequence, new domains of mathematical knowledge are becoming available to a greater cross-section of society, giving people the intellectual power to solve problems that were previously the domain of an elite minority. Two challenges remain, however, for mathematics education. First, there is the urgent need to increase access to technology in learning environments around the globe. The provision of adequate technological resources remains a major challenge for educational systems in both developed and underdeveloped nations. Second, when adequate technological resources are available, their power and potential must be exploited in mathematics education. Research that addresses the new and emerging representational infrastructures to which Kaput et al. refer is essential in informing curriculum development that enables learners to use, modify, and create new systems of expression.

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A society in which only a few have the mathematical knowledge needed to fill crucial economic, political, and scientific roles is not consistent with the values of a just democratic system or its economic needs. (National Council of Teachers of Mathematics, 2000, p. 5)

Can you imagine a society where most people are engaged in working for the good of society? Can you envision a place where students volunteer in social justice and equity projects because they want to rather than because it is a school requirement? Can you imagine having students who want you to explain a mathematics problem because it will help them to help others? In my view, this is the energy and outreach that democratic access to powerful mathematical ideas generates. An ideal education in which students have democratic access to powerful mathematical ideas can result in students having the mathematical skills, knowledge, and understanding to become educated citizens who use their political rights to shape their government and their personal futures. They see the power of mathematics and understand that they can use mathematical power to address ills in our society. Education of this sort addresses political aspects of democratic schooling, the social systems of nations, and often has as its focus the social betterment of nations and the world (Beyer, 1996). The crux of democratic access to mathematics is our understanding and researching new ways to think about mathematics teaching and learning that has a moral commitment to the common good, as well as to individual needs. This is democratic education.
Democratic education is important for social justice and equity in our world where, at the present time, they do not always prevail. An additional benefit of such education is that it provides mathematical access to all students because it is inclusive of all cultures and students rather than exclusive to cultures and students who historically have had access. Democratic education is collective in its goals and individual in opportunities for student participation. As a result, it is emancipatory for all students.

Even with all of its positive benefits, democratic access based on principles of democratic education is not prevalent in mathematics education. Why is this the case? Is it because it is too difficult, because we have not considered attempting it, or because we have little knowledge of it? Regardless of the reason, the uneven mathematics achievement and understanding of students worldwide implores mathematics educators to investigate new options for teaching mathematics. The old phrase—if we always do what we always did, we will get what we always got—beckons us to consider democratic education as a process to help students understand and use mathematics.

In this introductory chapter to the section on democratic access to powerful mathematical ideas, I ask readers to consider three benefits of democratic education in mathematics: inclusiveness, mathematics understanding, and application of mathematics to problems in social justice and equity. To do this, I will first present a brief history of the need and development of powerful mathematical ideas. Because I am a mathematics educator in the United States, the context for development will be from my worldview. Second, I will relate the development of powerful mathematical ideas to tenants of democratic education. Finally, I will discuss the benefits of social justice and equity to mathematics education.

**POWERFUL MATHEMATICAL IDEAS**

**Importance**

Democratic access to powerful mathematical ideas, as presented above, is achieved through the preparation of a populace for participatory citizenship; but it also addresses who receives the education and to what degree. The idea of children having democratic access to powerful mathematical ideas is a human right, and it is important to the future of our international society. Children, families, and teachers often establish goals in education that include intellectual growth or learning for personal intellectual benefits. Leonard (1968) captures the spirit of learning for intellectual benefits, explaining that education is “achievement of moments of ecstasy. Not fun, not simply pleasure . . . but ultimate delight” (p. 17). But learning mathematics for intellectual pleasure is not widespread; it tends not to happen for most children. Even though most first- and second-grade students identify mathematics as their favorite subject in school, many students reject it before they leave elementary school (K–5). They either remove themselves or are removed from challenging programs in mathematics by the end of middle school (6–8). Reasons that students reject or are rejected by mathematics are numerous including teacher or societal perceptions of ability, cultural discontinuity in learning and instruction, tracking, poverty and school finance, and low expectations (see Tate and Rousseau, chapter 12 of this volume). As a result, many students rarely experience the delight of or become enculturated into learning mathematics (Malloy & Malloy, 1998).

Instead of universal enculturation, students are educated in a world that concentrates on differences, which consciously and unconsciously separate the rich and poor, educated and noneducated, leaders and followers, and racial and ethnic groups. The dichotomy of difference is based on the colonial nature of education (Willinsky,
Overwhelmingly school politics and policies are based on and conducted in the name of imperialism’s intellectual interests—the interests of “gentlemanly capitalists” (Willinsky, 1998). These colonial legacies often lead to the practice of normalizing self and stereotyping others (Kubota, 2001), the practice of differentiating to keep some students outside of the mathematics mainstream. Often this practice is a systematic effort to reproduce injustices by influencing the thoughts and thus the beliefs of the people who will control and populate the West and East (Willinsky, 1998). This thinking results in elitism, not democratic access. In the United States, the preservation of differences results in minority, poor, and other disenfranchised groups having lower levels of tangible resources for education, reduced access to qualified teachers and high-quality mathematics teaching, an overrepresentation in low tracks and special education, limited access to powerful mathematical ideas, high rates of retention and dropouts, and dysfunctional school environments (Darling-Hammond & Ancess, 1996). Overcoming this history of limited mathematics opportunity for some populations and making schools work for these populations is the primary challenge educators face today (Seeley, 1999; Shade, 1997).

Many educators are aware of the inequities propagated through difference-based exclusion, causing inequities in mathematics education to be the impetus for reform in mathematics teaching and learning. Internationally, educators are trying to provide all students with access to strong mathematics programs by changing the culture of learning in the mathematics classroom. At the 1998 University of Chicago School Mathematics Project (UCSMP) International Conference on Mathematics Education, speakers from throughout the world spoke of initiatives their countries were establishing to ensure that all students have access to mathematics (Usiskin, 1999). Amit (1999), in proceedings of this conference, quotes a section of a report from the Israeli national committee on education that is indicative of many international documents:

> Mathematics, the natural sciences and technology are growing in importance, especially for future scientists in the coming millennium. Hence, it is our duty and privilege, as educators, to provide all students with mathematical knowledge and thinking processes, so that they may be fruitful, constructive citizens in a democratic society. (p. 23)

Certainly educational and governmental leaders worldwide understand a clear need for democratic access to powerful mathematical ideas.

**Development**

The development of powerful mathematical ideas in school mathematics has been a historic goal of U.S. governmental officials, mathematics educators, and parents since 1779, when mathematics was placed in the public school curriculum (Amit, 1999; Spring, 1994). At that time the U.S. president, Thomas Jefferson, proposed a plan in which male children would be educated in Latin and Greek languages, English grammar, geography, and numerical arithmetic including decimal fractions and square and cube roots. The assumption was that without understanding powerful mathematics of that time, the society and government could not be sustained and would not progress. In recent history, since the 1960s, this same assumption has been the basis of most reforms in mathematics education and impacts the mathematics offered to children in schools today. The “new math” movement of the 1960s, led mainly by mathematicians, was initiated to improve the precollege teaching and learning of mathematics for college-intending students and was used widely in schools. The intent was to increase the number of students who were prepared to enter college as mathematicians, thus helping to sustain and expand the number of mathematicians in the
United States. The “new math” vision of mathematics was a different conception of school mathematics, based on axiomatic methods. It triumphed for a decade; but technology and new conceptions of learners necessitated changes in the mathematics curriculum (McLeod, Stake, Schappelle, Mellissinos, & Gierl, 1996).

In the 1980s, because parents and teachers were afraid that the new mathematics was not giving children basic mathematical knowledge, “basic skills” became the public goal of the mathematics curriculum. Unfortunately, the teaching of basics narrowed the curriculum to the learning of facts, process, and algorithms. Schools were saddled with a curriculum that taught children how to compute but not to understand mathematics well enough to solve nonroutine problems. Additionally, this narrow view of teaching and learning was counter to a growing body of research on teaching and learning mathematics (McLeod et al., 1996).

In the United States and Canada, after considerable deliberation by mathematics organizations and individuals, mathematics educators envisioned a curriculum that would encompass both the learning and understanding of mathematics. The National Council of Teachers of Mathematics (NCTM) undertook the development of a curriculum that would deliver a powerful mathematics to children (see McLeod et al., 1996 for more information). The NCTM Curriculum and Evaluation Standards for School Mathematics (1989) were published with a vision of school mathematics that contained the processes necessary for the doing of mathematics, a curriculum that included geometry, statistics, and probability at all grade levels and stressed the connectivity of all mathematics. The development of the standards was influenced by the work of international mathematics educators from England, Australia, Brazil, and The Netherlands, to name a few (McLeod et al., 1996).

Mathematics educators around the world were not alone in their concern for strong mathematics programs in schools. In the early 1990s, many governments, including those of Israel, Japan, China, Egypt, the United States, Canada, and South Africa, had national goals or programs to deliver strong mathematical programs to all students (Amit, 1999; Ebeid, 1999; Er-sheng, 1999; Hashimoto, 1999; Volmink, 1999). These and other countries also reacted with reforms in mathematics education after the results of the Third International Mathematics and Science Study (TIMSS) were distributed. In 2000 NCTM released the second version of standards, the Principles and Standards for School Mathematics, a document that broadened the vision of school mathematics and challenged educators to work to educate all students.

The mathematics taught to school children today is based on the varied emphases of mathematics reforms or shifts in the past. The reoccurring themes of international mathematics programs presented at the 1998 UCSMP conference include basic skills, conceptual understanding, numeracy, meaningful mathematics, creativity, positive disposition, reasoning, representation and modeling, communication, application, social development, cultural context, integration of technology, and general mathematics literacy. The topics covered in mathematics curricula include variations of number, algebra, geometry, measurement, and statistics and probability. Internationally there seems to be general agreement on the essentials of strong mathematics curricula (Usiskin, 1999).

Some Components of Powerful Mathematical Ideas. It would be presumptuous to believe that one mathematics educator can determine the components of powerful mathematical ideas; however, using the emphases in international mathematics programs and NCTM Standards as a collective reference, it is possible to posit some of the components. In the United States, the NCTM Standards (2000) proposes student proficiency in basic skills and conceptual understanding of those skills across the mathematical topics of number, algebra, geometry, measurement, and data and statistics as a powerful mathematics curriculum. It recommends the use of the
mathematical processes of problem solving, reasoning (including recognition of patterns, conjecturing, generalizing, and formalized proof), connections among topics within and outside of the field mathematics, communication about mathematics, and different representations in mathematics and across content areas. Implied throughout this document and similar international proposals is the need for students to understand the mathematics they use, to have a positive disposition toward the discipline, to understand and use technology to help solve problems, and to have habits of mind or mathematical ways of thinking that focus on making sense of the world (Cuoco, 1998; NCTM, 2000, Usiskin, 1999). Not to be excluded from the list are mathematical experiences that allow students to be creative and flexible, that help students become critical thinkers and decision makers who value others’ opinions, and that show students the utility of mathematics by using mathematics in context (Malloy & Jones, 1998; NCTM, 2000; Romberg & Kaput, 1999).

DEMOCRATIC ACCESS

The commitment to accomplish the goals of educators and governments raises a plethora of broad questions regarding democratic access to powerful mathematical ideas: Can we teach all children powerful mathematics? What does teaching mathematics for democratic access to powerful mathematical ideas look like? Is it possible to identify all powerful mathematical ideas? Are social justice and equity viable methods for teaching powerful mathematics? How these questions are answered will certainly affect the ability of children to gain access to powerful mathematical ideas and the understanding, skills, and processes necessary to sustain themselves in the 21st century. Democratic education, in classrooms that promote social justice and equity, must be considered to help answer these questions. Democratic education is accessible to all students, provides students with an avenue through which they can learn substantial mathematics, and can help students become productive and active citizens.

Democratic education is a process where teachers and students work collaboratively to reconstruct curriculum to include everyone. Each classroom will differ in its attributes because the interactions of democratic classrooms are based on student experiences and community and educational context. Just as this occurs in democratic classrooms, it occurs in mathematics classrooms. There is no one way or context through which mathematics is taught. There are concepts, topics, and processes that must be taught and learned, but individual teachers and learners will approach mathematics based on their needs, preferences, and experiences.

The literature on democratic education consistently identifies distinguishing qualities of democratic classrooms to include (a) problem-solving curriculum, (b) inclusivity and rights, (c) equal participation in decisions, and (d) equal encouragement for success (Beyer, 1996; Pearl & Knight, 1999; Wilbur, 1998). These qualities do not define the curriculum, but they are the rationale for classroom interactions and discussions of overriding issues and questions through the use of specific and integrated knowledge of content areas. Below, I briefly describe four qualities through a compilation of the work of Beyer (1996), Pearl and Knight (1999), and Wilbur (1998) in terms of the mathematics.

1. Problem-solving curriculum. Students should be presented with a curriculum in mathematics that allows them to draw on their accumulated knowledge to solve problems important to their lives and to society. They should have experiences that help them to locate relevant information and visualize multiple representations to access new meanings. Through a process of collaboration, they should have experiences that develop their ability to analyze, critique, and evaluate mathematical options.
2. **Inclusivity and rights.** Students should be taught using approaches that provide a range of opportunities for accessing and processing mathematical ideas. Mathematics should be presented from multiple perspectives affirming individuals and groups of the worth of diverse experiences and approaches in solving problems.

3. **Equal participation in decisions that affect students’ lives.** Students should be able to use the mathematics classroom as a forum for public discussion of issues and ideas, because through such discussion students are able to create, clarify, and reevaluate their ideas and understand the ideas of others. Students should be adept at communicating their mathematical ideas to others in a process of accuracy, persuasion, and negotiation.

4. **Equal encouragement for success.** Students should have access to materials that engage them actively in the learning of mathematics. They should be encouraged equally as they develop the habits of mind to draw conclusions and critically evaluate implications from mathematical data for personal and social action.

The learning experiences and processes in these four qualities resemble the goals of many reformed mathematics programs. Democratic education does not concentrate on just the social studies curriculum and exclude teaching and learning of basic concepts and processes of mathematics. It requires the teaching of basic skills and understanding across mathematical topics. Democratic education requires that a democratic citizen is mathematically literate. Pearl and Knight (1999) clearly stated the importance of mathematical knowledge: “It is impossible to be a democratic citizen and not be proficient in mathematics. Every decision that a citizen must make requires complicated calculations” (p. 119). The important addition is that it demonstrates the utility of mathematics through problem solving.

**THE BENEFITS OF SOCIAL JUSTICE AND EQUITY IN MATHEMATICS EDUCATION**

Mathematics education can benefit from a democratic approach to education in at least three important ways: inclusiveness, mathematics understanding, and application of mathematics to problems in social justice and equity. An example of these three benefits was experienced by the Algebra Project, developed and run by Bob Moses. Moses (1993) spoke of the needs of African American students in the United States and explained that in a time when mathematics literacy is a must for survival, it is necessary to raise the ceiling of opportunity for all students. He explained that giving students the opportunity to learn algebra is a civil rights issue. Without this knowledge children are limited in their opportunity to become full participants in the world. More important, he spoke of the difficulty he had convincing students of the importance of learning algebra and other academic mathematics courses. They saw no value in learning algebra. He was aware that social action initiatives could mobilize students by answering questions such as, “Why do I need to learn this?” Using a metaphor from the history of social justice in the United States during the 1960s Civil Rights Movement, he was able to help students understand their responsibility to gain access to powerful mathematical ideas. The metaphor was embedded in the mathematics of “one-person one vote.” He explained that the process of convincing large numbers of sharecroppers that their votes could change the political landscape in Mississippi was difficult, but he and other civil rights workers were successful, and the dream of equal participation was realized. Using problem solving as a motivation for learning and applying mathematics, students understood that mathematics was a tool to help learn and solve problems of the poor and powerless.

After students understand the power of mathematics and how it can be applied to other situations, they are able to expend their ability to use mathematics for social
justic and equity. Secada and Berman (1999), writing of equity in understanding school mathematics, stated that primary students can and do use the school store to learn about addition and subtraction with money, maintaining an inventory, and computing profit and loss in relation to expenses. But they also suggest primary students could consider a different context to learn addition and subtraction. They could compute the cost of baby-sitting, childcare, or tutoring charged by a social agency that uses a sliding scale based on ability to pay. The mathematics is similar, but the problem used is moved to a context that many students’ families experience.

High school and middle-grade students can investigate the economics of politics resulting in the placement of dumps or hazardous waste plants in poor neighborhoods or in the transference of farmers’ lands in developing countries to large international corporations for agribusiness (Apple, 1996; Malloy & Malloy, 1998). Pearl and Knight (1999) itemized several pressing social issues that could become the problem-solving curriculum of democratic schools, including (a) an ecologically sustainable society; (b) an economy that meets human needs while achieving full, fair, and gratifying employment; (c) elimination of world poverty; and (d) marshalling technology for socially useful purposes. These are large difficult problems, but each issue could be broken into subproblems based on the needs of students’ communities and investigated by students in school mathematics courses.

Mathematics educators must learn about democratic education and move it into their classrooms. Darling-Hammond and Ancess (1996) stated, “Education for democracy requires more than equal access to technical knowledge. It requires access to social knowledge and understanding forged by participation in a democratic community” (p. 166). Within the democratic classroom, children should see themselves in the curriculum and link mathematics to their everyday lives; they should see that mathematics is connected to social needs of the community and that mathematics can expand and deepen the democratic possibilities for equity in mathematics (Hanson, 1997; Ladson-Billings, 1994; Malloy & Malloy, 1998; Mark & Hansen, 1992; Tate, 1994; Woodrow, 1997). These are benefits to the mathematics education and learning of all students.

Democratic access to mathematics for social justice confronts students with moral issues for a common good that are related to mathematics. As students become aware of social justice, we must present them with problems that not only tackle issues affecting their communities, but also reveal the motivations and the hidden agenda (curriculum) in their world. These agenda are prevalent and support the social structures within all communities. When students use and apply mathematical knowledge in such situations, they are learning to think critically about world issues and their environment through mathematics. Through this process students will have an understanding of inequities in society and will be able to critique the mathematical foundations of social situations—a skill that they will take through their lives. The critique leads to emancipation—mathematics as a tool to use the present to shape the future instead of the future to shape the present. Emancipation helps students to become aware of social inequities and to understand the motivation for policy decisions and solutions. This is the beginning of social action.

Clearly the intent of exploring research for democratic access to powerful mathematical ideas, implies that we are searching for ways to enable mathematics teachers, teacher educators, and researchers to pursue new avenues to provide children with a strong and worthwhile mathematical education that will serve them throughout their lifetimes. This is necessary because although mathematics educators and related governmental bodies throughout the world have been working to provide children in their countries with a strong mathematics education, the results of the TIMSS indicate that all students are not being afforded a mathematical education that accomplishes this goal. If this continues to happen, we will continue to have disenfranchised students who do not want to learn or use mathematics and capable students who do not
know how to use the mathematics they know to solve social justice problems. We are confronted with a situation that requires a different approach.

Democratic access to powerful mathematical ideas is a politically charged right of every child. Mathematics educators must ensure that it occurs. We must have goals for educational programs, processes that motivate reform in mathematics education, specific actions that enable children to think critically about the use of mathematics in their lives, and mathematics for social change. Students should receive a mathematics education that is inclusive and prepares them for tomorrow; they should receive an education that enables them to learn powerful mathematics and to be citizens in a society where their knowledge—especially their mathematical knowledge—can help determine their futures and the future of their world.

Many issues have been raised in this chapter, and many more are raised in section 2 of this handbook. The authors hope these chapters will provide readers with increased understanding of the mathematical ideas required to help learners solve problems in their lives and in their workplaces, to help them develop scientific and technical advancements, to guide them in making decisions about economic and social justice issues, and to help them understand and appreciate the mathematics indigenous to their culture. These are the powerful mathematical ideas that require democratic access for all students.

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During the closing years of the 20th century, a number of books and articles were published describing the status of research in mathematics education and discussing possible ways to increase its usefulness (Romberg, 1992; Sierpinska & Kilpatrick, 1998; Steen, 1999). Most of these publications focused on summaries of past research, or, insofar as they shifted attention toward the future, they stated the authors’ views about problems or theoretical perspectives that (they believed) should be treated as priorities for future research. Should teachers’ decision-making issues be treated as higher priorities than the decision-making issues that confront policymakers or others who influence what goes on in classroom instruction? Should issues of equity be given priority over issues of content quality or innovative uses of advanced technologies? Should theoretical perspectives be favored (for funding, publication, or presentations at professional meetings) if they are grounded in brain research, or artificial intelligence models, or constructivist philosophies? Should quantitative research procedures be emphasized more than qualitative procedures (or vice versa)? My own prejudices about such issues are not central concerns of this paper. Instead, I’ll address the following question: “What kind of research designs have proven to be especially useful in mathematics education, and what principles exist for improving (and assessing) their usefulness, power, shareability, and cumulativeness?”

Special attention will be given to a category of research methodologies called “design experiments” (Frecktling, 1998), so called because the goal is for participants (whose ways of thinking are being investigated) to design thought-revealing artifacts using a process that involves a series of iterative testing-and-revising cycles. Thus, as participants’ ways of thinking evolve and become more clear, by-products of this design process include auditable trails of documentation that reveal important aspects about developments that occur. Examples will be given from Purdue’s Center for Twenty-first Century Conceptual Tools (TCCT), where a primary goal is to
investigate the following question: What mathematical abilities will be basics for success beyond school in a technology-based age of information?

**WHY FOCUS ON RESEARCH DESIGN?**

In general, the reviews of research that I’ve referred to above suggest that (a) mathematics education research has made far less progress than is needed, and (b) little attention has been given to many of the most important issues that are priorities for practitioners to address. I don’t dispute these claims. However, speaking as a firsthand witness to many of the most significant events in the birth of mathematics education as a field of specialized scientific inquiry, I am far more impressed with its achievements than concerned with its shortcomings. Also, I am impressed by how often criticisms leveled against research come from people who don’t do any empirical research precisely because their conception of research is that of an enterprise that is not worth doing or from people who have had more than their share of influence on the policies adopted by professional and governmental organizations that have restricted what kinds of research receives favored treatment in professional publications, conference presentations, and funded projects.

I believe that research should be, above all, about knowledge development. Furthermore, I consider it to be obvious that, during the past quarter of a century, if any progress has been made in projects aimed at curriculum development, software development, program development, or teacher development, then it is precisely because more is known (see Kelly & Lesh, 2000, Part VI). Or, in cases where little progress has been made, it is precisely because too few attempts have been made to increase the usefulness, shareability and cumulativeness of what is known. Consequently, I consider research to be a worthwhile enterprise; at the same time, I’m willing to admit that most research publications, most conference research presentations, and most funded research projects do little to advance what is known about the kind of complex systems that are the most important for mathematics educators to be able to understand and explain.

To anybody who was intimately involved in mathematics education research during the past two or three decades of the 20th century, it is difficult to see how they could fail to notice the following trends (see Kelly & Lesh, 2000, Part I).

- Enormous progress has been made concerning our collective understandings about the nature of children’s developing mathematical knowledge—and about the nature of effective teaching, learning, and problem solving in topic areas ranging from early number concepts, to rational number concepts, to early algebraic reasoning. Furthermore, it is obvious that these new ways of thinking have provided primary driving forces behind many of the most successful attempts at standards-based curriculum reforms. In fact, for populations of “students” ranging from children through adults (or teachers), and for subject matter areas ranging from arithmetic to calculus (or from science to social studies), the effectiveness of curriculum reforms has tended to be directly related to the depth and breadth of the research base on the nature of students’ knowledge. In particular, when the research base is least detailed and least extensive, curriculum reforms have tended to be least effective.

- Enormous progress has been made to shift beyond theory borrowing (from fields such as developmental psychology, artificial intelligence-based cognitive psychology, or more recent developments in brain research) toward theory building (where problems, tools, research literature, theoretical models, and research procedures are distinctive to the field of mathematics education). Whereas in the past, mathematics educators conducted Piagetian research, Vygotskian research (or research that is based on psychometric models, or information processing models, or artificial intelligence
3. RESEARCH DESIGN

models—where both the theoretical models and the research methodologies were borrowed from these other fields), today—in topic areas ranging from early number concepts, to rational number concepts, to early algebraic reasoning—examples abound in which mathematics educators have developed their own distinctive theoretical models, conceptions of critical problems, research literature, research tools and procedures, and (most important) communities of inquiry. A variety of examples can be found in a new book titled Beyond Constructivism: A Models & Modeling Perspective on Mathematics Problem Solving, Learning & Teaching (Doerr & Lesh, 2002), which includes chapters by more than 40 leading mathematics and science educators.

Unfortunately, the development of widely recognized standards for research has not kept pace with the development of new problems, new perspectives, and new research procedures. Consequently, there is a growing concern among active researchers in the field that a crisis has arisen that threatens to impede future progress. The crisis arises because, when there is a lack of clarity about appropriate principles for optimizing (or assessing) the quality of innovative research designs, three kinds of undesirable results are likely to occur when proposals are reviewed for funding or when manuscripts are reviewed for professional publications or presentations. First, potentially significant studies may be marred by methodological flaws. Second, high-quality studies may be rejected because they involve unfamiliar research designs, because inadequate space is available for explanation, or because inappropriate or obsolete standards of assessment are used. Third, conceptually flawed studies may be accepted because they employ traditional research designs, even though these research designs may be based on naive or inappropriate ways of thinking about the nature of teaching, learning, and problem solving or about the nature of program development, dissemination, and implementation.

To develop productive ways of dealing with preceding difficulties, the National Science Foundation (United States) recently supported a project that resulted in the Handbook of Research Design in Mathematics & Science Education (Kelly & Lesh, 2000). This handbook includes chapters written by more than 40 leading researchers in mathematics and science education, and it emphasizes research designs that have been pioneered by mathematics and science educators, have distinctive characteristics when used in mathematics or science education, or have proven to be especially productive in mathematics or science education. Examples of such research designs include several different types of teaching experiments, as well as distinctive types of clinical interviews, videotape analyses, and naturalistic observations, as well as a variety of action research paradigms in which participant-observers may include not only researchers-acting-as-teachers or teachers-acting-as-researchers but also curriculum designers, software designers, and teacher educators whose aims include both optimizing and understanding mathematics teaching, learning, or problem solving. In general, these new research designs draw on multiple types of quantitative and qualitative information; the knowledge-development products they produce often are not reducible to tested hypotheses or answered questions, and they often involve cyclic and iterative techniques in which participant-researchers include a variety of interacting students, teachers, and other mathematics educators. Finally, and most important from the point of view of this chapter, they often involve new ways of thinking about the nature of students’ developing mathematical knowledge and abilities and new ways of thinking about the nature of effective teaching, learning, and problem solving.

The purpose of the Handbook of Research Design was to clarify the nature of some of the most important experience-tested factors that should be considered to improve (or assess) the usefulness, power, shareability, and cumulativeness of the results that are produced when innovative research designs are included in proposals for research projects, publications, or presentations at professional meetings. Of course, from the
beginning of the project, participants were mindful of the fact that if obsolete or otherwise inappropriate standards are adopted, then the results could hinder rather than help. Nonetheless, as long as decisions must be made about funding, publications, and presentations, it is not possible to avoid issues related to quality assessments. Decisions will be made. Therefore, our goal was to attempt to increase the chances that appropriate issues will be considered and that productive decisions will be made.

TWO FACTORS INFLUENCE RESEARCH DESIGNS THAT ARE DISTINCTIVE IN MATHEMATICS EDUCATION

In the project described in the preceding paragraphs, two factors emerged as having especially strong influences on the kind of research designs that mathematics educators have pioneered. First, most of these research designs have been intended to radically increase the relevance of research to practice, often by involving practitioners in the identification and formulation of problems to be addressed, in the interpretation of results, or in other key roles in the research process. Second, there has been a growing realization that, regardless of whether researchers focus on the developing capabilities of students, groups of students, teachers, schools, or other relevant learning communities, the evolving ways of thinking of each of these “problem solvers” involve complex systems that are not simply inert and waiting to be stimulated. Instead, they are dynamic, living, interacting, self-regulating, and continually adapting systems with competencies that generally cannot be reduced to simpleminded checklists of condition–action rules. Furthermore, among the most important systems that mathematics educators need to investigate and understand are as follows: (a) many do not occur naturally (as givens in nature) but instead are products of human construction, (b) many cannot be isolated because their entire nature may change if they are separated from complex holistic systems in which they are embedded, (c) many may not be observable directly but may be knowable only by their effects, and (d) rather than simply lying dormant until they are acted upon, most initiate actions; when they are acted upon, they act back. In particular, when they’re observed, changes may be induced that make researchers integral parts of the systems being investigated. So, there may be no such thing as an “immaculate perception” (see Kelly & Lesh, 2000, Part II).

For the preceding kinds of reasons, in mathematics education—just as in more mature modern sciences—it has become necessary to move beyond machine-based metaphors and factory-based models to explain patterns and regularities in the behaviors of relevant complex systems (see Fig. 3.1). In particular, it has become necessary

From an Industrial Age
Using analogies based on hardware
Where systems are considered to be no more than the sum of their parts, and where the interactions that are emphasized involve no more than simple one-way cause-and-effect relationships.

⇒

Beyond an Age of Electronic Technologies
Using analogies based on computer software
Where silicone-based electronic circuits may involve layers of recursive interactions that often lead to emergent phenomena at higher levels that are not derived from characteristics of phenomena at lower levels.

⇒

Toward an Age of Biotechnologies
Using analogies based on wetware
Where neurochemical interactions may involve “logics” that are fuzzy, partly redundant, partly inconsistent, and unstable—as well as living systems that are complex, dynamic, and continually adapting.

FIG. 3.1. Recent transitions in models for making (or making sense of) complex systems.
to move beyond the assumption that the behaviors of these systems can be described using simple linear combinations of one-directional cause-and-effect mechanisms that are described using closed-form equations from elementary algebra or statistics.

According to ways of thinking borrowed from the industrial revolution, teachers were led to believe that the construction of mathematical knowledge in a child’s mind is similar to the process of assembling a machine or programming a computer. That is, complex systems were thought of as being nothing more than the sums or their parts, the parts were assumed to be defined operationally using naive checklists of condition–action rules, and each part was expected to be taught and tested one at a time, in isolation and out of context.

In contrast to the preceding perspectives, scientists today are investigating complexity theories in which the processes governing the development of complex, dynamic, self-organizing, and continually adapting systems are assumed to be quite different than those that apply to simple machines. Parts interact. Logic is fuzzy. Whole systems are more than the sums of their parts; and, when the relevant systems are acted on, they act back. To recognize consequences of these facts, we need only point to the fact that simple iterates of a quadratic function of a single variable can lead to chaotic data. A dozen or so iterations can yield enormously complicated phenomena, with many characteristics that are unpredictable in principle and that certainly cannot be predicted using simple closed-form algebraic equations or simple-minded “if–then” rules.

When the preceding views are adopted in mathematics education, it becomes obvious that students, teachers, classrooms, courses, curricula, learning tools, and minds are all complex systems—taken singly, let alone in combination. Consequently, these facts should have strong influences on the kind of research designs that are likely to be appropriate and productive in mathematics education, and they have equally strong influences on the nature of the criteria that are appropriate for assessing the quality of relevant research designs.

RESEARCH IS ABOUT THE DEVELOPMENT OF SHARED KNOWLEDGE

Dealing with complexity in a disciplined way is the essence of research design in mathematics education. Relevant perspectives include cognitive science, social science, mathematical sciences, and a wide range of other points of view. No single means of understanding is likely to be sufficient; no single style of inquiry is likely to take us very far; it is unlikely that relevant research can ever be reduced to a formula-based process.

Far from being a process of using “accepted” techniques in ways that are “correct,” research in mathematics education is a “no-holds-barred” process of developing shared knowledge about important issues. Doing it well involves developing a chain of reasoning that is meaningful, coherent, sharable, powerful, auditable, and persuasive to a well-intentioned skeptic about issues that are priorities to address.

In the preceding comments, I use the term research design rather than research methodology precisely because, in mathematics education, the design of research generally involves trade-offs similar to those that occur when other types of complex products (such as automobiles) need to be designed to meet conflicting goals (such as optimizing speed, safety, durability, and economy). Whereas the term research methodology tends to be associated with statistics-oriented college courses in which the emphasis is on how to carry out “canned” computational procedures for analyzing data, the kinds of situation and issues that are most important for mathematics educators to investigate seldom lend themselves to the selection and execution of off-the-shelf data-analysis techniques. Multistage combinations of qualitative and
quantitative approaches tend to be needed; it is not a choice of one versus the other. Furthermore, in addition to the stages of research that deal with data analysis, other equally important issues typically arise that involve (a) developing productive conceptions of problems that need to be solved, products that need to be produced, or opportunities that need to be investigated; (b) devising ways to generate or gather relevant information to develop, test, refine, revise, or extend relevant ways of thinking; (c) developing appropriate ways to sort out the signal from the noise in information that is available and to organize, code, and interpret raw data in ways that highlight patterns and regularities; or (d) analyzing underlying assumptions and formulating appropriate models to explain implications.

Often, the kind of research that is most needed in mathematics education is aimed at making a difference in theory or in practice. That is, it must go beyond simply providing additional accuracy or precision related to current ways of thinking and acting, and it must go beyond simply carrying out the Nth step in some ongoing program of investigation. Furthermore, even though it is aimed at solving real problems, it generally must involve more than simply problem solving (in the sense of producing unsharable solutions to isolated problems). In fact, it implicitly involves developing a community that has adopted a shared language, as well as shared models, metaphors, rules, tools, and principles. Consequently, many criteria that determine the quality of research focus on straightforward ways of assessing the extent to which its products of research are meaningful, useful, sharable, and cumulative.

PRODUCTS OF RESEARCH INCLUDE MORE THAN TESTED HYPOTHESES AND ANSWERED QUESTIONS

It often is said that “good research requires clearly stated hypotheses or research questions.” But, on close examination, this statement is at best a half-truth. For example, when we emphasize that research is about the development of knowledge, it is clear that what we know consists of a great deal more than tested hypotheses (stated in the form of “if–then” rules) and answered questions (using standardized tests, questionnaires, or other techniques leading to quantitative measures of relevant variables). Some of the most important products of research also include the following factors:

- **Models and conceptual systems** (e.g., descriptions and explanations) for constructing and making sense of complex systems. Truth and falsity may not be at issue as much as fidelity, internal consistency, and other characteristics similar to those that apply to quality assessments for painted portraits or verbal descriptions.
- **Tools** such as those that are intended to be used to increase (or document, or assess) the understandings, abilities, and achievements of students, teachers, programs, or relevant learning communities. The quality of such tools depends on the extent to which they are sharable, powerful, and useful for a variety of purposes and in a variety of situations. (Note: These tools may or may not involve measurement or quantification.)
- **Demonstrated possibilities** that may involve existence proofs (with small numbers of “subjects”) and that may need to be expressed in forms that are accompanied by (or embedded in) exemplary software, informative assessment instruments, or illustrative instructional activities, programs, or prototypes to be used in schools. Again, the quality of results depends on the extent to which these products are meaningful, sharable, powerful, and useful for a variety of purposes and in a variety of situations.

Similar products of research are familiar in the natural sciences. For example, in fields such as physics, chemistry, or biology, some of the most important products
of research involve the development of tools or explanatory models that describe, measure, or predict phenomena such as waves, fields, and black holes. In general, these descriptions go considerably beyond single-number characterizations that attempt to collapse all relevant attributes of a complex system onto a single-dimension number line. In fact, the models are often iconic and analog in nature, being built up from more primitive and familiar notions in which the visualizable model is a major locus of meaning for relevant scientific theories.

MODERN RESEARCH IN MATHEMATICS EDUCATION
PRESUPPOSES SOPHISTICATED INTERACTIONS
INVOLVING MANY LEVELS AND TYPES OF
RESEARCHERS AND PRACTITIONERS

It often is said that “good research should provide answers to teachers’ questions.” If the point of this statement is to emphasize that projects focusing on the development of knowledge should make a positive difference in mathematics teaching and learning, then I not only agree strongly but I’d also point out that a main driving force that has led mathematics education researchers to develop new research designs has been the desire to increase the relevance of research to practice. Nonetheless, the view that “teachers should ask questions and researchers should answer them” is quite naive. For example,

- In mathematics education, no clear line can be drawn between researchers and practitioners; there are many levels and types of both researchers and practitioners; and, the process of knowledge development is far more cyclic and interactive than is suggested by one-way transmissions in which teachers ask questions and researchers answer them. Teachers are not the only ones whose actions and beliefs have strong influences on what goes on in mathematics classrooms. Other influential individuals include policymakers, administrators, school-board members, curriculum specialists, textbook writers, test developers, teacher educators, and others whose knowledge needs are no less important than those of teachers (see Fig. 3.2). Also, in mathematics education, most people who are known as leading “researchers” also tend to have

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**FIG. 3.2.** Interactions between researchers and practitioners.
equally strong reputations as teachers, as teacher educators, as curriculum developers, or as software developers. Similarly, many people who are best known in these latter categories also are highly capable researchers.

• What people ask for isn’t necessarily a wise specification of what they need. Useful ways of thinking usually need to be developed iteratively and recursively, with input from people representing multiple perspectives. Also, statements of problems often are more like “ouches” (expressions of difficulty or discomfort) than well-formulated problems. For example, they often focus on “symptoms” rather than on underlying “diseases” (causes). One reason this is true was expressed by a teacher who participated in one of our recent projects. She said, “When I’m up to my neck in alligators, I haven’t got time to think about clever ways to drain the swamp!” On the other hand, what I really need isn’t just one quick fix after another. Somebody’s got to look ahead, and think more broadly and deeply. Consider the case of politicians who say, “Show me what works!” Such demands overlook the well-known facts that small innovations seldom lead to large results and large innovations seldom get implemented completely. Yet nearly every educational innovation works some of the time, in some situations, for some purposes, in some ways, and for some students. Therefore, unless it is known which parts work when, where, why, how, with whom, and in what ways, the pseudo-information that “This program (or policy) works!” isn’t likely to be useful to educational decision makers. Implementations of sophisticated programs and curriculum materials generally involve complex interactions, sophisticated adaptation cycles, iterative developments, and intricate feedback loops in which second-order effects often are more significant than first-order effects. Consequently, breakdowns occur in traditional distinctions between researchers and teachers, assessment experts and curriculum developers, observers and observed, and simple-minded conceptions of curriculum innovations are doomed to failure.

• Among the challenges and opportunities that are most important for mathematics educators to confront, most are sufficiently complex that they are not likely to be addressed effectively using results from a single isolated research study. Rather than thinking in terms of a one-to-one match between research studies and solutions to problems, it is more reasonable to expect that results from many research studies should contribute to the development of a theory (or model) that should have significant payoff over a prolonged period of time. This is why—in addition to factors such as usefulness, power, and shareability—cumulativeness is factor that determines the significance of research results. Nonetheless, cumulativeness is again a factor that tends to blur the lines of distinction between researchers and practitioners. For example, the projects conducted by productive knowledge development often must involve some form of curriculum development or program development. Similarly, productive curriculum development and program development projects often must involve knowledge development and teacher development. Such endeavors shouldn’t be artificially separated; the flow of information is bidirectional (see Fig. 3.3).
3. RESEARCH DESIGN

NEEDED RESULTS FROM RESEARCH SHOULD INCLUDE TOOLS TO SUPPORT RESEARCH ACTIVITIES

To understand important systems and to solve important problems in mathematics education, some of the kinds of research results that are needed most urgently are tools to support the research enterprise itself. Some of these tools include standard research designs that are modularized in ways that are easy to adapt for alternative conditions and purposes. Others include instruments for measuring or assessing important constructs. In particular, tools that are used to operationally define key constructs strongly influence the quality of research results.

Decisions about how to observe, classify, or quantify relevant information constitute informal “operational definitions” of the subjects and constructs being investigated, and useful operational definitions are needed as both tools and products of research. For example, even when data collection involves tools such as video recordings (which sometimes give the illusion of capturing “raw data”), researchers’ prejudices about the nature of relevant subjects strongly influence decisions about whom to observe, what to observe, when to observe, which aspects of the situation to observe, and how to filter, denote, organize, analyze, and interpret the information that is observed or generated. Thus, some of the most important factors that influence the quality of research focus on the extent to which tentative operational definitions are consistent with intended assumptions about the subjects being investigated. For example, consider the following complex systems that mathematics educators commonly investigate.

Common Assumptions About Students’ Developing Knowledge

Thinking mathematically involves more than computation with written symbols. It also involves, for example, mathematizing experiences by quantifying them, dimensionalizing them, coordinatizing them, or making sense of them using other kinds of mathematical systems. Consequently, to investigate students’ mathematical sense-making abilities, researchers often must focus on problem-solving situations in which interpretation is not trivial. In nontrivial situations, however, most modern theories of teaching and learning believe that the way learning and problem-solving experiences are interpreted is influenced by both (internal) conceptual systems and (external) systems students encounter. Therefore, different students are expected to interpret a given situation in fundamentally different ways, and a given student is not expected to perform in the same way across a series of similar tasks. These assumptions raise the following kinds of research design issues:

- When a given student is not expected to perform in the same way on a series of similar tasks, what does it mean to speak about “reliability” in which repeated measurements are assumed to vary around the student’s “true” (invariant) understandings and abilities that are assumed to apply equally to all tasks?

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1. A variety of levels and types of interpretations are possible, and a variety of representation systems may be useful (each of which emphasizes and deemphasizes somewhat different characteristics of the situations being described); different analyses often involve different “grain sizes,” perspectives, or trade-offs among factors such as simplicity and precision.

2. If the goal of a task involves constructing an interpretation (description, explanation) that is mathematically significant (useful, sharable, modifiable, transportable beyond the situation where it was developed), then the completion of such task often involves significant forms of learning.
When different problem solvers are expected to interpret a single problem-solving situation in fundamentally different ways, what does it mean to speak about “standardized” questions? Similarly, what does it mean to speak about “the same treatment” being given to two different participants or groups?

Such questions are not intended to suggest that notions of reliability (or validity or replicability) are irrelevant to modern research in mathematics education. Indeed, closely related criteria such as usefulness, meaningfulness, power, and shareability must be part of any productive knowledge development efforts in applied fields such as mathematics education. Nonetheless, the meanings of these terms must be conceived in ways that are not inconsistent with defensible assumptions about the systems and constructs being investigated. This means that off-the-shelf definitions that were appropriate in the past may no longer be so—especially if they were grounded in obsolete, machine-based models of teaching, learning, and problem solving.

Common Assumptions About Teachers

For teachers just as for other types of problem solvers and decision makers (including students), expertise is reflected not only in what they do but also in what they see. Alternatively, we could say that what teachers do is strongly influenced by what they see in given teaching and learning situations. For example, as teachers develop, they tend to notice new things about their students, about their instructional materials, and about the ideas and abilities that they are trying to help students learn. Consequently, these new observations often create new needs and opportunities that, in turn, require teachers to develop further. Thus, the teaching and learning situations that teachers encounter are not given in nature; they are, in large part, created by teachers themselves based on their own current conceptions of mathematics, teaching, learning, and problem solving. Thus, there exists no fixed and final state of excellence in teaching. In fact, continual adaptation is a hallmark of teachers who are successful over long periods of time. Furthermore, no teacher is equally effective for all grade levels (kindergarten through college), for all topic areas (algebra through statistics and geometry or calculus), for all types of students (handicapped through gifted), and for all types of settings (inner city through rural). No teacher can be expected to be constantly “good” in “bad” situations; not everything that experts do is effective, and not everything that novices do is ineffective. Characteristics that lead to success in one situation (or for one person) often are counterproductive in other situations (or for another person). Even though gains in students’ achievement should be considered when documenting the accomplishments of teachers (or programs), it is foolish to assume that the best teachers always produce the largest learning gains for students. (What if a great teacher chooses to deal with difficult students or difficult circumstances?)

The preceding observations suggest that expertise (for teachers, students, or other problem solvers) is plural, multidimensional, nonuniform, conditional, and continuously evolving. Therefore, if there is no single type of “best” teacher, if every teacher has a complex profile of strengths and weaknesses, if teachers who are effective in some ways and under some conditions are not necessarily effective in others, and if teachers at every level of expertise must continue to adapt and develop, then what does it mean to classify teachers into naive categories such as “experts” or “nonexperts” (as if complex profiles of capabilities can be collapsed to fixed positions on a single-dimensional “good–bad” scale)?
Common Assumptions About Programs, Materials, or Classroom Learning Environments

Because classroom learning environments, schools, and programs are not given in nature, constructs and principles that are used to construct, describe, explain, manipulate, or control them often appear to be less like “laws of nature” than “laws of the land” that govern a country’s legal system. Also,

- Researchers investigating such systems often are not simply disinterested observers, and they may be more interested in what’s possible than in what’s “real” (or typical). So, issues about the truth or falsity of given may be less relevant than issues about the consistency, transferability, power, or desirability of outcomes.
- Legislated programs, defined curricula, and planned classroom learning environments often are quite different than implemented programs, curricula, and classroom activities; complex programs, materials, and activities seldom operate as simple functions in which a small number of input variables completely determine a small number of output variables. Second-order effects (and other higher order effects) often have significant impacts; emergent phenomena resulting from interactions among variables often lead to results that are at least as significant as attributes associated with the variables themselves. In particular, tests often go beyond being objective indicators of development to exert powerful forces on the programs, curricula, or activities that are intended to assess. Consequently, if naïve pretest-posttest designs are based on tests that reflect narrow, shallow, or naïve conceptions of outcomes and interactions, then they often have strong negative impacts on outcomes. Therefore, researchers often must abandon assumptions about their own detached objectivity.

OPERATIONAL DEFINITIONS USUALLY SHOULD GO BEYOND CHECKLISTS OF BEHAVIORS OBJECTIVES

Because of the complex systemic nature of most of the “subjects” and “constructs” that mathematics educators need to investigate and understand, because many of the relevant systems are products of human construction rather than simply being given in nature, and because characterizations of behaviors seldom are modeled using simply functions or simple logical rules, it has become commonplace to hear mathematics education researchers talk about rejecting traditions of “doing science” as they imagine it is done in the physical sciences (where, it is imagined, researchers treat “reality” as if it were objectively given). When educators speak about rejecting notions of objective reality, however, or about rejecting the notion of detached objectivity on the part of the researcher, such statements tend to be based on antiquated notions about the nature of modern research in the physical sciences. For example, in mature sciences such as astronomy, biology, geology, or physics, when entities such as subatomic particles are described using fanciful terms such as color, charm, wisdom, truth, and beauty, it is clear that the relevant scientists are quite comfortable with the notion that reality is a construct. Or, when these scientists speak about principles such as the Heisenberg indeterminancy principle, it is clear that they are familiar with the notions that (a) the relevant systems act back when they are acted upon, (b) the observations researchers make often induce significant changes in the systems they observe, and (c) researchers often are integral parts of the systems they are hoping to understand and explain. Yet, such realities do not prevent these researchers from developing a variety of levels and types of productive operational definitions to deal with constructs such as black holes, neutrinos, strange quarks, and other entities for which existence is related to
systems with behaviors characterized by mathematical discontinuities, chaos, and complexity.

Consider the case of the neutrino, where huge vats of heavy water must be surrounded by photomultipliers to create situations in which the effects of neutrinos are likely to be observable and measurable. Even under these conditions, however, neutrinos cannot be observed directly and can be known only through their effects. Between the beholder and the beheld, elaborate systems of theory and assumptions are needed to distinguish signal from noise—and to shape interpretations of the phenomena under investigation. Also, small changes in initial conditions often lead to large effects that are essentially unpredictable, observations that are made induce significant changes in the systems being observed, and both researchers and their instruments are integral parts of the systems that scientists are hoping to understand and explain. Therefore, educators are not alone in their need to deal with systems with the preceding characteristics.

In many respects, the development and assessment of complex conceptual systems in education is similar to the development and assessment of complex and dynamic systems that occur in other fields, such as sports, arts, or business, in which coordinated and smoothly functioning systems usually have properties-as-a-whole that do not derive from the simple combination of constituent parts. For instance, it may be true that a great artist (or athlete or team) should be able to perform well on certain basic drills and exercises; nonetheless, a program of assessment (or instruction) that focuses on nothing more than checklists of basic facts and skills is not likely to promote high achievement. For example, if we taught (and tested) cooks or carpenters in this way, we’d never allow them to try cooking a meal or build a house until they memorized the names and skills associated with every tool at stores such as Crate & Barrel, Williams Sonoma, Ace Hardware, and Sears. In contrast, in education much more than in more mature sciences, it is common to treat low level indicators of achievements as if they embodied or defined the relevant understandings.

IN MATURE SCIENCES, OPERATIONAL DEFINITIONS TYPICALLY INVOLVE THREE PARTS

In mature sciences, measurement instruments seldom are based on the assumption that what’s being measured can be reduced to a checklist of simple condition–action rules. For example, when devices such as cloud chambers or cyclotrons are used to observe, record, and measure illusive constructs, the existences of which depend on complex systems, it is clear that (a) the relevant construct does not reside in the device, (b) being able to measure a construct does not guarantee that a corresponding dictionary-style definition will be apparent, and (c) even when a dictionary-style definition can be given (for a construct such as a black hole in astronomy), this doesn’t guarantee that procedures will be available for observing or measuring the construct. In spite of these facts, however, useful operational definitions typically involve three

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3 Even in everyday situations, thermometers measure temperature, yet it is obvious that simply causing the mercury to rise doesn’t do anything significant to change the weather. Clocks and wristwatches measure time without leading us to believe that they tell what time really is. Symptoms may enable doctors to diagnose a disease, yet it is clear that eliminating the symptoms is different than curing the disease.

4 Whereas behavioral objectives treat mathematical ideas as if they resided in specific problems or tasks, modern mathematics education researchers have turned their attention beyond analyses of “task variables” to focus on analyses of “response variables” where mathematical thinking is assumed to reside in students’ interpretations and responses, not in the situations that elicited these mathematical ways of thinking.
parts that are similar, in some respects, to the following three parts of traditional types of behavioral objectives (of the type that have been emphasized in past research in mathematics education).

**BEHAVIORAL OBJECTIVES INVOLVE THREE PARTS**

| GIVEN {specified conditions} | THE STUDENT WILL EXHIBIT {specified behaviors} | WITH IDENTIFIABLE QUALITY {perhaps specified as percents correct on relevant samples of tasks, or perhaps specified as a correspondence with “correct” prototypes}. |

Whereas behavioral objectives collapse three different kinds of statements into a single condition–action rule, more general types of operational definitions typically keep these components separate. For example, when researchers in fields such as physics deal with complex phenomena involving entities such as photons or neutrinos, minimum requirements for useful operational definitions usually require that explicit procedures must be specified for creating

1. *situations* that optimize chances the targeted construct will occur in observable forms,
2. *observation tools* that enable observers to sort out signal from noise in results that occur, and
3. *assessment criteria* that allow observations to be classified or quantified.

It is beyond the scope of this chapter to detail principles that mathematics educators can use to deal with each of the preceding three components of productive operational definitions, but examples can be found in some of the best standards-based “performance assessment instruments” that have been developed during recent years (Lesh & Lamon, 1993). In general, these performance assessment instruments involve (a) thought-revealing activities that require students, teachers, or other relevant “subjects” to express their ways of thinking in forms that are visible to both researchers and to the subjects themselves; (b) response analysis tools to classify alternative responses and to identify strengths, weaknesses, and directions for improvement; and (c) response assessment tools to evaluate alternative responses.

**AN EXAMPLE: PURDUE’S CENTER FOR TWENTY-FIRST CENTURY CONCEPTUAL TOOLS (TCCT)**

Many of the issues described in this chapter occur in investigations currently being conducted in Purdue’s Center for Twenty-first Century Conceptual Tools (TCCT). TCCT’s overall research mission is to investigate

- What is the nature of typical problem-solving situations in which elementary-but-deep mathematical or scientific constructs need to be used beyond school in a technology-based age of information?

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5 Even if it is impossible to reduce Granny’s cooking expertise to a checklist of rules for others could follow, it may be easy to identify situations where her distinctive achievements are required—and where many of the most important components of her abilities will be apparent.
• What is the nature of the most important elementary-but-powerful understandings and abilities that are likely to be needed as foundations for success in the preceding kinds of problem-solving situations? (Hoyles & Noss, 1998)
• How can we enlist the input, understanding, and support of parents, teachers, school administrators, community leaders, and policymakers during the process of generating answers to the preceding questions?

In particular, TCCT focuses on ways that the traditional “three Rs” (Reading, WRiting, and aRithmetic) need to be reconceptualized to meet the demands of the new millennium, as well as considering ways that these three Rs might be profitably extended to include four additional Rs: Representational Fluency, (Scientific) Reasoning, Reflection, and Responsibility.

To investigate such issues, TCCT goes beyond talking to “school people” to also enlist help from professors in future-oriented university programs and professional schools, parents and policymakers who care most about the success of their children, and leaders from business and industry who have firsthand experience about what kinds of abilities are most likely to succeed in desirable jobs and professions. To investigate these people’s views, TCCT often uses a type of research design that a recent National Science Foundation report refers to as design experiments (Frecktling, 1999). We don’t simply send out questionnaires; we don’t settle for opinions that are poorly informed or that are lacking sufficient reflection, and we don’t rely exclusively on our own abilities to observe “experts” in real-life situations. Instead, we enlist teachers, professors, parents, policymakers, and other relevant participants to work with researchers in collaborating teams of “evolving experts” that come together in weekly meetings where they repeatedly express, test, and revise or refine their collective beliefs during the process of developing thought-revealing case studies for kids that (the evolving experts believe) are simulations of the kind of “real-life” situations in which mathematical thinking will be required in the 21st century and tools for documenting and assessing the kind of understandings and abilities that students actually use when they’re successful in the preceding kinds of problem-solving situations.

To accomplish the preceding goals, TCCT often uses semester-long multitier design experiments (Kelly & Lesh, 2000) that involve cohorts of at least 15 to 20 teachers, parents, professors, and policymakers. These multitier design experiments were explicitly developed so that multiple researchers, working at multiple sites and representing multiple theoretical and practical perspectives, can collaborate to investigate the interacting development of three categories and three interacting levels of problem solvers and decision makers, each of which is understood only incompletely if the development of others is ignored (see Table 3.1).

Using TCCT’s multitier design experiments, teachers, parents, policymakers, professors, and researchers are all considered to be “evolving experts.” Each has important views that should be considered in discussions about What’s needed for success in the 21st century? Nonetheless, different experts often hold conflicting views; none have exclusive insights about truth, and all tend to evolve significantly during the process of designing relevant tools using a series of testing-and-revising cycles in which formative feedback and consensus building influence the final results that are produced. Thus, theory development and model development proceed hand-in-hand;

6TCCT’s case studies of kids are middle-school versions of the kind of “case studies” that are emphasized for both instruction and assessment in many of Purdue’s most future-oriented graduate programs and professional schools, in fields ranging from agricultural sciences, to business management, to aerospace engineering.
TABLE 3.1
Multitier Design Experiments

<table>
<thead>
<tr>
<th>Levels</th>
<th>Middle School</th>
<th>University</th>
<th>Business</th>
</tr>
</thead>
<tbody>
<tr>
<td>Evolving researchers</td>
<td>Teachers and parents</td>
<td>Professors</td>
<td>Professionals</td>
</tr>
<tr>
<td>Evolving teachers</td>
<td>6th- through 8th-grade students</td>
<td>University students</td>
<td>Trainees</td>
</tr>
</tbody>
</table>

A key feature of TCCT’s multitier design experiments, is that thought-revealing activities for students generally provide the basis for equally thought-revealing activities for teachers (and parents, administrators, school-board members, and other participants). For example,

• Thought-revealing activities for students often can be characterized as middle-school versions of the kind of case studies (i.e., simulations of real-life problem-solving situations) that are emphasized in both instruction and assessment in many of Purdue’s most future-oriented graduate programs and professional schools, in fields ranging from aeronautical engineering, to business administration, to the agricultural sciences. The goals of these problem-solving episodes is not simply to produce a short-answer response to someone else’s unambiguously formulated question; instead, it is to produce sharable and reusable conceptual tools (or conceptual systems) for constructing, describing, explaining, manipulating, predicting, or controlling complex systems.

• Thought-revealing activities for teachers (parents, policymakers) often involve developing assessment activities that participants believe to be simulations of real-life situations in which mathematics will be used in everyday situations in the 21st century. But, they also may involve developing sharable and reusable tools that teachers can use to make sense of students’ work in the preceding simulations of real-life problem-solving situations. For example, these tools may include
  • Observation forms to gather information about the roles and processes that contribute to students’ success in the preceding activities
  • Ways of thinking sheets to identify strengths and weaknesses of products that students produce and to give appropriate feedback and directions for improvement
  • Quality assessment guides for assessing the relative quality of alternative products that students produce
  • Guidelines for conducting mock job interviews based on students’ portfolios of work produced during case studies for kids and focusing abilities valued by employers in future-oriented professions

During the process of testing and revising the preceding kinds of tools, the following three distinct forms of feedback are available: (a) feedback from researchers (for example, when participants hear about results that others have produced in past projects, they might say, “I never thought of that.”); (b) feedback from peers (for example, when participants see results that other participants produce, they might say, “That’s a good idea, I should have done that.”); and (c) feedback about how the
tool actually worked (for example, when the tool is used with students, participants might say, “What I thought would happen, didn’t happen.”). Thus, formative feedback and consensus building are used to enable “evolving experts” to develop in directions that they themselves consider to be “better” without basing their judgments on anybody’s preconceived notion of “best.” Furthermore, as products are developed, ways of thinking evolve and auditable trails of documentation are generated that reveal important information about the nature of developments that occur. In other words, multitier design experiments automatically emphasize all three components that are needed to operationally define relevant constructs that are under investigation.

WHY DOESN’T TCCT RESEARCH SIMPLY ASK (OR OBSERVE) EXPERTS?

In the past, when mathematics educators have asked what understandings and abilities should be treated as basic, one approach that has been emphasized focuses on the development of “standards for curriculum and assessment” by organizations ranging from the National Council of Teacher of Mathematics (1999), to the American Association for the Advancement of Science (1993), to Indiana’s State Department of Education (1998). But, most of these documents mainly have been formulated by people representing schools and by university-based experts in relevant disciplines. Those whose views have been neglected include people whose jobs and lives do not center around schools and, in particular, scientists or professionals in fields such as engineering (or business or agriculture) that are heavy users of mathematics, science, and technology. Consequently, it is not surprising that “school people” have focused on finding ways to make incremental changes in the traditional curriculum, nor is it surprising that people whose views have been neglected often lead backlash movements proclaiming naive notions of “back to basics” as a theme to oppose changes recommended by more forward-looking documents describing standards for instruction and assessment.

Unlike most projects that have investigated the nature of basic understandings and abilities, TCCT research takes great care to enlist the thinking of more than “school people”; we also try to avoid becoming so preoccupied with low-level skills needed to avoid failure (in school) that we fail to give appropriate attention to deeper or higher order knowledge and abilities needed to prepare for success (beyond school). Similarly, TCCT tries to avoid becoming so preoccupied with minimum competencies associated with low-level or entry-level jobs (e.g., street vendors, gas station attendants) that we fail to consider powerful conceptual tools that are needed for long-term success in desirable professions and lives.

To investigate what kind of mathematical abilities are needed for success in a technology-based age of information, why not just ask (or observe) experts? Or, why not just observe them in “real-life” situations? (Greeno, 1997; Latour, 1987). When we’ve tried such approaches, the following questions arose, the answers of which seemed to us to depend too heavily on our own preconceived notions about what it means to “think mathematically”:

- When should we observe these people? When they’re calculating? When they are estimating sizes, distances, or time intervals? When they are working with numbers and written symbols? When they’re working with graphics, shapes, paths, locations, trends, or patterns? When they’re deciding what information
to collect about decision-making issues that seem to involve “mathematical” thinking? When they are describing, explaining, or predicting the behaviors of the preceding systems? When they are planning, monitoring, assessing, or justifying steps in the construction processes used to develop the preceding systems? When they are reporting intermediate or final results of the preceding processes?

- What should we count as mathematical activities? For example, in fields where mathematics is widely considered to be useful, a large part of expertise consists of developing “routines” that reduce large classes of tasks to situations that are no longer problematic. As a result, what was once a problem becomes only an exercise. Is an exercise with numeric symbols necessarily more mathematical than a structurally equivalent exercise with patterns of musical notes, or Cuisenaire rods, or ingredients in cooking?

Answers to such questions often expose preconceived notions about what it means to “think mathematically”—and unexamined assumptions about the nature of real-life situations in which mathematics is useful. Therefore, because it is precisely these assumptions that we want to question and investigate in TCCT research, it is not appropriate to begin by assuming that someone (who’s dubbed an “expert” based on preconceived notions about correct answers) already knows the “correct” answers. Instead, TCCT enlists input from a variety of evolving experts who include not only teachers and curriculum specialists but also parents, policymakers, professors, and others who may have important views that should not be ignored about What’s needed for success in the 21st century. Then, these evolving experts engage in a series of situations where their views must be expressed in forms that are tested and revised repeatedly.

TCCT’s approach recognizes that (a) different experts often hold significantly different views about the nature of mathematics, learning, and problem solving; (b) none of the preceding people have exclusive insights about “truth” regarding the preceding beliefs; and (c) all of the preceding people have ways of thinking that tend to evolve significantly if they are engaged in activities that repeatedly require them to express their views in forms that go through sequences of testing-and-revision cycles in which formative feedback and consensus building influence final conclusions that are reached. TCCT’s evolving expert methodologies also recognize that respecting the views of teachers, parents, professors, and others doesn’t mean that these views must be accepted passively, nor that they represent well-informed opinions that are based on thoughtful reflection. When it comes to specifying what kind of understandings and abilities will be needed for success in the 21st century, it is not reasonable to assume that anybody’s views (including our own) are ready to be carved in stone as being “the truth.” To investigate such issues, what’s needed is a process that encourages development at the same time that evolving views are taken seriously. Using TCCT’s approach, evolving experts are truly collaborators in the development of a more refined and more sophisticated conception of what it means to develop understandings and abilities of the type that will be most useful in real-life problem-solving situations (Anderson, Reder, & Simon 1996; Cobb & Bowers, in press; diSessa, Hammer, Sherin, & Kolpakowski, 1991).

WHAT KIND OF RESULT CAN BE EXPECTED FROM MULTITIER DESIGN EXPERIMENTS?

Purdue is proving to be an ideal place to investigate what’s needed for success in the 21st century because it has a distinctive identity as one of the leading U.S. research universities focusing on applied sciences, engineering, and technology, in future-oriented fields that range from aeronautical engineering, to business management,
agricultural sciences. Purdue not only stands for content quality and solutions that work, it also has pioneered sophisticated interactive working relationships among scientists and those who are heavy users of mathematics, science, and technology in their businesses and lives. Furthermore, ever since the time when Amelia Earhart (the famous early aviator) worked at Purdue to recruit women and minorities into the sciences and engineering, Purdue has provided national leadership in issues related to diversity and equity in the sciences. Most important for the purposes of the TCCT Center, Purdue is a university that is filled with content specialists whose job it is to prepare students for future-oriented jobs and who know what it means to say that the most important goals of instruction (and assessment) often consist of helping students to develop powerful models and conceptual tools for making (and making sense of) complex systems. Consequently, these leaders know that some of the most effective ways to help students develop the preceding competencies and conceptual systems is through the use case studies in which students develop, test, and refine sharable conceptual tools for dealing with classes of structurally similar problem-solving situations. Finally, these leaders know that when their students are interviewed for jobs, the abilities emphasized focus on communicating and working effectively within teams of diverse specialists; adopting and adapting rapidly evolving conceptual tools; constructing, describing, and explaining complex systems; and to coping with problems related to complex systems.

Purdue is a place where many leading scientists and professionals are concerned about the negative effects of teaching that is driven (almost exclusively) by students’ performances on standardized tests that focus on narrow and shallow notions of “basic skills.” Consequently, many of the preceding scientists have been willing to participate in semester-long evolving-expert experiments designed to help both them and us clarify our collective thinking about what kind of “mathematical thinking” is needed as preparation for success in future-oriented fields that are heavy users of mathematics. Results are showing that after participating in a semester-long multitier design experiment in which their views are expressed in forms that must be tested and revised repeatedly, participants are consistently reaching a consensus about the following claims:

• Some of the most important goals of mathematics and science instruction should focus on helping students develop powerful models and conceptual tools for making (and making sense of) complex systems. Although it is not common for K–12 educators to think of models as being among the most important goals of instruction, this fact has a long history of being treated as obvious by leaders in university graduate programs or professional schools that prepare students for future-oriented jobs in fields such as engineering, management, or medicine. Furthermore, some of the most effective ways to help students develop productive competencies and conceptual systems involve using case studies that are adult-level versions of TCCT’s case studies for kids.

• In problem-solving and decision-making situations beyond schools, the kind of mathematical and scientific capabilities that are in highest demand are those that involve the ability to work in diverse teams of specialists; to adapt to new tools and unfamiliar settings; to unpack complex tasks into manageable chunks that can be addressed by different specialists; to plan, monitor, and assess progress; to describe intermediate and final results in forms that are meaningful and useful to others; and to produce results that are timely, sharable, transportable, and reusable. Consequently, mathematical communication capabilities tend to be emphasized, as do social or interpersonal abilities that often go far beyond traditional conceptions of content-related expertise.

• Past conceptions of mathematics, science, reading, writing, and communication often are far too narrow, shallow, and restricted to be used as a basis for identifying
students whose mathematical abilities should be recognized when decisions are made about hiring for jobs or about admission to educational programs. Students who emerge as being especially productive and capable in simulations of real-life problem-solving situations often are not those with records of high scores on standardized tests—or even high grades in courses where students are seldom required to develop ways to construct (or make sense of) complex systems that are needed for complex pruposes. Therefore, new ways need to be developed to recognize and reward these students, and these new approaches should focus on productivity, over prolonged periods of time, on the same kind of complex tasks that are emphasize in case study approaches to instruction.

When middle school students work on TCCT’s case studies for kids (Doerr & Lesh, 2002), responses typically are showing that children who’ve been classified as “below average” (based on performance in situations involving traditional tests, textbooks, and teaching) often invent (or significantly revise or extend) constructs and ways of thinking that are far more sophisticated than anybody ever dared to try to teach to them. Furthermore, students who are especially productive in the context of such problems often are not those who have histories of high scores on traditional tests.

Comments in the preceding paragraphs do not imply that if we simply walk into the offices of random professors in leading research universities, then their views about teaching and learning (or problem solving) should be expected to be thoughtful or enlightened. In fact, it is well known that some university professors have been leading opponents of standards-based curriculum reforms. Nonetheless, if school curriculum reform initiatives make almost no effort to enlist the understanding and support of parents, policymakers, and others who are not professional educators, then it should be expected that these nonschool people often will end up opposing proposed curriculum reforms. On the other hand, research in Purdue’s TCCT Center is showing that if professors (or parents, policymakers, business leaders, or other taxpayers) participate as evolving experts whose views about teaching, learning, and goals of instruction must go through several test-and-revision cycles, then the views that these evolving experts ultimately express often become quite sophisticated and supportive for productive curriculum reform. In fact, they often push for changes that go considerably beyond what we are hearing from curriculum reforms that are exclusively school based—and definitely do not represent calls for more “business as usual.”

**SUMMARY AND CONCLUSIONS**

Traditional descriptions of research often characterize it as a rule-governed process that involves the list of steps as follows (Romberg, 1992, pp. 51–53). Yet, from the perspective of issues emphasized in this chapter, the list is similar to other half-truths that were described earlier. That is, it represents a highly inadequate characterization of what really happens in a large share of the most productive research in mathematics education.

1. **Identify phenomena of interest about which questions will be formulated and addressed.** These phenomena might concern the past, present, or future, and they may involve already existing situations or, often, situations to be created.
2. **Build a tentative model or description that helps sort out the key aspects of the phenomena in question, especially distinguishing those that seem most relevant from those that seem less so.**
3. **Relate the phenomena and model to others’ ideas and results, both among those who share your world view and those who do not.** That is, situate your work
in that of the larger community of scholars; know what they have done and written.

4. Ask specific questions or make reasoned hypotheses or conjectures, trying to get at the essence of the phenomena in a way that supports a chain of inquiry that eventually affords some kinds of answers to the questions or specific tests of the conjectures.

5. Select a general research strategy for gathering evidence that fits all that has been decided to date to examine an existing situation in detail, to manipulate variables in a situation under your control, to compare situations, to look at the situation over an extended period in great detail, to survey a population in some systematic way, and so forth.

6. Select specific procedures and plan data collection. It is here where the usual research methods course content comes into play. However, given the complexity of most research situations, a combination of procedures usually is required.

7. Collect the information. At this point, the procedure(s) should be well specified, although most substantive research involves pilot stages of data gathering in which two or more cycles may be needed.

8. Interpret the information collected in the light of all the previous steps. Again, it may be that this step occurs as part of a sequence feeding back into many of the prior steps before a final sequence occurs.

9. Share the results. Even this step may involve cycles with the others and involving other scholars outside your immediate working community. This is especially the case in an environment in which electronic communication supports rapid interchange and wide preliminary dissemination across a distributed community of working colleagues.

10. Anticipate the actions of others—other researchers, policymakers, practitioners, materials developers, and so forth. A given research activity virtually never occurs in isolation, and indeed a measure of its significance is the degree to which it spawns actions on the part of a wider community. The researcher should include as part of the working plan some anticipation of what comes next.

In what ways do the preceding steps represent a misleading conception of what’s needed to produce useful and sharable knowledge about most of the important systems that are priorities for mathematics educators to understand? First, the development of useful knowledge is not restricted to products consisting of answered questions and tested hypotheses. Second, the design of models, conceptual tools, and other products of research often involve cyclic and iterative processes in which the design principles that must be emphasized do not conform to the preceding one-way assembly-line characterizations of knowledge development. Third, the line between researchers and subjects is by no means as clear as suggested by the preceding list of steps; many levels and types of researchers and practitioners may be involved, and communication is not simply in one direction. Forth, the systems being investigated are complex, dynamic, and continually adapting—researchers often must abandon notions of detached objectivity and naive replicability. Yet, when attempts are made to assess or improve the quality of specific research projects, issues related to usefulness, sharability, and cumulativeness continue to be highly relevant to consider, even though naive notions of reliability and validity may need to be reconceived to avoid being inconsistent with modern theoretical perspectives.

Consider the case of research that is related to program development. One result of thinking of schools and programs as simple input–output machines is that it is commonly assumed that the best way to ensure progress toward new curriculum goals is to for a “blue-ribbon” panel of local teachers to convert national standards to “local”
versions that won’t be viewed as top–down impositions and to write new tests items or performance assessment activities that are aligned with these “local” standards. Then, to promote accountability, pressures are applied for teachers to teach to these tests. However, despite the apparent simplicity of these approaches, the following realities proved that they tended to be expensive, ineffective, and counterproductive to the intended goals of the projects leaders:

• The process is by no means straightforward for converting curriculum standards for teachers (or teaching) into performance standards for students. So, local standards often end up reconverting national standards back into checklists of behavioral objectives similar to those the national standards were intended to replace. Furthermore, asking a small committee of expert teachers to create “local” standards seldom encourages other teachers to assume meaningful ownership of these standards, and curriculum goals that are defined only by “school people” typically fail to enlist the understanding and support of parents, policymakers, or community leaders. So, many schools have experienced “back-to-basics” backlashes from concerned parents, policymakers, or community leaders whose input and understanding never was sought.

• Because of their highly restricted formats and time limitations, even though developed standardized tests tend to focus on only a narrow, shallow, and biased subset of the understandings and abilities that are needed for success in less restricted and more representative samples of problem-solving experiences. Therefore, teaching to such tests often has strong negative influences on what is taught and how it is taught.

• Teachers generally have been given far too little time and support to develop alternative assessments. Furthermore, the abilities that are needed to be a great developer of curriculum materials are not identical to those that make a great teacher. Therefore, locally developed alternative assessment programs seldom satisfy quality-assurance principles that teachers participating in our project have developed. Furthermore, the time that teachers spend writing curriculum materials takes them away from what they do best—working with students. These teachers often view such activities as intrusions on their main duties.

During TCCT-related research has that been conducted recently in the context of the National Science Foundation (USA)—supported systemic curriculum reform initiatives in Connecticut, Massachusetts, Minnesota, New Jersey, and Rhode Island (Schorr & Lesh, in press), it repeatedly became apparent that one of the most significant characteristics that differentiated “more successful” from “less successful” schools (districts, teachers) is that, as the most successful projects evolved, teachers and other participants develop much more clear and sophisticated notions about (a) the kinds of problem-solving situations that will be especially important for students to master as preparation for success in a technology-based age of information;

7It is one thing to state that goals such as “emphasizing connections among ideas” should be emphasized in instruction, and it is quite another to specify how these abilities can be revealed and assessed in students’ work. Similarly, saying that instruction should focus on deep treatments of a small number of big ideas does not make it clear what these “big ideas” are, what it means to understand them deeply, nor how higher order understandings are related to the mastery lower level facts and skills that traditional instruction has treated as prerequisites. For example, specifying what is meant by terms such as understanding tends to be especially problematic when modifiers are added such as concrete, abstract, symbolic, intuitive, situated, higher order, instrumental, relational, deeper, or shared.

8Based on our experiences working with hundreds of expert teachers who were trying to write performance assessment activities, we estimate that it requires not less than one person-month of full-time work to develop one single activity that satisfies minimum standards that these teachers helped to develop to ensure quality.
important kinds of elementary-but-powerful understandings and abilities that are likely to be needed for success in the preceding problem-solving situations; (c) what it means to focus on deep treatments of a small number of big ideas rather than trying to superficially cover a large number of small facts and skills; (d) how to document students’ achievements that involve deeper and higher order understandings of the preceding big ideas; (e) what kind of relationships exist between students’ development of these big ideas and their mastery of facts and skills that traditional textbooks and tests treat as prerequisites. Conversely, hallmarks of schools (districts, teachers) that were least successful were that (a) projects began by reducing their goals to a checklist of behavioral objectives (local standards), (b) tests were adopted or created that had strong negative influences on what was taught and how it was taught, (c) teaching was locked into initial simplistic conceptions of goals and naive pretest-posttest assessment designs that were completely insensitive to the most significant achievements of students and teachers.

To deal with the preceding realities, TCCT’s evolving expert methodologies recognize that (a) even the most insightful “expert teachers” (or other participants) continue to develop in significant ways; (b) there is no single formula for being a successful teacher (excellent teachers have diverse profiles of abilities and styles); (c) excellence in teaching cannot be reduced to a simplistic checklist of principles or rules; (d) one of the most effective ways to improve teaching is to help teachers develop more sophisticated “ways of thinking” about the nature of mathematics, learning, problem solving, and the ways that mathematics is useful in a technology-based society; and (e) some of the most effective ways to influence teachers’ ways of thinking is to shift attention beyond writing standards (or assessment activities based on these standards) toward interpreting standards (that already exist), while analyzing the strengths and weaknesses of students’ work (in the context of thought-revealing activities that already are available). Our research also shows that when teachers, parents, and others interact to interpret existing national standards in insightful (and local) ways, results tend to go far beyond demands for “basics” from an industrial age.

REFERENCES


CHAPTER 4

Developing New Notations for a Learnable Mathematics in the Computational Era

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Not for the first time we are at a turning point in intellectual history. The appearance of new computational forms and literacies are pervading the social and economic lives of individuals and nations alike. Yet nowhere is this upheaval correspondingly represented in educational systems, classrooms, or school curricula. In particular, the massive changes to mathematics that characterize the late 20th century, in terms of the way it is done and what counts as mathematics, are almost invisible in the classrooms of our schools and, to only a slightly lesser extent, in our universities.

The real changes are not technical, they are cultural. Understanding them (and why some things change quickly and others change slowly) is a question of the social relations among people, not among things. Nevertheless, there are important ways in which computational technologies are different from those that preceded them, and in trying to assess the actual and potential contribution of these technologies to education, it will help to view them in a historical light.

The notation systems we use to present and re-present our thoughts to ourselves and to others, to create and communicate records across space and time, and to support reasoning and computation constitute a central part of any civilization’s infrastructure. As with infrastructure in general, it functions best when it is taken for granted, invisible, when it simply “works.” This chapter is being prepared as the United Kingdom’s ground transportation system has, because of a number of additive causes,

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almost totally failed and when the electricity production and distribution system of California is likewise in disarray. When the infrastructure either fails or undergoes changes, the disruptions can be major. Furthermore, they tend to propagate, so that one change causes another in tightly interconnected systems—when the electricity goes out, lots of other things go out, too. The same is true on the positive side: When a new technological infrastructure appears, such as the Internet, many things change, often in unpredictable ways as sequences of new opportunity spaces open up and old ones close down. Entire industries are born, old ways of doing things change, sometimes in fundamental ways, how people participate in the economy changes, the physically based means for defining and controlling ownership of intellectual property are challenged, and indeed the means by which innovation itself is fostered changes. Lastly, these kinds of infrastructural changes are typically not the result of systematic planning or central control. They emerge in unpredictable ways from the mix of existing circumstances.

These general questions of representational infrastructures, may seem far removed from the apparently more mundane task of learning mathematics; but the central challenge of mathematical learning for educators is surely the design of learnable systems. Such systems depend for their learnability (or lack of it) on the particularities and interconnectedness of the representational systems in which they are expressed. These, as we stated at the outset, are undergoing rapid change. To understand these changes more fully, we wish to examine the longer term sweep of representational infrastructure change across several important examples to provide a long-term perspective on the content choices and trends embodied in school mathematics. As we shall see, mathematics enjoys a particularly interesting role in this story.

THE EARLIEST QUANTITATIVE NOTATION SYSTEMS

Most representational infrastructures develop in response to the social needs of one or more groups, where the needs might be broad and involve the whole society, as was the case with the development of writing, or they may serve a smaller subgroup such as mathematicians and scientists, who needed to express and reason with general quantitative relationships and hence developed what we now know as the algebraic system. Indeed, the earliest, prephonetic written language coevolved with mathematics in the cradle of Western civilization some 6,000 to 8,000 years ago to record physical quantities through a gradual process of semiotically abstracting the physical referents into systems of schematic representations of those referents (Kaput & Shaffer, in press).

The systematicity initially took the form of separating inscriptions denoting object-types from inscriptions denoting their properties (identity of owner, size, color, vintage, etc.), and, gradually over hundreds of years, the numerosity property. Inscriptions denoting numerosity gradually condensed, from a repetition model where four instances of an item were represented by four tokens for that type of item, then four tallies adjacent to a single token for that type of item, to a modifier model employing symbols denoting numerosity, that is, a symbol for “four,” replacing the tally marks. This last step required the coevolution of the concept of counting number, mainly through the work of those specialists who were the scribes responsible for producing the records. There is little indication that such early numbering systems were used for computation other than incrementing and decrementing.

Distinct from, but certainly not unrelated to, the notational system was the physical system in which it was instantiated: primarily clay, which afforded the means to impress tokens of objects (sheepskins, jars of olive oil, etc.) first onto clay envelopes containing the tokens and then tablets, when the tokens came to be regarded as redundant (Schmandt-Besserat, 1978, 1992). The medium was temporarily inscriptable and then hardened to provide stability that enabled the inscriptions to act as records,
indeed somewhat mobile records. In this way, evolutionary limitations of human biological memory were finally overcome through the use of “extracortical” records (Donald, 1991).

**THE EVOLUTION OF NOTATION SYSTEMS SUPPORTING QUANTITATIVE COMPUTATION**

We now examine the evolution of a second representational infrastructure. In the several millennia that followed, and across several different societies in which urbanization and commerce developed, various number systems evolved to support ever better and more compact ways of expressing quantities and abstract numbers, particularly to express the large numbers required for calendar purposes and for tracking quantities in the city-states and empires—Babylonian, Egyptian, and eventually Roman. Important for our purposes, although they typically embodied grouping structures, they tended not to be neither rigorously positional nor fully hierarchical. The most tightly structured and efficient system was the Babylonian (semi) sexagesimal (base 60) system. It employed a mix of additive and multiplicative methods of representing numbers as there were no common symbols for smaller numbers (as with the Hindu-Arabic numerals). It did, however, use position to denote powers of 60. Hence a number would be represented as an array of symbols for units, tens, and powers of 60 (using cuneiform or wedge-based signs). As is widely appreciated, this system supported a rich practical mathematics that served many aspects of society for more than two millennia, although with the lack of a zero for a placeholder (in its later years a placeholder system did develop), it did not support efficient multiplication or division. Further, the lack of compact numerals for the first nine numbers meant that it was considerably less efficient for writing numbers in the hundreds and thousands than our current system—and even less efficient, relatively, for larger numbers. Of course, the contemporaneous writing systems were likewise ideographic and difficult to learn and hence the tool of specialists—the scribes (Walker, 1987). The later Roman system was less structured and less multiplicative in its organization and hence even less efficient for multiplication and division.

How did these systems survive in supporting the extensive calculational tasks they were called on to serve? The answer is clear: Only a very small minority of the respective societies were needed to do such calculations, and these scribes were specially trained in the art of manipulating the symbol systems. In this respect, the role of the physical medium (e.g., the marks made on clay and so on) were crucial in supporting the prodigious amounts of human processing power that would otherwise be engaged. Although the structural features of the notational system were not particularly tuned to calculational ends (anyone who has ever tried to do long division with Roman numerals will testify to this), the combination of the physical instantiation of symbols, together with human processing power on the part of a few, was sufficient to sustain powerful empires over hundreds and thousands of years. Furthermore, the existing static record-keeping capacity could be used to record methods, results (especially in the case of the Babylonians, who made wide use of tables of all sorts to record quantitative information and mathematical relationships, make astronomical predictions, and so on), and even instructional materials by which expertise could be extended across generations (Kline, 1953).

Another big representational transformation had roots several centuries earlier, in the 8th to 11th centuries, preceding Fibonacci’s importation of the Hindu-Arabic numerals into Europe in the early 13th century. More efficient methods of computing developed, based on systematic use of specially marked physical “counting tables” on which physical tokens were manipulated. In this way, the technology of the counting tables externalized some of the knowledge and transformational skill that would
otherwise have existed only in the minds of individuals: The physical instantiation of these skills directly supported not only the limited processing capacity of human brains, but the affordance of the notational system for achieving results.

These results of computations were recorded first in Roman numerals, but then gradually more often in Hindu-Arabic numerals. These methods are typically referred to as abacus-style computations based on the Greek word for slab, on which the procedures took place. At the same time, and then more intensely during the 13th century, new and more efficient ways of computing evolved based on manipulation of the readily inscribable Hindu-Arabic numerals in the positional and hierarchical number system that Fibonacci had described. These were referred to as algorithm-style computations based on a Latinized version of the name Abu Ja’far Muhammed ibn Musa al-Khwarizmi, a mathematician from Baghdad who wrote an arithmetic book describing some of the early computational schemes using Hindu-Arabic numerals. Clearly, al-Khwarizmi conceived algebra as a way of solving pressing practical problems of the Islamic Empire. Similarly, in response to burgeoning commerce in the 14th and 15th centuries in northern Italy and elsewhere, the algorithms were refined and gradually displaced the abacus methods, although not without controversy. The efficiency payoff of a positional and exponentially hierarchical system was enormous because it allowed a person to compute simply by writing and rewriting the small set of 10 symbols according to certain rules (the algorithms) and, on the basis of the quantitative coherence of the notation system, be assured of a correct answer based on the rules alone. Computational skill became encoded in syntactically defined rules on a symbol system.

The algorithmic methods were put forward (anonymously) in what amounts to the first arithmetic text, the Treviso, named after the city outside Venice where it was published in 1478, less than 40 years after Gutenberg’s introduction of moveable type (itself a response to the pressing need to find a way of salvaging religious texts that contained mistakes, without destroying the entire work). These algorithms, exploiting the physical positional structure of the notation system and the paper medium, are essentially the same forms for addition, subtraction, multiplication, and division that have dominated school mathematics up to the present. In the book (translated and appearing in Swetz, 1987) they were illustrated within the contexts of commerce and currency exchange and were passed along from generation to generation as a body of practical knowledge in what amounted to professional schools for “reckoners,” the accountants of the time.

Interestingly, the Treviso was written in the vernacular, as opposed to Latin, and thus was one of the first printed mathematics books intended to serve a “nonacademic” (Dantzig, 1954) public—or at least that public that needed to know these special techniques. The new representational infrastructure helped democratize access to what had previously been the province of a small intellectual elite because up to that time numerical computations beyond addition and simple subtraction were a scholarly pursuit undertaken at the universities. Recall the oft-cited anecdote from Dantzig (1954): “It appears that a [German] merchant had a son whom he desired to give an advanced commercial education. He appealed to a prominent professor of a university for advice as to where he should send his son. The reply was that if the mathematical curriculum of the young man was to be confined to adding and subtracting, he perhaps could obtain the instruction in a German university; but the art of multiplying and dividing, he continued, had been greatly developed in Italy, which, in his opinion, was the only country where such advanced instruction could be obtained” (pp. ).
Indeed, the question of inclusion of these same algorithms in school mathematics to support basic shopkeeper arithmetic, currency exchange, and other simple arithmetic tasks continued to be the subject of vigorous debate through the 20th century and into the 21st. Of course, in the intervening five centuries, the practical role of arithmetic has broadened with the increasing complexity of modern life, especially in the workplace, and it has assumed a cornerstone position in school mathematics. Indeed, the skills of arithmetic are seen in almost all developed societies not only as essential for the efficient operation of economies, but as an entitlement of an educated individual. We shall return to this issue below, but for the moment it is worth delineating two separate skills that arithmetic teaching in the 20th century came to serve: the one concerned with obtaining answers quickly and correctly and the other as a backdrop against which the process of executing algorithms could be performed, number relationships learned, and manipulative methods practiced. Although execution was the preserve of the human mind, this distinction hardly arose, but as we shall see it becomes more central in this computational era.

THE EVOLUTION OF ALGEBRAIC NOTATIONS

We now examine a third example of a representational infrastructure. Algebra began in the times of the Egyptians in the second millennium BC, as evidenced in the famous Ahmes Papyrus, by using available writing systems to express quantitative relationships, especially to “solve equations”—to determine unknown quantities based on given quantitative relationships. This is the so-called rhetorical algebra that continued to Diophantus’ time in the fourth century of the Christian era, when the process of abbreviation of natural language statements and the introduction of special symbols began to accelerate. Algebra written in this way is normally referred to as syncopated algebra. By today’s standards, achievement to that point was primitive, with little generalization of methods across cases and little theory to support generalization. Indeed, approximately two millennia produced solutions to what we would now refer to as linear, quadratic, and certain cubic equations (of course they were not written as equations), often based in contrived and stylized concrete situations and not much more. Indeed, it appears that, in the absence of a systematic symbol system, the stylized situation provided a kind of semiabstract conceptual scaffolding for the quantitative reasoning that constituted the methods. The accumulated skill was encoded in illustrative examples rather than in syntactically defined rules for actions on a symbol system.

Then, in a slow, millennium-long struggle involving the coevolution of underlying concepts of number (see especially Klein, 1968), algebraic symbolism gradually freed itself from written language to support techniques that increasingly depended on working with the symbols themselves according to systematic rules of substitution and transformation, rather than the quantitative relations for which they stood. Just as the symbolism for numbers evolved to yield support for rule-based operations on inscriptions taken to denote numbers, so the symbolism for quantitative relations likewise developed. Bruner (1973) refers to this as an “opaque” use of the symbols rather than a “transparent” use: The former implies attention to actions on the inscriptions, whereas the latter implies that actions are guided by reasoning about the entities to which the inscriptions are assumed to refer.

In effect, algebraic symbolism gradually freed itself from the (highly functional) ambiguities and general expressiveness of natural language. The newly developed and systematic semiotic structures embodied hard-won understandings of general mathematical relations and, by the 17th century, functions. This symbol system also embodied forms of generality (particularly through the use of symbols for variables) and the dual use of operation symbols (so symbols such as + could be applied to symbols
for variables ranging over sets of numbers as well as for the numbers themselves). Hence general statements of quantitative relations could be expressed efficiently.

Nonetheless, the more important aspects of the new representational infrastructure are those that involved the rules—the syntax—for guiding operations on these expressions of generality. These emerged in the 17th century as the symbolism became more compact and standardized in the intense attempts to mathematize the natural world that reached such triumphant fruition in the “calculus” of Newton and Leibniz. In the words of Bochner (1966), “Not only was this algebra a characteristic of the century, but a certain feature of it, namely the “symbolization” inherent to it, became a profoundly distinguishing mark of all mathematics to follow. . . . This feature of algebra has become an attribute of the essence of mathematics, of its foundations, and of the nature of its abstractness on the uppermost level of the “ideation” à la Plato” (pp. 38–39).

Beyond this first aspect of algebra, its role in the expression of abstraction and generalization, he also pointed out the critical new ingredient: “that various types of ‘equalities,’ ‘equivalences,’ ‘congruences,’ ‘homeomorphisms,’ etc. between objects of mathematics must be discerned, and strictly adhered to. However this is not enough. In mathematics there is the second requirement that one must know how to ‘operate’ with mathematical objects, that is, to produce new objects out of given ones” (p. 313).

Indeed, Mahoney (1980) pointed out that this development made possible an entirely new mode of thought “characterized by the use of an operant symbolism, that is, a symbolism that not only abbreviates words but represents the workings of the combinatorial operations, or, in other words, a symbolism with which one operates” (p. 142).

This second aspect of algebra, the syntactically guided transformation of symbols while holding in abeyance their potential interpretation, flowered in the 18th century, particularly in the hands of such masters as Euler. At the same time, this referral of interpretation led to the further separation of algebraic and natural language writing and hence the separation from the phonetic aspects of writing that connect with the many powerful narrative and acoustic memory features of natural language. Indeed, as is well known from such examples as the “Students–Professors Problem” (Clement, 1982; Kaput & Sims-Knight, 1983), the algebraic system can be in partial conflict with features of natural language.

Thus, over an extremely long period, a new special-purpose operational representational infrastructure was developed that reached beyond the operational infrastructure for arithmetic. However, in contrast with the arithmetic system, it was built by and for a small and specialized intellectual elite in whose hands, quite literally, it extended the power of human understanding far beyond what was imaginable without it. In the hands of an extremely small community over the next 250 years, the expressive and operational aspects of this narrowly scoped representational infrastructure made possible a science and technology that irreversibly changed the world, as well as of our views of it and of our place in it.

**CALCULUS AND THE IDEA OF “A CALCULUS”**

Our last example of representational infrastructure evolution involves calculus. The notion of an automatic computing machine to carry out numeric calculations, as we have seen, is very old. Leibniz, however, wanted to go further and be able to compute logical consequences of assumptions through an appropriate symbol system. He understood, perhaps more clearly than anyone before him, not only that choice of notation system was critically important to what one could achieve with the system, but also and more specifically, that a well-chosen syntax for operations on the notation system could support ease of symbolic computation. Hence, as reflected in correspondence with contemporaries, especially Huygens (Edwards, 1979), he was careful in the
design of a notation to represent his findings regarding how a function was related to what we now call its derivative or integral. His goal was that his new notation would support a "calculus" for computing such new functions in the general sense that the word "calculus" was used in those times. His nicely compact and mnemonic notation also allowed a direct expression of the relations between derivatives and integrals, relations expressed in the fundamental theorem of calculus.

Indeed, diSessa (2000), reminds us that Leibniz’s notation, which dominated the way calculus was used more than 300 years later, was at least as important as the insights that it encoded. After all, these ideas were also created by Newton. But Newton’s brilliant insights and methods have come to be learned and used for generations in Leibniz’s notation, and the reluctance of his British followers to adopt Leibniz’s notation was likely a significant factor in the century-long lag of British mathematics behind that of the Continent (Boyer, 1959; Edwards, 1979). diSessa pointed out that Leibniz’s notation became “infrastructural” (p. 11) in the same sense that we have been using the phrase “representational infrastructure” in this chapter. Incidentally, diSessa pointed out that the achievement of infrastructural status for Leibniz’s notation was in no small part due to the fact that it was easier to teach.

So once again, as in the case of arithmetic and then algebra, the development of a compact, efficient notation system turned out to be a critical factor in what followed.

COMPUTATIONAL MEDIA AND THE SEPARATION OF OUTCOME FROM PROCESS

The foregoing provides a brief overview of the structural changes in the semiotics of mathematical expression over time, leading to the emergence of complex and strongly supportive systems that sustain and expand the possibilities of human calculation and manipulation—at least for the few who were inducted into its use. We will argue that there are two key developments in a computational era: First, human participation is no longer required for the execution of a process, and second, access to the symbolism is no longer restricted to a privileged minority. To elaborate on the points, we will need just a little more historical perspective before we focus our attention on the digital media themselves. As recounted in Shaffer and Kaput (1999), the development of computational media required three elements: the existence of discrete notations without fixed reference fields (that is, the idea of formalism), the creation of syntactically coherent rules of transformation on such notations, and a physical medium in which to instantiate these transformations outside the human cortex and apart from human physical actions. Hence in the 20th century a profound shift has occurred, from operable notation systems requiring a suitably trained human partner for execution of the operations, to systems that run autonomously from a human partner.

ON CHANGING REPRESENTATIONAL INFRASTRUCTURES

Our starting point is the assertion that the extent to which a medium becomes infrastructural is the extent to which it passes as unnoticed. This is fine, until one needs to be aware of the structural facets of the medium to learn either how to express oneself within it or understand what might be expressed within it (or both). From the point of view of the learner, this can be confusing. For example, Leibniz’s notation

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2As a way of computing (derived, of course, from the Roman word for pebble because pebbles were used for computation, ironically, because the Roman numeral system was so inefficient).
is a second-order notation built on top of algebra because it guides actions on expressions built in algebraic notation (a fact that confuses many students even today who do not distinguish between “taking the derivative” and simplifying the result—after all, they are both ways of transforming strings of symbols into new strings of symbols). Because the ideas, constructions, and techniques of calculus are written in the language of algebra, knowledge of calculus has historically depended on knowledge of algebra. This in turn means that this knowledge has historically been the province of a small intellectual elite, despite the fact that the key underlying ideas concerning rates, accumulations, and the relations between them are far more general than the narrow algebraic representations of them in most curricula (Kaput, 1994).

Representational forms are often transparent to the expert. Musicians do not “think” about musical notation as they play an instrument any more than expert mathematicians have to (except when they are constructing a new notation or definition). But when one is learning or constructing something new, one needs to think explicitly about the representational system itself; we require, in other words, that the representational system is simultaneously transparent and opaque. This “coordinated transparency” (Hancock, 1995) represents a synthesis of meaning and mechanism, a situation (desirable but not always easily achievable) in which fluency with and within the medium can temporarily be replaced by a conscious awareness of its (usually invisible) internal structures. Grammar checking (human rather than computational) is a good example.

As noted above, the development of algebraic representational forms which generated fluency among the cognoscenti, took place within the semiotic constraints of static, inert media, and largely without regard to learnability outside the community of intellectual elite involved. Over the past several centuries this community’s intellectual tools, methods, and products (the foundations of the science and technology on which we depend) were not only institutionalized as the structure and core content of school and university curricula in most industrialized countries and taken as the epistemological essence of mathematics (Bochner, 1966; Mahoney, 1980) but in most countries became the yardstick against which academic success was defined. Thus the close relationship of knowledge and its culturally shared preferred representations, precisely the coupling that has produced such a powerful synergy for developing scientific ideas since the Renaissance, became an obstacle to learning, even a barrier that prevented whole classes from accessing the ideas the representations were so finely tuned to express.

Although the execution of processes was necessarily subsumed within the individual mind, decoupling knowledge from its preferred representation was difficult. But as we have seen, this situation has now changed. The emergence of a virtual culture has had far-reaching implications for what it is that people need to know, as well as how they can express that knowledge. We may, in fact, have to reevaluate what knowledge itself is, now that knowledge and the means to act on it can reside inside circuits that are fired by electrons rather than neurons. Key among these implications is the recognition that algorithms, and their instantiation in computer programs, are now a ubiquitous form of knowledge, and that they—or at least the outcomes of their execution—are fundamental to the working and recreational experiences of all individuals within the developed world.

Many individuals and social groups have suffered a massive deskilling of their working lives precisely because of this devolution of executive power to the machine.

3But not impossible: Indeed, a considerable amount of mathematical education research has tried to study—and encourage—the ways in which people form conceptual images of mathematical ideas independently of, and sometimes in conflict with, the preferred algebraic or formal representation.
But not all. Indeed many occupations (or at least parts of them) have become more challenging and enriching because of the introduction of digital technologies. What is common, however, is that the relationship between computational systems and individuals has become much more intimate than was ever envisaged. In part, this is a simple maturation of the technology—three of the more obvious and striking aspects are its miniaturization, the power of graphical displays and, of course, its connectivity (in 1982, the idea of communication as a central functionality of computers was the preserve of only a few experts in universities).

These technical facets of computer systems and the ways we use them, have re-shaped our relationship with them. On the one hand, they have reduced even further the necessity for “users” to make sense of how computational systems do what they do. The intimacy that, for example, a painter has with her brush—or a perhaps more relevant analogy, the relationship between a musician and her instrument—is rarely (currently) possible with the computer, despite the close proximity and personal relationship that many people have with their machines, especially handheld ones.

An accepted (but, as we shall see, fundamentally false) pedagogical corollary is that because mathematics is now performed by the computer, there is no need for “users” to know any mathematics themselves (for a well-publicized but disappointing set of arguments propounding this belief, see Brammall & White, 2000). Like most conventional wisdoms, this argument contains a grain (but only a grain) of truth. Purely computational abilities beyond the trivial, for example, are increasingly anachronistic. Low-level programming is increasingly redundant for users, as the tools available for configuring systems become increasingly high level. Taken together, one might be forgiven for believing that the devolution of executive power to the computer removes the necessity for human expression altogether (or at least, for all but those who program them).

In one sense, this is true. Precomputational infrastructures certainly make it necessary that individuals pay attention to calculation, and generations of “successful” students can testify to the fact that calculational ability can be sufficient (e.g., for passing examinations) even at the expense of understanding how the symbols work. In fact, quite generally the need to think creatively about representational forms arises less obviously in settings where things work transparently (cogs, levers, and pulleys have their own phenomenology). Now the devolution of processing power to the computer has generated the need for a new intellectual infrastructure; people need to represent for themselves how things work, what makes systems fail and what would be needed to correct them. This kind of knowledge is increasingly important; it is knowledge that potentially unlocks the mathematics that is wrapped invisibly into the systems we now use and yet understand so little of. Increasingly, we need—to put it bluntly—to make sense of mechanism.

Yet the need to make sense of mechanism is not fundamentally new. Indeed, the syntax of the numerical, algebraic, and calculus representation systems can be regarded as mechanisms, and the bulk of mathematics schooling has been devoted to teaching and learning that form of mechanism. There is a further complexity to the present situation, however. It is true that fewer and fewer people need to program computers, at least in the usual sense of the term “program.” But more and more people need to know something of how the machines and the systems (social, professional, financial, physical) operate—not just the few who are responsible for building them. We cannot adduce evidence for this assertion here (for a convincing selection of papers on this theme, see Hoyles, Morgan, & Woodhouse, 1999).

We share a vision of a mathematics curriculum that assumes mathematical understanding should be built around the construction and interpretation of quantitative and semiquantitative models, where students explore mathematical technologies and
analyze methods in contexts that show how they can be used and why they work in the way they do. We can also refer the interested reader to a series of papers, which have studied the mathematics of professional practices in a number of areas (aviation pilots, nurses, bank employees, and, most recently, engineers). (See, for example, Noss & Hoyles, 1996; Pozzi, Noss, & Hoyles, 1998; Noss, Pozzi, & Hoyles, 1999; Hoyles, Noss, & Pozzi, 2001.)

We restrict ourselves to two observations. First, at critical moments of their professional practice, people try to make sense out of complex situations by building mental models, or, if they do not have access to the raw material of model building, by circumventing them. Circumvention (ignoring inconvenient data) can be a dangerous strategy. To gain access to underlying models, to make them visible, is to focus on the quantities that matter and on the relationships among them. To gain such a sense of mechanism, one needs interpretative knowledge about, for example, graphs, parameters and variables, continuity, and a broad range of representational abilities that are different from, but no less important than, calculational and manipulational skills we have inculcated in young people until now.

The second observation concerns the complexity of interaction between professional and mathematical practices. It is true that more and more professional practice devolves calculational expertise to the computer. But it is not true (or if it is, it is dangerously so) that the computers can be left to make judgment (one of our examples concerns a life-and-death decision on a pediatric ward; see Noss, 1998). Judgment in the presence of intimate computational power requires new kinds of representational knowledge—distinguishing between what the computer is and is not doing; what can be easily modified in the model and what cannot; what has been incorporated into the model and why; and what kinds of model have been instantiated. As examples, we may consider the difference between parallel and serial computational models, how different kinds of knowledge are encoded with them, and what kinds of interpretation they allow; and, not least, the communicative value of representational knowledge in terms of sharing knowledge with others who interact with other parts of the same system or other, linked systems.

The new element in the situation is, of course, that the systems that control our lives are now built on mathematical principles. This is a major—perhaps the major—property of the virtual culture. The devolution of execution to the machines means more than this: Not only do the machines now do mathematical execution, it implies that any consequential appreciation of what the machines do must itself be based on mathematical principles. If an individual does not have the means formally to relate his or her intellectual model of the mathematical principles with those inside the machine, then appreciation of the model must necessarily be partial.

Of course, this does not mean that such models need to be expressed in the same languages as used inside the machine. Quite the reverse. It means that we have to find ways to help people to capture the dynamics of the system, so that they can follow the consequences of particular actions while maintaining a realistic sense of the structures of relationships between them. We now turn to some examples that begin to address the issues raised and then show how students can be stimulated to explore mathematical mechanisms and in so doing rebuild the synergy of knowledge and representation.4

4The former provides a putative enhancement of experiential phenomena and thus a richer base for intuitive knowledge. This is hardly unique to digital technologies: When mechanized transport was first invented, people for the first time found it “obvious” that centrifugal force was something to do with changing direction (the fact that it feels like centrifugal force rather than centripetal acceleration just shows that intuitions don’t always give the whole picture!).
A NEW REPRESENTATIONAL INFRASTRUCTURE FOR CARTESIAN GRAPHS COUPLED WITH EMBEDDED DERIVATIVES AND INTEGRALS LINKED TO PHENOMENA

Over the past two decades, the character string approaches to the mathematics of change and variation have been extended to include and to link to tabular and graphical approaches, yielding the “the Big Three” representation systems: algebra, tables, and graphs frequently advocated in mathematics education. However, almost all functions in school mathematics continue to be defined and identified as character-string algebraic objects, especially as closed form definitions of functions, built into the technology via keyboard hardware. In the SimCalc Project, we have identified five representational innovations, all of which require a computational medium for their realization but which do not require the algebraic infrastructure for their use and comprehension. The aim in introducing these facilities is to put phenomena at the center of the representation experience, so children can see the results, in observable phenomena, of their actions on representations of the phenomenon, and vice versa. These are as follows:

• The definition and direct manipulation of graphically defined and editable functions, especially piecewise-defined functions, with or without algebraic descriptions. Included is “snap-to-grid” control, whereby the allowed values can be constrained as needed (to integers, for example), allowing a new balance between complexity and computational tractability. This facility means that students can model interesting change situations while avoiding degeneracy of constant rates of change and postponing (but not ignoring!) the messiness and conceptual challenges of continuous change.

• Direct, hot-linked connections between functions and their derivatives or integrals. Traditionally, connections between descriptions of rates of change (e.g., velocities) and accumulations (positions) are mediated through the algebraic symbol system as sequential procedures employing derivative and integral formulas, which is the main reason that calculus sits at the end of a long sequence of curricular prerequisites.

• Direct connections between these new representations and simulations to allow immediate construction and execution of variation phenomena.

• Importing physical motion-data (via microcomputer-based lab or calculator-based lab [MBL/CBL]) and reenacting it in simulations and exporting function-generated data to define LBM (Line Becomes Motion) to drive physical phenomena (including cars on tracks).

We also employ hybrid physical–cybernetic devices embodying dynamical systems, the inner workings of which are visible and open to examination and control and the quantitative behavior of which is symbolized with real-time graphs generated on a computer screen.

We risk real danger by providing grayscale snapshots of colorful, dynamic, interactive lessons, especially by superimposing multiple problems and solutions on the same graphs. We provide some basic activities to illustrate concretely, albeit thinly, how this new representational infrastructure can work. First note that the various graphs appearing in the figures below are created piecewise simply by clicking, dragging, and/or stretching segments, although in other activities it is also possible to specify the graphs algebraically, by importing data, or by (partially constrained) drawing. (A similar set appears in Kaput, 2000.)
Variation, Area, Average, Approximation, Slope, Continuity, and Smoothness for Jerky Elevators

Suppose we are given a staircase velocity function (see Fig. 4.1), which drives the motion of the left-hand elevator to its left (these are color-coded in the software). The following kinds of lesson snippets are usually preceded by context-rich work that involves moving elevators around to accomplish various tasks, such as delivering pizzas to various floors and so forth.

1. How will the elevator move if driven by Plot 1 (the piecewise downward staircase velocity function in Fig. 4.1), and where will it end its trip? (It starts on the 0th floor.)
2. Does there exist a constant velocity function for the second elevator (just to the right of the first) that gets to the same final floor at exactly the same time as the first? If so, build it. (Plot 2: the one-piece constant velocity function)
3. Make a linear velocity function for the third elevator that provides a smoothly decreasing velocity approximating the motion of the first (staircase) elevator. Before running it, predict how far apart the first and third elevators will finish their trips. (Plot 3: the linear decreasing velocity function)
4. For the staircase velocity function (Plot 1) in Fig. 4.1, what is the corresponding position graph, and what is its slope at 3.5 sec? (Plot 4: the piecewise increasing position function)
5. Because the average-velocity function for the staircase has exactly the same area under it as the staircase, what is an easy way to draw its corresponding position graph? (Plot 5: the linear increasing position function)
6. What is the key difference between the position graph for the staircase and the position graph of the third velocity function? (Plot 6: the quadratic position function)

Question 1 involves interpreting variation via a variable velocity function with an integral that, to determine the final position, can be determined with whole number arithmetic. Such step-wise varying rate functions are intensively used in SimCalc.
4. LEARNABLE MATHEMATICS IN THE COMPUTATIONAL ERA

Instructional materials to build the notion of area as accumulation. Such functions also raise issues of continuity, acceleration, and physical realizability of simulations that are explored in depth using Motion Becomes Line (MBL) and Line Becomes Motion (LBM) technologies. Of course they also occur in economic situations with great frequency—tax rates, pay rates, telephone rates, and so forth.

Question 2 introduces the key idea of average, which, via our ability to use snap-to-grid to control the available number system also enables examination of when the average “exists” and whether it must inevitably equal the value of the varying function at some point in its domain. In traditional instruction, most students only experience continuously changing rates and hence never really confront the issue because the average always hits the continuously varying intermediate values.

Question 3 points up the reversal of the usual relationship between step functions and continuous ones (usually the former are used to approximate the latter) and highlights the integration of fraction and signed number arithmetic in the Math of Change and Variation (MCV). Position graphs and linearly changing velocity are developed over many lessons in many ways, including the differences between physical motion (including force–acceleration issues) and economic functions, so the glimpse here may be misleading in its abruptness.

Question 4 introduces the idea of slope as height of velocity segment and is part of an extensive study of slope as rate of change.

Question 5 illustrates the power of a “second opinion” because the position description of the average velocity motion is merely a straight line joining the start and end of the position graph. These two ways of describing change phenomena are treated as complementary throughout SimCalc instructional materials.

Question 6 deals with smoothness and is part of an extended introduction to quadratic functions as accumulations of linear ones that weaves back to issues of acceleration and physical motion and their physical realizability. An accompanying set of investigations examine nonphysical motion, for example, price or other money rates that change discontinuously such as tax rates, telephone rates, royalty rates, and so forth.

**Activities Linking Velocity and Position Descriptions of Motion in the Context of Signed Numbers and Areas**

The earlier parts of the next lesson, from which these snippets are taken, involve students in creating graphs to move Clown and Dude around, switching places at constant speed, coming together, and then returning to their original positions, and so on. (Only step-wise constant velocities have been made available here, although other function types could have been used, and in fact are used in SimCalc materials.)

**Challenge:** Clown and Dude are to switch their positions so that they pass by each other to the left of the midpoint between them and stop at exactly the same time. First, after marking off a line about 12 feet long, you and a classmate walk their motions! Now make a position graph for Clown and a velocity graph for Dude so that they can do this.

The student needs to construct graphs similar to Plot 1 (on the position graph) and Plot 2 (on the velocity graph) in Fig. 4.2. We have also shown the respective corresponding velocity and position graphs, Plots 3 and 4, which can be revealed and discussed later. Note that velocity and position graphs are hot-linked, so changes in the height of a velocity segment are immediately reflected in the slope of the corresponding position segment, and vice versa. Importantly, the activity requires interpretations of positive and negative velocities and hence provides meaningful work with
signed number arithmetic, as well as the representation of simultaneous position—paving the way for simultaneous equations. Later activities involve a storyline where Dude is patrolling the area (periodic motion) and Clown gets “interested” in Dude, follows him at a fixed distance, “harasses” him, and eventually, they dance—where the student, of course, is responsible for making the dance.

**Extensions to MBL and LBM:** The above representational innovations can be combined with the principals illustrated by Questions 5 and 6, mentioned earlier, to create opportunities to study the math of change and variation. For example, we can import and display motion data in the classic MBL/CBL ways, but in addition, we can now attach this physically based data to the objects in a simulation and replay their motion and compare it with motions defined synthetically, so that a student can perform and import a physical motion that can lead an entire group of dancers whose motions are created synthetically. Furthermore, a student can define a motion using a mathematical function (position or velocity) in any of the ways one might care to define a function and then “run” it physically in a linked LBM miniature car on a track. The forms of learning supported by these kinds of devices and activities, especially how they relate to one another and to physical intuition, are under active investigation, and the study of this richly populated space of interrelated inscriptions—and the new connections among physical, kinesthetic, cybernetic and notational phenomena—will continue for years to come. It also will be instantiated in increasingly networked contexts (Kaput, 2000; Nemirovsky, Kaput, & Roschelle, 1998).

The result of using these systems, particularly in combination and over an extended period of time, is a qualitative transformation in the mathematical experience of change and variation. Short term, however, in less than a minute using either rate or totals descriptions of the quantities involved (or even a mix of them) a student as early as sixth to eighth grade can construct and examine a variety of interesting change phenomena that relate to direct experience of daily phenomena. In more extended
investigations, newly intimate connections among physical, linguistic, kinesthetic, cognitive, and symbolic experience become possible.

**Preliminary Reflections on the New Representational Infrastructure**

A key aspect of the above representational infrastructure is revealed when we compare how the knowledge and skill embodied in the system relates to the knowledge and skill embodied in the usual curriculum leading to and including calculus. At the heart of the calculus is the fundamental theorem of calculus, the bidirectional relationship between the rate of change and the accumulation of varying quantities. This core relationship is built into the infrastructure at the ground level. Recall that the hierarchical placeholder representation system for arithmetic, and the rules built on it embody an enormously efficient structure for representing quantities (especially when extended to rational numbers), which in turn supports an extremely efficient calculation system for use by those who master the rules built on it. This is true of the highly refined algebraic system as well. Similarly, this new system embodies the enormously powerful idea of the fundamental theorem in an extremely efficient, graphically manipulable structure that confers on those who master it an extraordinary ability to relate rates of change of variable quantities and their accumulation. In a deep sense, the new system amounts to the same kind of consolidation into a manipulable representational infrastructure an important set of achievements of the prior culture that occurred with arithmetic and algebra.

**DEVELOPING A SENSE OF MECHANISM**

In this, penultimate section of our chapter, we focus on a corpus of work that is emerging from the Playground project (see www.ioe.ac.uk/playground). Like the SimCalc examples above, our interest focuses on new ways to express mathematical relationships, bringing children into contact with mathematized descriptions of their realities at ages much younger than we would normally countenance with static technologies. Unlike the SimCalc example, which typically involves students beginning at ages 11 or 12 (although it also is used at the university level), we are trying to explore what might be gained by younger children (aged between 4 and 8 years) building their own executable representations of relationships. In effect, we are redefining the idea of programming.

The rationale for programming has a long and distinguished history, stretching back some 30 years or more (see, for example, Feurzeig, Papert et al. 1969). We have no intention of rehearsing the argument here (see Noss & Hoyles, 1996b, for a history and rationale for programming in the context of mathematical learning). What is new is that programming has begun to change its character, having been expressed in various forms: as text (still the dominant form) as icons, and now, as we shall see, as animated code. We believe that this last change of expressive form marks a significant shift in what is possible for young children.

Our central focus is to open possibilities for children to design, construct, and share their own video games. We are designing computational environments for children to build and modify games using the formalization of rules as creative tools in the constructive process. We call these environments “Playgrounds.” We are working with two new and evolving programming systems, ToonTalk, an animated programming language (Kahn, 1999), and Imagine a concurrent object-oriented variant of the Logo programming language (Blaho, Kalas, & Tomcsányi, 1999; note, at this point, the language was named “OpenLogo”). Each of our two Playgrounds represents a layer
we have built on top of these platforms, incorporating elements that allow multiple entry points into the ideas of formalizing rules. In this chapter, we concentrate on our work with ToonTalk.

Our objective has a strong epistemological rationale. The challenge is to find ways for young children to use nontextual means to express and explore the knowledge underpinning the genre of video games, that is, what it means for objects to collide, how one can think about two-dimensional motions of an object (or a mouse or joystick), the construction of animation, and the hundreds of little pieces of knowledge that make up the workings of video games. We see this as an instantiation of a much broader class of knowledge, which, quite simply, we call developing a sense of mechanism.

Our choice of video games builds on established work by, for example, Kafai (1995) in that it has chosen a domain that seems to be naturally attractive for many children. We have no ulterior pedagogical or epistemological motive: We do not ask the children with whom we work to design games for any purpose other than their own amusement. Testing our intuitive belief that games themselves form a sufficiently rich backdrop against which to explore mathematical relationships forms part of our studies with children and form a central element of our design brief.

ToonTalk is a world in which animations themselves are the source code of the language; that is, programs are created by directly manipulating animated characters, and programming is by example (see Cypher, 1993). A full description can be found in Kahn, 1999). ToonTalk is constructed around the metaphor of a city, populated by houses (in which programs or methods are built), trucks are dispatched to build new houses (new processes spawned), robots are trained (for new programs or methods), and birds fly to their nests (message passing). A helicopter allows the user to navigate around the city or to hover above it watching trucks move around (as an aside, and to emphasize that ToonTalk is a Turing-equivalent language, it is both instructive and surprising to watch a city recursively grow and shrink as a quicksort is executed).

Robots are trained to carry out tasks inside houses (defining the body of a method). A user trains a robot by entering its “thought bubble” and controlling it to work on concrete values (see Fig. 4.3). The robot remembers the actions in a manner that

FIG. 4.3. A robot is trained to add one value to another.
easily can be abstracted to apply in other contexts by later removing detail from the 
robot’s thought bubble (see Fig. 4.4). Message passing between methods (robots) is 
represented as a bird taking a message to her nest, and changing a tuple is achieved 
based on taking items out of compartments of a box and dropping in new ones.

It is quite difficult in text and static graphics to convey the feel of programming 
with ToonTalk. The metaphor of moving around in a city is pervasive, and the sense 
of object-oriented programming in an environment by direct manipulation is a novel 
experience for those of us who believed that symbolic formalism of programming 
made interaction on a textual level inevitable.

The nature of the platform is paramount. Our choice of ToonTalk implied that any 
layers we built above it had to mesh with the metaphors of the platform. Our aim 
was to design a permeable abstraction barrier between ready-made pieces of open 
code with multimodal representations (we call these “behaviors”; some examples 
will be given below) and the ToonTalk language itself lying underneath. This stands 
in marked contrast to some modern programming languages such as Java and C++, 
which by default enforce these abstraction barriers and do not allow programmers 
using predefined objects to discover their underlying implementation. But the crucial 
dimension, which dictated the design of the playground layer, was that of openness. 
At any level of granularity, an element should be decomposable into smaller pieces 
down to the lowest level of the animated ToonTalk programming language. Indeed, 
as we began to see children decomposing the games and sharing their parts across 
sites and countries, it became clearer that we were working in a design paradigm 
akin to component software architecture (CSA). Although some (but not all) of the 
component community are concerned to a greater or lesser extent with the adaptability 
of their components, for us it is central. We are concerned with designing software 
for investigating mechanism; individual components therefore need to have intuitive 
windows to their workings and a means for modification. (For more information on 
the role of behaviors in playgrounds, see Hoyles, Noss, & Adamson, 2001).

In the design of an environment where the opening of mechanisms is the pri-
mary objective, it is desirable not only that pieces are easily opened but that they
afford access to their workings through an intuitive interface. In traditional CSA, the
user interacts with the interface model provided by the architecture but not the im-
plementation of individual components. In our open component model, we require
both. Users should be able to work at several levels simultaneously: (a) composing
components where necessary as wholes relying solely on the interface for compo-
nent manipulation and (b) opening a component to reveal the source code whenever
modification or inspection of the component is desired. To facilitate this, we need to
ensure that components interoperate at a technical level but also that manipulation
at interface and implementation levels is made intuitive by a high degree of semantic
interoperability. In other words, if users are to use, share, and manipulate components
in the construction of larger pieces of software, consistency of interface and multiple
ways of accessing the functionality become important criteria in their design.

An Illustration: The Space Behaviors Game

We start with a game based loosely on the “space invaders” genre of shooting games
(Fig. 4.5). The player controls a space ship that can fire white bullets in four directions.
If a white bullet hits an invader, it blows up, but if an invader hits the spaceship, the
spaceship is destroyed. The aim is to destroy the three “invaders” before they hit you.

The two boys in our case study wanted to change the appearance of the spaceship.
Working at the surface level, that is, changing the appearance of objects, is relatively
simple. Objects and behaviors are interoperable, so they simply had to select their new
object and transfer all the behaviors across by placing the old object on the back of the
new one. They chose a Poké-mon character called Pikachu as their new representation
of the space ship. Pikachu is associated with lightning so the boys wanted it to shoot
bolts of lightning rather than white bullets. Turning over Pikachu to reveal its robots
and behaviors, they could immediately identify the firing mechanism through its
visual representation (see Fig. 4.6). The specific representation here is quite subtle, as
the actual white bullets are not immediately visible. A modular, visually represented
architecture is required to give the clues for further inspection (see Fig. 4.7).

![FIG. 4.5. The original “space behaviors” game.](image-url)
FIG. 4.6. The behaviors on the spaceship.

FIG. 4.7. (A) The complete firing mechanism with four firing components. (B) The firing up component only.
At each level, the level below is visible. In Fig. 4.7, the firing behavior is shown in successive stages of exposure.

Taking apart the behavior down to the lowest level reveals the white bullet (Fig. 4.8). We are hardly in a position to claim that the boys fully understood the meaning of all the inputs to the firing robot or how the robot actually worked, but they could simply home in on the bullet. They knew that its functionality had to be taken over by lightning. The boys removed the bullet picture and put it on the back of a lightning picture and did this four times, once for each direction. Having modified the input to these four behaviors in this way and put them all back together on the back of Pikachu, the boys were ready to try out their modified game (Fig. 4.9).

But would it work? The boys thought so, but, as it turned out, they were wrong. The changes to the spaceship worked as expected, and Pikachu could fire lightning in four directions, but now the boys noticed that the “invaders” appeared to be indestructible. They were not sure why and guessed that it was because the destroy behavior had somehow gone missing from the back of the invaders. So they decided to check this out and removed the blue “invader” from the scene and turned it over to investigate its behaviors (Fig. 4.10 shows the back of the invader).

They recognized the relevant behavior by first noticing the explosion icon and the written rule, “I disappear when I touch a white bullet.” They removed this behavior from the back of the invader so they could study the next level and look at the actual code (see Fig. 4.11). They would then “see” the concrete representation of the condition for the robot, that is, it would perform its action (destroy object) when hit by a white bullet.

Seeing how the rule worked helped the boys debug what had gone wrong: The lightning picture had to appear in the place of the white bullet. Again, without having to appreciate how all the pieces of the mechanism worked, the boys could make this replacement and so achieved their rule change (see Figs. 4.11 and 12).

It might be that some readers will be wondering what this has to do with mathematics. Our reply is that it is about rules expressed formally and their implications and that, as we argued earlier, this is a central aspect of what it means to think mathematically
FIG. 4.9. The game changes after changes are made at the surface and input levels.

FIG. 4.10. The back of the blue invader.
in the computational era. At a more detailed level, this claim breaks down into two subclaims. First, it is about children learning there are rules that have implications for what is modeled and what is observed, and these rules are something over which they have some control. Second, these rules embody and are built on a previously constructed representational infrastructure that offers extraordinary power to those
who master it. Over the 2 years of the Playground project we have collected numerous examples of children taking apart a scene and exploring how it worked, why it worked, and how it could be changed. Often, a teacher was involved; in fact, a key aspect of the claim is that such an approach was more teachable than other programming environments because the things that mattered are visible and easily manipulable and the granularity of the pieces customized by the teacher for the learner. As with the SimCalc infrastructure, increasing learnability and expressive power for all students are fundamental goals.

CONCLUSIONS

In this chapter we have attempted to show how the evolution of representational infrastructures and associated artifacts and technologies have, over long periods of time, gradually externalized aspects of knowledge and transformational skill that previously existed only in the minds and practices of a privileged elite. We have sought to show how changes in representational infrastructure are intimately linked to learnability and the democratization of intellectual power. We have illustrated this point by reference to the development of number and algebraic notation, calculus, SimCalc graphs, and ideas of open and manipulable mechanisms. In each example the physical instantiation of these notations directly enlarged the limited processing power of human minds as well as affording experience of new domains of knowledge to solve new problems among populations that previously had no access to that knowledge and intellectual capacity. Computational media have provided a next step in the evolution of powerful, expressive systems for mathematics.

We have endeavored to illustrate our major contention: Mathematicians and mathematics educators need to turn their attention to defining these newly empowering representational infrastructures for children. In the past, beginning with writing itself (Kaput, 2000) more powerful representational infrastructures have been a source of intellectual and mathematical power, but at the price of learnability and hence access. These structures therefore tended to remain the province of an elite minority who were inducted into their use. New computational media offer the opportunity to create democratizing infrastructures that will redefine school knowledge (for a fuller discussion of these issues, see Noss & Hoyles, 1996). Viewed optimistically, these will exploit the processing power of the new media while ensuring that students maintain an intuitive feel for the central knowledge elements at work and how they relate to each other. Yet if the power and potential of computers is to be exploited in school mathematics, attention must be paid to this level of representational infrastructure. A companion need is to develop sustained curricula and modes of teaching and learning that incorporate and exploit these new representations and that encourage students to develop their meta-representational abilities (diSessa, 2000) so they become fluent with new systems of expression as they arise, can create and modify such systems themselves, and can make wise choices among them as these systems proliferate in the coming decades.

Thus we wish to challenge our community to focus attention on the design and use of representational infrastructures that intimately link to students’ personal experience. This is a necessary step if we are to move away from a 19th-century school mathematics concentrating on isolated skills based on static representational systems in a tightly defined curriculum (with only a minority able to engage in independent problem solving). Our contention is that knowledge produced in static, inert media can become learnable in new ways and that new representational infrastructures and systems of knowledge become possible, serving both the learnability of previously constructed knowledge and the construction of new knowledge.
REFERENCES


SECTION II

Lifelong Democratic Access to Powerful Mathematical Ideas
PART A

Learning and Teaching
CHAPTER 5

Young Children’s Access to Powerful Mathematical Ideas

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It has taken a long time for mathematics educators and mathematics education researchers to realize—in any concerted way—that young children, especially those who had not yet started school, were capable of anything but the most rudimentary mathematical development. One of us presented a paper to a mathematics education research conference in the 1970s on mathematics in preschools (Perry, 1977). We recall that it was greeted with disdainful remarks such as, “Does this mean that kindergarten [the first year of school] will become a remedial year?”

Of course, it is not only mathematics educators and researchers who have struggled with the notion that meaningful learning might occur before children start school. In 1991, the president of the United States declared that “all children in America will start school ready to learn” (National Education Goals Panel, 1991). This statement has generated a great deal of criticism because of its inherent bias against education in the home and other prior-to-school settings (Kagan, 1992; Shore, 1998). It is hoped that the notion that children’s learning starts when they come to school has been put to rest by the many efforts to demonstrate the value of learning in the preschool years.

CHAPTER OVERVIEW

We start this chapter with an overview of the characteristics of the early childhood years and children’s learning in these years. A historical perspective is used to link the more general discussion of early childhood with the notions of mathematics learning and teaching in these years. In the second part of the chapter, we consider the powerful mathematical ideas that research tells us are accessible to young children. We illustrate young children’s access to these ideas with examples of learning in both school and nonschool settings. From these and other examples, we move, in the third part

1 We use the term young children to designate people between the ages of 0 and 8 years (the early childhood years).
2 The term “prior to school” includes preschool learning centers, daycare, and other settings.
of the chapter to consider young children’s learning and use of these powerful mathematical ideas. In particular, we report on issues of importance to their mathematics learning as children start school. As well, we discuss some issues around teaching mathematics to young children. Finally, we make some suggestions for the future of early childhood mathematics education research which, it is hoped, will lead to the further enhancement of learning opportunities for our young children.

Learning and Young Children

In the documentary *Twice Five Plus the Wings of a Bird* (BBC Enterprises, 1985), the late Hilary Shuard implored viewers not to consider young children as “empty vessels.” In fact, she reminded us that we sometimes not only think of them as “empty” but also “leaky vessels” when it comes to their mathematics learning. It is important for us to reject this view of learners completely and, instead, treat all children as capable learners who know a great deal and who can learn a great deal more.

In an internationally accepted definition, *early childhood* refers to the period of a child’s life between birth and 8 years of age (C. Ball, 1994; Bredekamp & Copple, 1997; Organisation Mondiale pour l’Education Prescolaire, 1980; Schools Council, 1992). The definition of the early childhood period equates roughly with the first two stages of cognitive development as described by Piaget (1926, 1928): the sensorimotor stage and the preoperational stages. Although the link to Piagetian stages has resulted in the development of some significant programs, materials, and approaches to early childhood education (such as Early Mathematical Experiences in the United Kingdom (Schools Council, 1978) and the Bank Street and High/Scope programs in the United States of America (Cohen, 1972; Hohmann, Banet, & Weikart, 1979), it also has meant that young children, until about age 8, have been considered lacking in logical representational ability and incapable of using logical and abstract thought, resulting in the perception that children in the early years are “cognitively deficient” (Berk, 1997, p. 232).

Challenges to this position have cited the nature and complexity of the tasks employed (Donaldson, 1978; Gelman, 1972; Newcombe & Huttenlocher, 1992), observations of children’s competence in naturally occurring social interactions (Gelman & Shatz, 1978), understanding of appearance-reality contrasts (Woolley & Wellman, 1990), their construction of naive theories (Wellman & Gelman, 1992), and use of categorization (Keil, 1989) as evidence that children are anything but deficient in terms of understanding situations that matter to them. With this increasing awareness of children’s learning and children’s thinking (for example, Case, 1998; Gelman & Williams, 1998; Siegler, 2000), there is now a trend to regard children as possessing some logical ability in a range of circumstances. This is not to suggest that young children have the same understandings as older children or adults. Rather, it suggests that mature understandings develop gradually and that the beginnings of such understandings are to be found in the early childhood years (Schwitzgebel, 1999). Furthermore, it is clear that such understanding is to be found in social and cultural contexts that make sense to the children involved. In this vein, Berk (1997) described the early childhood period as a time in which “children rely on increasingly effective mental as opposed to perceptual approaches to solving problems” (p. 235).

The focus on the social and cultural contexts of children highlights a growing awareness of the impact of these areas not only on what children learn, but also on how it is learned and how it is taught. For example, Rogoff (1998) has emphasized that “learning involves not just increasing knowledge of content but also incorporation of values and cultural assumptions that underlie views about how material should be taught and how the task of learning should be approached” (Siegler, 2000, p. 27). A shift toward a consideration of Vygotskian principles relating to the social mediation of knowledge has prompted a focus on not only what it is that children are
capable of on their own (for example, as assessed through Piagetian tasks), but also what they are capable of achieving with the assistance of more knowledgeable others through scaffolding and through teachers developing and implementing tasks that target the zone of proximal development (Berk & Winsler, 1995; Bodrova & Leong, 1996; Dockett & Fleer, 1999).

Bredekamp and Copple (1997, p. 97) noted that the preschool years are “recognised as a vitally important period of human development in its own right, not as a time to grow before ‘real learning’ begins in school.” Although there remains a body of research to suggest that children undergo a significant cognitive shift between the ages of about 5 and 7 years (Flavell, Miller, & Miller, 1993), resulting in a greater ability to reason in more adultlike ways, it should not be assumed that this ability is totally lacking in younger children. The developments that occur in the early childhood years are remarkable for their speed, comprehensiveness and complexity. This is evident in all areas of development and learning. Although the focus of this chapter is young children’s mathematical skills, abilities, understandings, and dispositions, it is important to remember that all areas of development and learning undergo rapid change in the early years and each influences the other:

as children develop physically . . . the range of environments and opportunities for social interaction that they are capable of exploring expands greatly, thus influencing their cognitive and social development . . . Children’s vastly increased language abilities enhance the complexity of their social interactions with adults and other children, which in turn, influence their language and cognitive abilities . . . . Their increasing language capacity enhances their ability to mentally represent their experiences (and thus, to think, reason and problem-solve), just as their improved fine-motor skill increases their ability to represent their thoughts graphically and visually. (Bredekamp & Copple, 1997, p. 98)

SOME HISTORICAL PERSPECTIVES

Ideas about the importance of early childhood education are not new. Comenius, writing in the 17th century (1630), identified the early childhood years as particularly important for setting the directions of future learning:

If we wish him [sic] to make great progress in the pursuit of wisdom, we must direct his [sic] faculties towards it in infancy, when desire burns, when thought is swift, and when memory is tenacious. (Keating [English translator], 1910, p. 59)

More recently, educators in Reggio Emilia (a city in northern Italy) emphasized that “the image of the child as rich, strong and powerful . . . [with] potential, plasticity, the desire to grow, curiosity, the ability to be amazed, and the desire to relate to other people and to communicate.” (Rinaldi, 1993, p. 102)

These comments highlight the importance of learning in the early years of life, both in terms of the preparation this provides for future learning and of the value it has in its own right.

Historically, early childhood curricula (at least in the Western world) have evolved with a focus on the use of concrete materials and the value of children’s play. The same elements can be seen in many modern early childhood programs.

The use of concrete materials was critical in Pestalozzi’s (1894) curriculum, which proceeded “from the concrete to the abstract, from the particular to the general [as] . . . a way of adjusting instruction to the child’s order of development” (Weber, 1984, p. 30). Building on this work, Froebel (1896) implemented an educational approach grounded in conceptions of universal order. The emphasis on concrete materials remained strong. Froebel’s curriculum—based on a series of “gifts” (a set of 10 manipulative materials) and “occupations” (learning activities)—reflects this.
Since Froebel viewed knowledge as being achieved through the grasp of symbols, the Froebelian curriculum consisted of activities and the use of materials that had the larger meanings Froebel considered important symbolically embedded within them. (Spodek, 1973, p. 40)

Froebel also emphasized the value of children’s play, although his definition of what constituted play differs somewhat from modern conceptions (see Dockett & Fleer, 1999).

Froebel’s gifts and occupations included many pertinent to the development of mathematical concepts and processes. For example, the third gift, is “a two-inch block . . . divided once in each dimension producing eight smaller cubes” (Wiggin & Smith, 1896, p. 10). The cube and its subparts, were described as promoting both arithmetical and geometric understanding. The reason for this “gift” is explained in that “the rational investigation, the dissecting and dividing by the mind—in short, analysis—should be preceded by a like process in real objects. . . . Division performed at random, however, can never give a clear idea of the whole or its parts (Wiggin & Smith, 1896, p. 11). As the child uses these cubes, “new revelations . . . come at every turn” (Wiggin & Smith, 1896, p. 11).

As in Froebel’s kindergarten, programs developed by Montessori included a strong focus on concrete materials as a means of “isolat[ing] a general principle or concept. A child manipulates them, performing actions, and in the meantime, through this sensorimotoric experience, gets acquainted with the principle or concept involved” (Montessori, 1976, p. 65). Many of the materials developed by Montessori involve comparisons of size, quantity, or both—the Long Stair is one example. This apparatus consisted of

red and blue rods that are scaled to a decimal system based upon the unit of a decimeter. Children can learn to compare these in size and find multiples of the smaller ones. These units are then given the number names. The written names are presented in sandpaper . . . with children asked to say the name of the number as they trace it with their finger. (Spodek, 1973, p. 53)

Montessori (1973) described the Long Stair as easing children’s “entrance into the complex and arduous field of numbers” by making the experience easy, interesting and attractive by the conception that collective number can be represented by a single object containing signs by which the relative quantity of unity can be recognised, instead of by a number of different units. For instance, the fact that five may be represented by a single object with five distinct and equal parts instead of by five distinct objects which the mind must reduce to a concept of number, saves mental effort and clarifies the idea. (p. 205)

In addition to materials promoting the operations of addition, multiplication, division, and subtraction, Montessori developed a series of materials for teaching and learning geometry. The Montessori approach and the materials have the aim of developing children’s independent mastery of specific tasks. The materials, and their supposedly embodied concepts, were designed to match children’s interests, to support their independent use and to be self correcting. Many of the materials remain in present-day early childhood education settings, whether or not those in these settings espouse a Montessori approach to education and whether or not educators understand the mathematical bases for the materials.

Educational programs for young children flourished in many countries during the 20th century. Several influential programs adopted Piagetian theory as their basis. One of these, the High Scope program, implemented initially in Ypsilanti, Michigan,
has exerted significant influence on early childhood education programs across the world. Within this program, and reflecting the perspectives of Piaget, great import is attached to young children’s development of logico-mathematical knowledge. Hence, objectives and key experiences are outlined for classification, seriation, number, and space and time (Hohmann et al., 1979; Kamii, 1973).

Much of the basis of the High Scope program has been reiterated within the concept of developmentally appropriate practice (DAP; Bredekamp, 1987; Bredekamp & Copple, 1997), which has had a substantial impact on teaching and learning in early childhood settings in recent years. A focus on a predictable pattern of development, influenced by individual variability, characterizes this approach. The pattern of development used as a basis for DAP is essentially Piagetian. Despite this, DAP is not explicit about the nature of logico-mathematical experiences and practices appropriate for young children. In trying to avoid a “push-down” academic curriculum (Elkind, 1987), there is a sense of avoiding “hard topics” such as mathematics all together. For example, appropriate practice for infants and toddlers makes no mention of mathematical interactions. Only when children reach 3 to 5 years of age is there recognition of the need to plan a variety of concrete learning experiences with materials and people relevant to children’s own life experiences and that promote their interest, engagement in learning and conceptual development. Materials include, but are not limited to, blocks and other construction materials, books and other language-arts materials, dramatic-play themes and props, art and modeling materials, sand and water with tools for measuring, and tools for simple science activities. (Bredekamp & Copple, 1997, p. 126)

Once again, there is emphasis on young children’s use of concrete materials. This is still the case in one of the most influential educational approaches of recent years—Reggio Emilia. Reggio Emilia has a municipal early childhood system based on the distinctive philosophical base of promoting children’s intellectual development through symbolic representation (Edwards, Gandini, & Forman, 1993). A reliance on concrete materials is incorporated within this. In this approach, however, relationships are regarded as the key to a successful learning and teaching experience. Learning takes place in relationships—with adults and children each making appropriate adjustments if the interactions are to continue: “the way we get along with children influences what motivates them and what they learn” (Malaguzzi, 1993, p. 61). Relationships are not seen just as a warm protective envelope, but rather as a dynamic conjunction of forces and elements interacting toward a common purpose. . . . We seek to support those social exchanges that better insure the flow of expectations, conflicts, cooperations, choices, and the explicit unfolding of problems tied to the cognitive, affective and expressive realms. (Malaguzzi, 1993, p. 62)

Children in Reggio Emilia settings are described as learning through communication as well as concrete materials, with “the system of relationships ha[v]ing in and of itself, a virtually autonomous capacity to educate . . . it is a permanent living presence always on the scene, required all the more when progress becomes difficult” (Malaguzzi, 1993, p. 63). It is within relationships that children make meaning. There is regard for children as autonomous meaning makers, with the emphasis that meanings are never static, univocal, or final; they are always generative of other meanings.” Within relationships, the adult role is described as one of activating, “the meaning-making competencies of children as a basis of all learning” (Malaguzzi, 1993, p. 75). The importance of relationships in early childhood learning and teaching
is a recurrent theme in recent research. It is revisited in this chapter in our discussion of learning and teaching mathematics.

WHAT POWERFUL MATHEMATICAL IDEAS ARE ACCESSIBLE TO YOUNG CHILDREN?

Consider the following examples of children interacting with mathematics.

Example 1

In the context of a clinical interview on statistical thinking, a seven-year-old Vietnamese/Australian child, Chi, was asked to find the average of five single-digit scores in a well-known children’s game. After thinking through the question for about 30 seconds, Chi gave the correct answer and explained it by saying, “Made the average, plussing all together and divide by 5. I learned that at Vietnamese school” (Putt, Perry, Jones, Thornton, Langrall, & Mooney, 2000, p. 524).

Example 2

Six-year-old Jeremy drew a shape on a deflated balloon and blew it up.

Jeremy: It’s gone, ‘cause I blew it up too much and the ink’s gone, it’s fade.
Teacher: Why has it faded?
J: It’s fade cause it goes stretches and the ink disappears. The ink stretches and leaves little dots and then it disappears. It gets smaller and smaller and it disappears.
T: How comes this happens?
J: Because it was very long and once it grows they get to be little dots and then it disappears. Then it gets disappearing.
T: What makes it disappear?
J: Because it’s stretching. Because it’s growing bigger, cause we’re blowing air into it. Air.
T: Does air make things grow bigger?
J: Yes. Because it’s stretching it inside and if you stretch it inside it grows bigger on the outside as well (Dockett & Perry, 2000).

Example 3

A six-year-old boy, Joshua, and one of the authors, were solving mathematical problems when we came to this one: “I’m going to have a party, and at this party, I plan to invite two friends. I have already bought the lollies for the party and in the packet there are sixteen lollies. . . . How many lollies would each of us get?”

After a great deal of mental arithmetic, counting on his fingers, counting by twos and by threes and fours and fives and sixes and so on, Joshua declared that there was no answer, that the problem was impossible, and that he had done it every way imaginable. I even invited him to use counters. So he counted out by twos until he had . . .

3“Vietnamese school” refers to school experiences supplementary to and quite separate from the child’s elementary schooling. For the most part, such schools are for the maintenance of the home language and the learning of English. However, there is obviously some mathematics taught as well.
16 counters in front of him. He then proceeded to share these into three collections, one for each of the party goers. Of course, there was one left over. I asked him what he might do with that one and his answers were quite intriguing.

| Joshua:    | Well you could share out the lollies before all the friends came and have the extra one yourself, or you could give the extra one to your mother. |
| Perry:     | Yes, are there other things you could do? |
| Joshua:    | Yes, you could cut it into two pieces and give each one of your friends half each. |
| Perry:     | Right, but anything else you could do? |
| Joshua (after some thought): | Yeah, you could cut it into quarters and you could each have a quarter. |
| Perry:     | Would that use all of the lollies? |
| Joshua:    | Yes, well, really not quarters, no, they’re sort of halfway between a half and a quarter. (Perry, 1990, 451–452) |

Example 4

A four-and-half-year-old girl, Jovalia, and an adult had just spent some time singing and playing the song about “Five little ducks went out one day.” Jovalia drew a picture and proceeded to talk about it.

“Mother duck is in the middle of the pond.”
“And what are the little marks around the edge of the pond?”
“They are the little ducks, silly. You are looking at them from above.” (Adapted from Perry & Conroy, 1994, p. 69)
Example 5

Four-year-old Jessica is standing at the bottom of a small rise in the preschool yard when she is asked by another four-year-old on the top of the rise to come up to her. “No, you climb down here. It’s much shorter for you.”

Example 6

A two-year-old boy, Will, is traveling to day care with his mother when he notices a plane flying overhead in roughly the opposite direction to that in which the car is traveling. “Mummy, that plane is going backward.”
“What do you mean by backward?”
“It is going behind my back.”
“Is that the same thing as going backward?”

All of these examples show clear evidence of the young children involved using mathematical ideas in meaningful and relevant ways. They provide some useful starting points for a discussion of young children’s access to powerful mathematical ideas. As a cautionary note, we stress that it is unlikely that anyone can be comprehensive about listing the particular powerful mathematical ideas that are accessible to all—or even some—young children because children have a habit of surprising whenever we think we have the whole story. One aspect is clear, however: Mathematical ideas that are genuinely powerful for young children have much more to do with the processes used to interact with and do mathematics than with particular items of mathematical knowledge. Hence, having a sense of number and a collection of strategies for dealing with numerical problems can be much more important to a young child than being able to recite the basic addition facts (Cobb & Bauersfeld, 1995; De Lange, 1996; Heuvel-Panhuizen, 1999; Kamii & DeClark, 1985). Similarly, being able to connect certain pieces of mathematics to situations that are relevant to the children and to use certain mathematics to help resolve such situations is much more important than knowing the “correct” mathematical terminology or notation (Cobb, Yackel, & McClain, 2000; Yackel & Cobb, 1996). This is not to say that mathematical “facts” are irrelevant. Rather, that they are not necessarily uppermost in the minds of children as they engage in mathematical experiences. However, there do seem to be certain processes that constitute mathematical power for young children.

Powerful Mathematical Ideas

In this section, we discuss the evidence of access to a number of powerful mathematical ideas by young children. Not surprisingly, many of these powerful ideas are also canvassed by Jones, Langrall, Thornton, and Nisbet in chapter on elementary students in this volume. In this rendition, however, we concentrate on the early childhood years and describe what we believe to be impressive evidence of access to these ideas by even the youngest of children.

The particular powerful mathematical ideas to which we suggest young children have access include:

- Mathematization
- Connections
- Argumentation
- Number sense and mental computation
- Algebraic reasoning
Spatial and geometric thinking
Data and probability sense

Each of these is begun and must be nurtured in the early childhood years.

Mathematization

Mathematization is a term coined by the eminent Dutch mathematics educator Hans Freudenthal in the 1960s to signify the process of generating mathematical problems, concepts, and ideas from a real-world situation and using mathematics to attempt a solution to the problems so derived. Two forms of mathematization are distinguished. The first is horizontal mathematization, where “students come up with mathematical tools that can help to organize and solve a problem set in a real-life situation” (Heuvel-Panhuizen, 1999, p. 4). The other is vertical mathematization which “is the process of reorganization within the mathematical system itself” (Heuvel-Panhuizen, 1999, p. 4). De Lange (1996, p. 69) expanded on these components of mathematization in the following way:

First we can identify that part of mathematization aimed at transferring the problem to a mathematically stated problem. Via schematizing and visualizing we try to discover regularities and relations, for which it is necessary to identify the specific mathematics in a general context. . . .

As soon as the problem has been transferred to a more or less mathematical problem this problem can be attacked and treated with mathematical tools: the mathematical processing and refurbishing of the real world problem transformed into mathematics.

Mathematization always goes together with reflection. This reflection must take place in all phases of mathematization. The students must reflect on their personal processes of mathematization, discuss their activities with other students, evaluate the products of their mathematization, and interpret the result. Horizontal and vertical mathematizing comes about through students’ actions and their reflections on their actions. In this sense the activity mathematization is essential for all students—from an educational perspective.

The critical and central role of mathematization is further expanded by Gravemeijer, Cobb, Bowers, and Whitenack (2000, p. 237):

the goal for mathematics education should be to support a process of guided reinvention in which students can participate in negotiation processes that parallel (to some extent) the deliberations surrounding the historical development of mathematics itself.

The heart of this reinvention process involves mathematizing activity in problem situations that are experientially real to students.

Examples 2 and 3 featured earlier show that young children mathematize. This is also clear from numerous studies with children in the first years of school (English, 1999; English & Halford, 1995; Jones, Langrall, Thornton, & Mogill, 1997; Jones, Langrall, Thornton, Mooney, Watres, Perry, Putt, & Nisbet, 2000; Jones, Thornton, & Putt, 1994; Yackel & Cobb, 1996). Jeremy’s explanation, in Example 2, for the disappearance of the shape from the balloon has involved his translation of the physical drawing into a mathematical model of lines consisting of dots, as well as a clear understanding of the links between the interior and exterior of objects. Joshua, in Example 3, uses his mathematical understandings to solve the continually evolving problem of the remainder in division, having first translated the real-world problem of lollies into a numerical problem using physical counters. He draws mathematics from the problem and uses it to suggest a solution. Joshua clearly has an embryonic understanding
of what a fraction is and how it might be illustrated through a model. Further opportunities for him to work with similar problems and to pose such problems for others will continue to enhance the development of this understanding (Bobis, Mulligan, Lowrie, & Taplin, 1999; English & Halford, 1995).

The examples with the children who have not started school—Examples 4, 5, and 6, above—also show that these children can and do undertake mathematization. Josephina is clearly using the mathematical concept of perspective to help explain her drawing, whereas Jessica has adopted a developing concept of comparison of length to solve—at least for her—the physical dilemma of having to walk up the rise. Even two-year-old Will has translated his observation of the direction in which the airplane is flying into a mathematical problem in which he holds a central role. It has been claimed that even children younger than Will are capable of using mathematical ideas to assist their purposes in real world situations (Simon, Hespos, & Rochat, 1995; Wynn, 1992, 2000), although these claims have been disputed recently (Wakeley, Rivera, & Langer, 2000).

Connections

The question of connections—mathematics learning being related to learning in other areas or mathematics learning being relevant to the contexts in which the child is working or playing and learning in one area of mathematics being related to learning in another area of mathematics—is clearly pertinent in the early childhood years, both in prior-to-school and school settings, where children are beginning to develop their knowledge and skills in mathematics while applying them to their own contexts. In some instances, these connections are enhanced by integrated curriculum. For example, a child learning to count will use this to find the answers to questions of “how many” in many meaningful situations. The development of the knowledge and skill go hand-in-hand with their application. Just as mathematics is learned “in context” so it is used “in context” to achieve some worthwhile purpose.

When students can connect mathematical ideas, their understanding is deeper and more lasting. They can see mathematical connections in the rich interplay among mathematical topics, in contexts that relate mathematics to other subjects, and in their own interests and experience. . . .

Mathematics is not a collection of separate strands or standards, even though it is often partitioned and presented in this manner. Rather, mathematics is an integrated field of study. Viewing mathematics as a whole highlights the need for studying and thinking about the connections within the discipline. (National Council of Teachers of Mathematics, 2000, p. 64)

In many parts of the world, the notion of mathematical connections is strongly related to other concepts with labels such as numeracy, mathematical literacy, or quantitative literacy (Department of Education, Training and Youth Affairs, 2000; Devlin, 2000; Hughes, Desforges, & Mitchell, 2000; Wright, Martland, & Stafford, 2000). A succinct description of numeracy is that it involves using “some mathematics to achieve some purpose in a particular context” (Australian Association of Mathematics Teachers, 1997, p. 13), whereas mathematical literacy has been described as having components including “thinking, talking, connecting, and problem solving” (Liedtke, 1997, p. 13). At the early childhood level, numeracy, mathematical literacy, and mathematics go hand in hand (Liedtke, 1997; Perry, 2000) as children, for example, strive to satisfy all of their friends by sharing out their lollies evenly to avoid social turmoil or teachers or parents use timers to ensure that children playing with a computer program can be assured of having a fair turn. The application of mathematics to a contextual problem or challenge confronts young children throughout their day in prior-to-school settings, schools, homes, and shopping centers, to name just a few.
contexts. To solve these problems and meet the challenges, young children need not only to have developed their mathematical skills and knowledge but also their dispositions and self-confidence so that they are willing to apply these in novel situations. The contextual learning and integrated curriculum apparent in many early childhood—particularly prior-to-school—settings ensures that there is little distinction to be drawn between numeracy, mathematical literacy, and aspects of mathematical connections with the children’s real worlds.

Some instantiations of this can be seen in the examples given above. Chi, in Example 1, made clear connections between the mathematics that she has been doing at the Vietnamese school and the questions asked by the interviewer—different contexts but connected by mathematics. Jovalia linked her literature experience with mathematics by using the mathematical idea of perspective to help record her experience. Jessica has clearly linked early measurement ideas to her need to avoid too much physical exertion by walking up the rise, whereas Will tried to explain the motion of the airplane through the use of the direction words with which he is familiar.

One of the clearest links between mathematics learning and children’s contexts occurs when we consider children’s literature. For example, Ginsburg and Seo (2000) highlighted the many mathematical ideas that can be introduced to children through “reading” books in prior-to-school settings. Lovitt and Clarke (1992) suggested that using books, stories and rhymes to stimulate thinking about mathematics and to develop and reinforce mathematical concepts enhances children’s understanding, promotes their enjoyment of the subject and develops their view of mathematics as an integral part of human knowledge. The context of the story gives a framework for the exploration of mathematical ideas. (p. 439)

There are many fine examples of children’s literature and numerous suggestions as to how these might be used by teachers (see for example, the Links to Literature section of recent numbers from *Teaching Children Mathematics*). Literature can provide a useful link between mathematics and something that most children seem to enjoy, although care should be taken to ensure that the joy of the literature is not lost through the overly zealous pursuit of the mathematics—or vice versa.

In both prior-to-school and school settings, one powerful way in which the mathematics children learn can be connected to them and their knowledge base is through consideration of cultural aspects of learning mathematics (see, for example, Barta & Schaelling, 1998; Perry, 1990; Perry & Howard, 2000; Perso, 2001). One activity the authors have found quite useful in celebrating the diversity of cultures that occur in the classroom is that of Honest Numbers (Bezuska & Kenney, 1997). This activity encourages the celebration—in a fun way—of the cultures the children bring to the classroom and shows that there is more to mathematics than the canonical Western curriculum that has become so dominant in schools around the world (Nebres, 1987; Shuard, 1986).

There are clear connections between different aspects of mathematics that need to be developed in and understood by children. Young children have access to some of these links as well. In Example 2, Jeremy used aspects of geometry—the notion that a line is made up of many parts—and the topological idea that changing the inside of a shape will affect the outside to attempt an explanation of why the shape fades on the inflated balloon. Joshua has clearly developed useful links between his understandings of whole and rational number. Another connection within mathematics is that between number and measurement ideas. Recent research (Cobb, Stephan, McClain, & Gravemeijer, 1998; McClain, Cobb, Gravemeijer, & Estes, 1999; Outhred & McPhail, 2000; Outhred & Mitchelmore, 2000; Stephan, 2000) suggests that measurement ideas are dependent on the notions of unitizing and of composite units, thus linking the two mathematical areas in terms of their underlying processes.
Argumentation

For many people, arguing is a feature of everyday life as they try to justify actions, negotiate situations, and implement compromises. Krummheuer (1995, p. 229) described argumentation as a “social phenomenon, when cooperating individuals [try to] adjust their intentions and interpretations by verbally presenting the rationale of their actions.” The process allows children, and other participants, to justify not only their own mathematical thinking but also to distinguish between the strengths of arguments and whether the mathematics being constructed within the arguments is actually different from previous mathematical arguments that have been interactively constructed (Voigt, 1995; Yackel, 1995, 1998; Yackel & Cobb, 1996). Based on the work of Piaget (Inhelder & Piaget, 1958/1977), the ability to argue logically was placed within the stage of formal operations and so was considered beyond the realms of young children. Recent work, in mathematics education and in other areas of cognitive development, suggest that this is not necessarily so (Dockett & Perry, 2000; Horn, 1999; Krummheuer, 1995; Leitao, 2000; Maher & Martino, 1996a, 1996b; Perry & Dockett, 1998; Pontecorvo & Pirchio, 2000; Yackel, 1998; Yackel & Cobb, 1996), with many examples of young children interactively constituting argumentation. As Joshua, in Example 3, explained how his thinking was developing toward an understanding of fractions, he demonstrated the value of argumentation in the mathematical development of young children. Similarly, the scaffolded discussion between the teacher and Jeremy in Example 2 resulted in a strong argument from Jeremy as to why his drawing had faded.

Quite young children are capable of dealing logically with their lives and their mathematics. It may not necessarily appear to adults that a child is using logic, but it will be coherent and logical to the child:

> the preschool child has a solid explanatory basis for his [sic] everyday life, within which, on the one hand, the facts are not generally accepted but are interpreted by his [sic] own ‘logic’ and, on the other, the motives of actions and facts are clear and comprehensible. (Tzekaki, 1996, p. 58)

A telling example of the “logic” that young children might use is provided in the following excerpt from a transcript involving two girls aged four-and-a-half years playing in the family area of their preschool:

Stella placed her hands on her hips and sighed. Jane adopted a similar stance and called loudly, “I’m the mother, I’m the mother.” She then moved closer to Stella, stood straight, and added “I’m the mother! See, I’m bigger than you!” Stella also stood up straight saying, “I’m bigger! And I’m gonna tell my Mum!” “No, I’m bigger,” replied Jane, “I’ll show you.” She stood right next to Stella and said, “Look! See, I’m bigger!” Stella looked, and stretched as high as she could. “And I’m big!” Jane looked again and complained, “Don’t stand on tippy toes, that’s not fair!” When Stella did not react, Jane added, “I’m gonna see my Daddy.” (Perry & Dockett, 1998, p. 8)

Even though both Stella and Jane have decided that (different) higher authorities are required to resolve a conflict situation, there are some points of agreement emerging. In particular, from a mathematical point of view, both seem to have determined that size, interpreted as height, is the critical factor in determining who should be ‘Mum.’ The use of argumentation in such young children points to this powerful mathematical idea being accessible to children much younger than Piaget would have suggested and even younger than might have been recognized by later
researchers. Given that argumentation will form the basis of mathematical proof in later years, it is important for us to realize the early genesis of this process.

**Number Sense and Mental Computation**

Number sense is “a person’s general understanding of numbers and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful and efficient strategies for dealing with numbers and operations” (McIntosh, Reys, & Reys, 1997, p. 322). In Example 3, Joshua provides an excellent example of a young child’s number sense as he explained his solution to the party lollies problem. Because almost all the mathematics that children encounter in elementary school, and much of what they encounter beyond that level, is firmly based in number, the importance of sound number sense cannot be overstated.

The Piagetian notion that classification, conservation, and ordering of number were foundational aspects on which many other aspects of number had to wait may have acted as a deterrent to the recognition and development of the extensive number repertoire of many young children (Davies, 1991; Gifford, 1995; Verschaffel & De Corte, 1996; Young-Loveridge, 1987). Hughes (1986), for example, clearly showed that before attending school, children understood concepts such as subtraction and that they could represent number and operations with these numbers when they were linked to concrete objects, even if these were hidden. Gifford (1995) and Ewers-Rogers and Cowan (1996) also noted young children’s use of idiosyncratic symbols for number. Bertelli, Joanni, and Martlew (1998) showed that 3-year-olds are able to reason about number, answering questions about more and less, even before they have mastered counting. Sophian and Vong (1995) have noted the use of part–whole relationships in number by 4- and 5-year-olds.

Young children perceive and use numbers in almost every context they experience. Their play can provide many of these experiences. Play activities such as making appointments and shopping (Gifford, 1995) taking the bus, using phones, and playing cards (Ewers-Roger & Cowan, 1996) are examples. Encounters with stories, rhymes, and other children’s literature (Copley, 2000; Ginsburg & Seo, 2000; Whitin, 1994, 1995) also can involve young children in meaningful number experiences. Many young children enjoy talking about “big” numbers and about fractions such as “half” (Gifford, 1995; Hunting & Davis, 1991).

The importance of counting to young children’s number development is well known (see, for example, Carraher & Schliemann, 1990; Steffe, Cobb, & von Glasersfeld, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983; Verschaffel & De Corte, 1996). Many early number programs are now based on the enhancement of children’s counting skills, including access to the forward and backward number–word sequences, skip counting, and counting in realistic situations (Wright et al., 2000). The need for facility in the use of the composite unit in base ten representations of number is seen to be a critical aspect of this approach to number (Cobb & Wheatley, 1988; Pengelly, 1990; Thomas & Mulligan, 1999; Wright, 1994). Certainly this facility is one well within the reach of children in the first years of school, if not earlier for some (Beishuizen, van Putten, & van Mulken, 1997; Jones et al., 1994; Menne, 2000; Tang & Ginsburg, 1999; Yackel, 1995).

Mental computation is an integral part of young children’s learning about number. It can be used as a tool to facilitate the meaningful development of mathematical concepts and skills and to promote thinking, conjecturing, and generalizing based on conceptual understanding (Reys & Barger, 1994). Mental computation is closely linked to the development of number sense and enables a “focus on strategies for computing with whole numbers so that students develop flexibility and computational
fluency” (National Council of Teachers of Mathematics, 2000, p. 35). Chi, in Example 1, demonstrated facility with mental computation when she calculated the average by “plussing all together and divide by 5,” whereas Joshua did a lot of mental calculation before declaring that there was no answer, that the problem was impossible, and that he had done it every way imaginable.

**Algebraic Reasoning**

Algebraic reasoning in the early childhood years often comes in the guise of patterning activities and challenges, where relationships of equality and sequence and of argument are developed.

Much of this patterning has to do with number and the development of a flexible, sound number sense. Many of the strategies developed by young children, including both inductive and deductive reasoning, will be useful in later years as the children work with number, especially in the development of their place value ideas and of their facility with counting (Schifter, 1999; Tang & Ginsburg, 1999). In particular, Schifter (1999, p. 80) made

the case for an emphasis on the development of operation sense as crucial to this preparation [for algebra instruction]. . . . once the teaching of elementary school arithmetic is aligned with reform principles—when classrooms are organised to build on students’ mathematical ideas and keep students connected to their own sense-making abilities—then children so taught will be ready for algebra.

In Example 1, Chi’s approach to the calculation of average suggests that she has a particular rule, in the form of an equation, that can be applied to the problem regardless of the numerical values occurring. Another example brings to the fore the importance of children’s cultural context in their learning of mathematics, and of patterning in particular.

In Taiwan, young children are taught to applaud success by clapping in socially appropriate ways. While it is clear that clapping in time involves some measurement skill, the patterns used are also mathematical. For instance, clapping in the following way: clap-clap/clap-clap-clap/clap-clap-clap-clap/clap-clap means “cheering with love” in Taiwan. Desirable social attributes can be integrated into mathematics learning. (S. Leung, personal communication, November 3, 2000).

Despite the example of Joshua given earlier, clear examples of proportional reasoning are rarely found among young children. Hence, it is mentioned only in passing here. Reporting the work of Resnick and Singer, English and Halford (1995) suggested that children may know about “covariation” before they come to school. For example, they may realize that bigger people wear bigger clothes or eat bigger meals. However, although this is clearly a precursor to proportional reasoning, it falls well short of even the beginnings of a comprehensive understanding. It is well known that proportional reasoning is an advanced mathematical idea. Lesh, Post, and Behr (1988, p. 93) called it “the capstone of children’s elementary school arithmetic.” Hence, it is not surprising that it does not appear, except in its most embryonic forms, in the early childhood years. Nonetheless, researchers have found many examples of mathematical reasoning among young children, and it seems appropriate to conclude this section by celebrating this and warning of the dangers of assuming that young children, from whatever background, are not capable learners.

Despite some opinion to the contrary, low-income minority children are capable of complex mathematical reasoning. They arrive in school with considerable capability for abstract thought and potential for learning mathematics. Indeed, potential for learning
mathematics may well be universal. Virtually all young children may well be capable of the kinds of reasoning we have described. Yet educators often fail to recognize, nourish, and promote mathematical abilities, particularly those of the disadvantaged. As a result, poor children’s subsequent inferior performance in later school mathematics should be attributed more to our failures in educating them than to their initial lack of ability. (Tang & Ginsburg, 1999, p. 60)

Spatial and Geometric Thinking and Data and Probability Sense

The areas of data and probability, space and measurement all feature in the early childhood years both before and during primary school. Data plays a critical role in modern society. Much information uses statistical ideas and is transmitted through graphs and these tables. Children at all levels of schooling need to be able to deal with these data in a sensible way. In the same way that they need to develop a sense about number, they need a sense about data. They need to be able to treat reports of data critically and to establish the veracity of claims for themselves—or, at least, to test this veracity when claims are made. The work of Watson and Jones and their teams (see, for example, Jones et al., 2000; Watson & Moritz, 2000) established in Australian contexts the need for children to develop such a data sense from an early age. Complementary work in other parts of the world has reinforced this notion of building data sense (Cobb, McClain, & Gravemeijer, 2000; Curcio, 1987; McClain, Cobb, & Gravemeijer, 2000; Shaughnessy, 1997; Shaughnessy, Garfield, & Greer, 1996).

Almost everyone has chance (probability) experiences every day. We regularly meet the language of probability—we hear 2-year-olds talk about the chance that it will rain, or that they will receive a lollypop as a result of being good, for example. Early introduction of probability language and experiences can assist in the avoidance of misconceptions in problems where intuition alone is insufficient to solve them (Bright & Hoefner, 1993; Jones et al., 1997; Shaughnessy, 1992; Way, 1997). There is a need to give children the opportunity to develop their thinking about chance and its quantification so that they are able to build on the informal chance experiences they will have in their lives and are in a position to make sensible decisions in situations of uncertainty (Borovcnik & Peard, 1996; Peard, 1996).

Spatial thinking involves processes such as recognition of shapes, transforming shapes, and seeing parts within shape configurations. It also involves spatial conceptualizing and the interaction of visual imagery with these concepts. Children in the early childhood years begin to reason about shapes by considering certain features of them. Spatial thinking plays a role in making sense of problems and in representing mathematics in different forms such as diagrams and graphs. The use of manipulatives in the development of mathematical ideas can require some spatial awareness. Spatial ideas—usually called geometry—was one of the first areas of mathematics to be systematically taught to young children. Many of Froebel’s “gifts” mentioned earlier in this chapter were based in geometry. More recently, in a study designed to see whether preschoolers could think analytically about space, Feeney and Stiles (1996) showed that by age four and a half, children were able to distinguish wholes and parts of simple designs such as plus or cross signs. They could do this by construction, by perception, by selecting from a picture, and by drawing. Clearly, young children have access to many spatial ideas. For example, in a class of 6 year olds, Perry and Dockett (2001) described a play session in which a group of children used large wooden shapes designed to assist teachers in drawing on the board to create patterns, construct images of local buildings, and make roads and maps. The children found that two semicircles could be put together to make a circle and that triangles could fit together to cover an area.
Much of the number research, particularly that dealing with the concept of fractions and the notion of iterable composite units, is pertinent to measurement (McClain et al., 1999; Stephan, 2000). We illustrate this here through a discussion on the topic of length. Traditionally, measurement of length has been taught through a sequence of activities described by Clements (1999c, p. 5) as “gross comparisons of length, measurement with nonstandard units such as paper clips, measurement with manipulative standard units, and finally measurement with standard instruments such as rulers.” This sequence is often repeated with other measurement constructs such as area, volume, and mass. However, this may need to be reconsidered in the light of research which has gone beyond that of Piaget and his colleagues (see, for example, Piaget, Inhelder, & Szeminska, 1960).

There is some evidence to suggest that using informal units in early measurement lessons may make the activity one of counting, with little concept of what is being measured or why counting results in a measure rather than a number (Bragg & Outhred, 2000; Clements, 1999c; Owens & Outhred, 1998). As well, there is evidence (Boulton-Lewis, Wilss, & Mutch, 1996; Clements, 1999c) that the use of rulers may facilitate the development of length measurement ideas and may be preferred by many children. Clements (1999c, p. 7) suggested that

> using non-standard units early so that students understand the need for standardization may not be the best way to teach. If introduced early, children often use unproductive and misleading strategies that may interfere with their development of measurement concepts.

Cobb and his colleagues (Cobb et al., 1998; McClain et al., 1999; Stephan, 2000) have found that the introduction of an informal unit in an appropriate context not only can make the task of linear measurement more interesting for the children but also can strengthen the links between the number and measurement through the development of “unitizing” in the measurement context. In one example, children in a Year 1 teaching experiment not only used nonstandard units—both perceptual and conceptual—but also created from these iterable composite units that they could use to develop their measurement knowledge. In short, they created their own “rulers” using these units and used them to measure and “to interpret their activity of measuring as the accumulation of distance” (Stephan, 2000, p. 4).

**WHAT MATHEMATICAL IDEAS DO CHILDREN BRING TO SCHOOL?**

Many mathematics education researchers have reported on the vast array of mathematical knowledge, skills, and dispositions young children do bring to school (Aubrey, 1993; Baroody, 2000; Bobis & Gould, 1999; English & Halford, 1995; Ginsburg, 2000; Hunting & Davis, 1991; Suggate, Aubrey, & Pettitt, 1997; Tang & Ginsburg, 1999). This research corpus suggests that many children will have access to much mathematical power by the time they start elementary school. Some examples of this power include strategies for carrying out arithmetical operations—how long will children have been sharing numbers of objects before they get to divide 8 by 2 in a formal sense?—basic shapes and their properties, knowledge that a ruler marked in units is used to measure lengths, patterning and tessellations, and notions of fairness and fractions. Much of this learning has been accomplished without the “assistance” of formal lessons and with the interest and excitement of the children intact. This is a result that teachers would do well to emulate in our children’s school mathematics learning. Baroody (2000, p. 66) summarized these thoughts in the following way:
Preschoolers are capable of mathematical thinking and knowledge that may be surprising to many adults. Teachers can support and build on this informal mathematical competence by engaging them in purposeful, meaningful, and inquiry-based instruction. Although using the investigative approach requires imagination, alertness, and patience by teachers, its reward can be increasing significantly the mathematical power of children.

WHAT DO WE KNOW ABOUT YOUNG CHILDREN’S LEARNING OF MATHEMATICS AND ITS TEACHING?

In the first section of this chapter, we offered an outline of learning and teaching in the early childhood years from both cognitive and historical perspectives. In this section we link these general comments with mathematics education in particular by considering issues that are at the forefront of current thinking about how children can be assisted in accessing the powerful mathematical ideas discussed in the previous section.

Neuroscience

Recent advances in neuroscience have provided a substantial boost in acknowledging the value and significance of learning in the early years and its impact on later learning. Arguments about the relative importance of nature and nurture have been addressed by the recognition that both inherited and environmental features have the potential to influence the “hard-wiring” of the brain (Shore, 1997). This work is significant in many ways. First, it recognizes the profound changes that occur within the early years. Second, it emphasizes the importance of early experiences for brain development. Third, it highlights the social element of development and learning, regarding relationships as central. Finally, it focuses on “the powerful capabilities, complex emotions and essential social skills that develop during the earliest months and years of life” (Shonkoff & Phillips, 2000, p. 383). Warm, responsive relationships are reported to help children develop and learn and to increase young children’s resilience in the face of difficulties (Shonkoff & Phillips, 2000; Shore, 1997). The stimulation provided within such relationships has a direct effect on the development and maintenance of neural pathways and in the amelioration of anxiety or trauma (Shore, 1997). In short, warm, responsive relationships set the context for meaningful interactions.

Relationships

Rogoff (1998) highlighted the importance of the sociocultural context in learning. Within this context, there is increasing focus on relationships and the quality of relationships as contexts for learning. We mentioned the importance placed on relationships within the pedagogy of Reggio Emilia programs; however, not only early childhood programs such as Reggio Emilia have this focus. Bronfenbrenner’s (1979) ecological model nests the child and family within a series of overlapping and intersecting contexts and recognizes that these contexts are both influenced by and influence the interactions that occur within them. For example, a child who believes he may be “no good” at mathematics could well disrupt the mathematics classroom, demand extra time of the teacher, and distract other children, influencing the context of the mathematics lesson. The context probably also has an effect on him, reinforcing his inability to complete the mathematics but reassuring him of his ability to attract attention in other ways.

Relationships between family members, children, and educators also have a substantial influence on learning, including the learning of mathematics. Studies have
shown a positive relationship between parental involvement in their children’s schooling and the achievement of these children in areas including mathematics (Brown, 1989; Civil, 1998; Greenberg, 1989; Reynolds, 1992; Young-Loveridge, 1993; Young-Loveridge, Peters, & Carr, 1998). Similar connections have been described between levels of parent involvement in prior-to-school settings, children’s academic attainment, and their social adjustment (Arthur, Beecher, Dockett, Farmer, & Death, 1996).

**Play and Mathematics**

One of the key ways in which children learn is through play. The “warm, responsive relationships” that have been identified as important in this learning can be supported through and within play (Dockett & Fleer, 1999). There is much more to play than this, however. Young children’s play can be complex in terms of theme, content, social interactions, and the nature of the understandings displayed and generated. In addition, they can have many mathematical experiences during play. For example, Ginsburg (2000) identified mathematical experiences in 42% of all the observed play experiences among a group of 4- and 5-year-old preschoolers. The value of block play in the development of many mathematical ideas is well known (Rogers, 1999, 2000), whereas water, sand, and dramatic play all provide opportunities for the development of mathematical ideas (Perry & Conroy, 1994).

Teachers who are most effective in promoting their children’s learning through play adopt the role of provocateur (Edwards et al., 1993) through which they observe and assess the understandings demonstrated by individual children and then generate situations that challenge these. This may involve asking questions, introducing elements of surprise, requiring the children to explain their position to others and working with children to consider the logical consequences of the positions they adopt. Teachers who use play in their classrooms have opportunities to observe what it is that children know and then to plan learning experiences which follow. For this to occur, the children need to feel comfortable in their classroom. They must feel free to interact with their peers about their mathematical ideas, and they must feel comfortable in taking risks with their learning. This process can begin in early childhood—both prior-to-school and in the first years of school—when teachers recognize the importance of play as one context in which children can safely explore understandings, make and test conjectures, and communicate these to others. There are many reports in which such a context has been used very successfully in the mathematical development of young children (Oers, 1996, 2000; Perry & Dockett, 1998; Yackel, 1998). In summary, Griffiths (1994, pp. 156–157) noted that.

Maths and play are very useful partners. If we want children to become successful mathematicians, we need to demonstrate to them that maths is enjoyable and useful, and that it can be a sociable and cooperative activity, as well as a quiet and individual one. We must be careful, too, to remember that play is not just a way of introducing simple ideas. Children will often set themselves much more difficult challenges if we give them control of their learning than if it is left up to the adults.

**Challenge**

Humans learn when they are simultaneously put into positions of “not knowing” and “wanting to know.” Little of value is learned through the rote recitation of multiplication tables or the mindless practice of addition and subtraction algorithms—except, perhaps, just how boring this sort of “mathematics” can be. We know that children learn a great deal of mathematics and possess powerful mathematical ideas by the time they start school. They have been challenged and have challenged themselves to learn. Can we do better when these children get to school? We do not believe that enough is expected of our young children in the first few years of school and
that much greater mathematical challenges should be put before them. We are not
talking here about “harder sums and more of them” but, rather, greater challenge
in terms of problems that are presented to the children. One of the key differences
between mathematics education in Japan and much of the Western world is that
children in developed Western countries are asked to do many more repetitive ex-
ercises than Japanese children, are expected to do them quickly, and are assessed
on the number that are completed correctly (Stevenson & Stigler, 1992; Stigler &
Hiebert, 1999). It is not unusual for children in Japanese primary schools to work
on one problem for many lessons, using time in between lessons to investigate the
problem from particular angles or to find particular information that may be help-
ful. For this to happen, the tasks children are given to investigate or the problems
they are given to solve must be much “richer” than is typically the case in most
Western school mathematics lessons. We must challenge our young school children
to work with these rich tasks and to move gradually over time toward a solution.
Anyone who has spent any time at all in a prior-to-school setting knows that young
children are capable of persevering with such tasks, provided they are interesting,
relevant, and challenging to the child. The same can happen in schools if such an
approach is encouraged and the teachers feel confident in letting the children “run
a little” with realistic problems. Some curriculum approaches that facilitate this aim
follow.

Curriculum Approaches

Play is a particularly important aspect of emergent curriculum, child-initiated cur-
riculum, and the project approach. Each of these approaches to planning for young
children emphasizes children as the source of curriculum. These approaches are men-
tioned in this chapter as they provide a context in which the teaching and learning
of mathematics can be promoted. It has already been noted that young children have
remarkable facility with some elements of mathematical understanding in situations
that make sense to them and that matter to them. The curriculum approaches listed
above emphasize these characteristics.

Emergent curriculum (Jones & Nimmo, 1994) is a responsive approach to curricu-
ulum. Rather than a totally preplanned curriculum, emergent curriculum relies on
the ability of educators to observe children closely to respond to their interests and
experiences. Within an emergent curriculum, there are opportunities to focus in ar-
eas of interest for as little or as long as is appropriate. Emergent curriculum can be
child-initiated, that is, the child can have “an active role in the initiation of interests,
questions and hypotheses and remain a collaborator in the process and form of sub-
sequent inquiry, exploration and creative expression” (Tinworth, 1997, p. 25). One
can generate a child-initiated curriculum from children’s questions, explanations, or
problems. In such curriculum, children make some decisions and work with adults to
explore and investigate issues that are relevant and meaningful. There is remarkable
potential within such a curriculum for children to pose and solve multiple problems
in multiple ways.

Adults have a critical role to play in creating an environment that stimulates questions
and exploration and that provides opportunities for children to express their questions
and challenge their understandings. The environment that is created must be safe, in
both the physical and the psychological sense. Children who feel safe are more likely to
take risks: more likely to ask questions when they don’t know the answer, more likely
to persist in their search for answers and more likely to share this with others, including
the teacher. (Dockett, 2000, p. 206)

The project approach (Katz & Chard, 1989) has a similar emphasis on children’s
active involvement:
A project is an in-depth investigation of a topic worth learning more about. The investigation is usually undertaken by a small group of children within a class, sometimes by a whole class, and occasionally by an individual child. The key feature of a project is that it is a research effort deliberately focused on finding answers to questions about a topic posed either by the children, the teacher or the teacher working with the children. (Katz, 1994, p. 1)

Although this is not a new approach, the flexibility it provides for teachers and children to pursue issues that matter to them can be refreshing in a broader context of predetermined curriculum outcomes. Rather than the topics of investigation being preplanned, the project approach has a structure based on introducing children to problem posing and problem solving based on research (Helm & Katz, 2001). As one example, Helm and Katz (2001) detailed the “fire truck project,” which involved children researching fire trucks to build one. Experiences such as generating questions they needed to answer to find out about fire trucks, visiting a fire station and recording relevant information (e.g., drawing the fire truck from different perspectives), graphing materials they wanted to research (e.g., the number of fire hoses and ladders on the truck) were important. After returning from the visit, children used the information they had collected to plan their construction of a fire truck. The opportunities for developing mathematical understanding in this one project were staggering.

The importance of connections in young children’s developing mathematical understandings has been mentioned previously. We want to emphasize, too, the importance of teachers facilitating such connections through an integrated approach to curriculum. The project approach is one means teachers have to achieve this. In addition, projects provide opportunities for children to pose and work toward solving problems, become physically and mentally engaged with the topic, and to plan and revisit ideas and experiences.

Models and Analogues

The use of manipulatives in mathematics education is well established, particularly in the early childhood years, and they have been shown to have great value in many aspects of mathematics, particularly in the development of place value ideas and written algorithms with whole numbers (Bohan & Shawaker, 1994; Cobb & Bauersfeld, 1995; Cobb, Wood, & Yackel, 1991; National Council of Teachers of Mathematics, 2000; Sherman & Richardson, 1995). Nonetheless, there is a deal of evidence to suggest that such manipulatives are not automatically helpful in the development of children’s mathematical ideas (Ball, 1992; Baroody, 1989; Cléments, 1999a; Howard & Perry, 1999; Perry & Howard, 1994; Price, 1999; Thompson, 1992). Part of this problem stems from children’s inabilities to argue cogently from the analogies that are formed through the manipulatives or to be overcome by these analogies to such an extent that it is the manipulatives, not the mathematics, that becomes most important (English, 1999). The Realistic Mathematics Education (RME) approach from the Netherlands has suggested an alternative way of thinking about models. It is suggested, in contrast to the common approach, where

the students are to discover the mathematics that is concretized by the designer, ... in the RME approach, the models are not derived from the intended mathematics. Instead, the models are grounded in the contextual problems that are to be solved by the students. The models in RME are related to modeling; the starting point is in the contextual situation of the problem that has to be solved. ... The premise here is that students who work with these models will be encouraged to (re)invent the more formal mathematics. (Gravemeijer, 1999, p. 159)
This approach to modeling allows a development of the notion of a “model of” mathematical activity becoming a “model for” mathematical reasoning. For example, problems about sharing pizzas were modeled by the students by drawing partitioning of circles that signify pizzas (model of). Later, the students used similar drawings to support their reasoning about relations between fractions (model for). (Gravemeijer, 1999, p. 161)

In Example 3, Joshua initially used the lollies, as represented by the counters available to him, as the model of the problem but moved quickly toward using his own understandings of fractions as the model for the relationships he was building. In Example 4, Jovalia drew a model of the story and her perspective on it but then used the drawing to explain her understandings of perspective (that is, provide a model for learning about perspective).

Language

The importance of language in the development of mathematical ideas is well documented (see, for example, Ellerton, Clarkson, & Clements, 2000). Without sufficient language to communicate the ideas being developed, children will be at a loss to interact with their peers and their teachers and therefore will have the opportunities for mathematical development seriously curtailed (Cobb et al., 2000). The importance of language is demonstrated particularly in our Examples 2, 4, 5, and 6 in which the children experimented with mathematical terms by playing with the ideas and the language that supports both the ideas and the children’s learning of them. In short, children need sufficient language to allow them to understand their peers and their teachers as explanations are presented and to allow them to give their own explanations. This has particular ramifications for those children for whom the language of instruction is not their first language. Examples abound of children starting school and not understanding even the most basic instructions when they are given in a language other than their home language. This situation is often exacerbated in the development of mathematical ideas because of its specialized vocabulary and its use of “common” words to have specialized meanings.

Language is important in young children’s mathematical development in other ways as well (Perry, VanderStoep, & Yu, 1993). For example, we all recognize the behavior of children trying to change their answers when asked by the teacher “Are you sure?” This is a perfectly reasonable question to ask, however, and, given appropriate sociomathematical norms in the classroom (Yackel & Cobb, 1996), could be asking the children to justify their answers, not necessarily to change them. A number of researchers (Krummheuer, 1995; Oers, 1996; Yackel, 1998) have demonstrated the power of this question in the development of argumentation among young children.

Mathematical symbols are another important form of language that needs to be considered. There has been a great deal of work done on symbolization, which has particularly important ramifications for early childhood mathematics learning and teaching (see, for example, Cobb et al., 2000; Kieran & Sfard, 1999; Sfard, 1991). There is no doubt that, eventually, children should be able to express their mathematical ideas using the standard mathematical symbols that have become socially accepted. It is unnecessary, however, and even counterproductive, to expect this level of symbol use among many young children who often have developed their own system of symbols and can use this consistently until another, more standardized system can be taken on board (Hughes, 1986). Children can be encouraged to use their own symbols, and, in fact, their own names for mathematical entities, and teachers should
become familiar with these. Just as we would want to encourage teachers at all levels of early childhood education to encourage the use of the children’s own strategies and methods, we would also want to encourage the use of their own language, at least in the stages where their concepts are being formed.

Technology

At the same time as the influence of information and communication technology becomes more and more pervasive in society, it is becoming an important aspect of the learning and teaching of mathematics at all levels, including early childhood. Clements (1999b) suggested that almost every preschool in the United States has a computer to which young children have access. This is not the case in many other countries, including some developed countries such as Australia (Dockett, Perry, & Nanlohy, 2000). In some countries, young school children have no access to computers. Similarly, access to other forms of technology that could be helpful in the development of mathematical ideas—such as calculators—is often limited.

Despite the extensive literature (see summaries of studies in Groves, 1996, 1997; Groves & Stacey, 1998; Hembree & Dessart, 1992; Shuard, 1992; Stacey & Groves, 1996) on the value of using calculators from early schooling, their use is still not as frequent as it could be for effective teaching (for example, Anderson, 1997; Sparrow & Swan, 1997a, 1997b). Sparrow & Swan (1997a, 1997b) suggested that an emphasis on standard written algorithms, and the generally reserved and negative attitudes and beliefs of teachers, obstructs the use of calculators. Groves (1996) illustrated well how calculators expand children’s knowledge of number when they are used in a range of different ways. Marley, Skinner, and Kenny (1998) also emphasized the value of calculators in the first year at school. Despite the success on a number of calculator projects in helping to develop number ideas in young children (see, for example, Groves, 1996, 1997; Ruthven, 1996), there does not seem to have been a great enthusiasm for them in the early school years and almost no recognition of their value in prior-to-school settings. One way this could be rectified, at least in part, is through the introduction of calculators into young children’s play.

On the other hand, computer technology is seen to have great value in young children’s learning through aspects such as

- Social and cognitive gains
- Children interacting within an individually appropriate learning environment over which they have some control
- A sense of mastery
- The development of representational competence
- Encouraging children to create and explore in a variety of ways not otherwise possible (Dockett et al., 2000, p. 50)

Clements (1999b) described a computer package that he has shown to be useful in the development of young children’s mathematics. This package, Building Blocks—Foundations for Mathematical Thinking, Pre-Kindergarten to Grade 2, is designed to assist young children in their construction of mathematical knowledge and, in particular, in the development higher order thinking.

Clements claimed that Building Blocks models an appropriate way in which computers might be used by young children because it provides

a manageable, clean manipulative; offering flexibility; changing arrangement or representation; storing and later retrieving configurations; recording and replaying students’ actions; linking the concrete and the symbolic with feedback; dynamically linking multiple representations; changing the very nature of the manipulative; linking the specific to the general; encouraging problem posing and conjecturing; scaffolding problem
solving; focusing attention and increasing motivation; and encouraging and facilitating complete, precise, explanations. (Clements, 1999b, p. 100)

The Playground project, addressed in chapter 4 of this volume (Kaput, Noss, & Hoyles), is another exciting new learning experience involving computer environments for children aged 4 to 8 years. In recognizing the mathematical potential of young children, the project enables participants to play, design, and create their own video games. In building their own executable representations of relationships, the children are coming into contact with mathematical ideas that would normally be reserved for much older students.

The potential for the use of computer technology by young children is enormous and ever increasing. It seems that the constraints to the use of this technology lie not with mathematics, nor with the learner but, most often, with the adults interacting with the young children involved. Both parents and early childhood educators—in both prior-to-school and school settings—need to develop the knowledge and confidence to allow their children to run with the technology, even if they run well beyond the adults (Dockett et al., 2000).

Role of the Adult

Adults—prior-to-school educators, school teachers, parents, and others—have an important role to play in young children’s mathematics learning. They can make a real difference. Through their actions and words, adults can encourage children to persevere with a problem, think about it in different ways, and share possible solutions with peers and other adults. They can challenge children to extend their thinking or the scope of their investigations. They can also hinder any or all of these. It is difficult to know when to intervene in a child’s activity and to know when “support” feels more like being “taken over.” This is a delicate balance and one that can be learned only through experience and by getting to know well the children with whom one is working.

In mathematics in the past, one of the roles of the adult was to hold the knowledge and to dispense it in small enough “doses” to ensure that most of the children absorbed it. Especially in the early childhood years, but we would argue at any age, there is little place for such an approach. We believe that children must construct their own knowledge in and from the social contexts in which they live. Adults form an important part of these contexts and can provide much needed scaffolding for children as they develop their mathematical ideas. Both of the adults in Examples 2 and 3 above have assisted Jeremy and Joshua to build on their current understandings to help them solve the particular problems they face. Neither adult has indicated whether the children’s answers are correct and neither have they proffered answers of their own—which the children would of course take to be correct and would have the effect of suggesting to the children that there was no further need for them to think. In both cases, however, the adults do know the most acceptable answers and are helping the children reach these.

One important point to make here is that if adults are to play the role of the “knowing assistant and supporter,” they need to know the mathematics with which their children are dealing. Not only do they need to be able to handle the questions posed, or at least be able to see a route toward a solution, but they also need to have what Ma (1999, p. 124) called a “profound understanding of fundamental mathematics” and which she defined in the following way:

Profound understanding of fundamental mathematics (PUFM) is more than a sound conceptual understanding of elementary mathematics—it is the awareness of the conceptual structure and basis attitudes of mathematics inherent in elementary mathematics and the ability to provide a foundation for that conceptual structure and instil
those basic attitudes in students. A profound understanding of mathematics has breadth, depth, and thoroughness. Breadth of understanding is the capacity to connect a topic with topics of similar or less conceptual power. Depth of understanding is the capacity to connect a topic with those of greater conceptual power. Thoroughness is the capacity to connect all topics. (Ma, 1999, p. 124)

Many teachers of young children do not have such a profound understanding of mathematics. In fact, many of these teachers have chosen not to precisely because they lack the confidence and knowledge in mathematics that would help them gain such an understanding. This presents a major challenge for mathematics educators, teacher educators, and mathematics education researchers if we are to support our young children in their development of powerful mathematical ideas.

CONCLUSION

Early childhood education, especially at the prior-to-school level, has had a long history of attempting to provide “purposeful, meaningful, and inquiry-based instruction” (Baroody, 2000, p. 66) for young children. Influenced by the nurturance of strong and positive relationships among all concerned, some of the approaches used in early childhood education provide models for what mathematics education for young children might look like in a wide range of educational settings. In this chapter, we have argued that young children have access to powerful mathematical ideas and can use these to solve many of the real-world and mathematical problems they meet. These children are capable of much more than their parents and teachers believe. Programs such as that emanating from Reggio Emilia have shown the power of the young mind and what can be achieved when children are placed in a supportive, challenging environment. The biggest challenge for mathematics educators and mathematics education researchers is to find ways to utilize the powerful mathematical ideas developed in early childhood as a springboard to even greater mathematical power for these children as they grow older.

The powerful mathematical ideas highlighted in this chapter are all processes used by young children in their everyday lives. They are processes that will be used in later mathematics education but that have a real purpose for the children, even when they are young. Although the developments in the prior-to-school years have been the province of many researchers over the years, only a small proportion of these have been mathematics education researchers. If we are to understand how young children develop their mathematical ideas and to use this effectively in the teaching of mathematics, there is a need for a lot more mathematics education research at the prior-to-school level.

Curriculum approaches that free the children to explore and investigate problems important to them are becoming more popular in mathematics education although there is still some reluctance to give up the traditional transmission approaches, which had been almost universal in schools up until the 1980s. In some aspects, the educators of young children can show the way. We need to continue to investigate learning and teaching alternatives, many of which could be based on the approaches used in early childhood settings for a long time.

One of the biggest challenges for mathematics education researchers is in the area of learning how to develop a “profound understanding of fundamental mathematics” in the adults who interact with the young children in their schools and prior-to-school settings and, indeed, in these children as well. One of the tensions in mathematics teaching and learning in the early childhood years is that although children demonstrate remarkable facility with many aspects of mathematics, many early childhood teachers do not have a strong mathematical background. At this time when children’s
mathematical potential is great, it is imperative that early childhood teachers have the
competence and confidence to engage meaningfully with both the children and their
mathematics. Until early childhood teaching is seen to be as prestigious a career as
elementary teaching—and it is in some countries—teachers who may have neither a
positive attitude toward mathematics nor a profound understanding of fundamental
mathematics will affect our young children. There is a broad range of research projects
begging to be completed in this area.

Young children are capable of dealing with great complexity in their mathematics
learning. Teachers are capable of dealing with great complexity in their facilitation of
children’s learning. These complexities can be harmoniously linked if teachers build
relationships with the children in their class, ascertain what mathematics they know,
how they know this, and how they can use it to solve realistic problems. Using this
and the children’s interests as a basis, teachers can plan challenging and complex
experiences for young children with the aim of helping them reach their potential in
mathematics learning.

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The elementary school is the educational environment where all children are expected to begin the process of accessing powerful mathematical ideas. Although the expectation for elementary students to learn powerful mathematical ideas has been universally accepted, there has been ongoing debate as to what constitutes powerful mathematical ideas for the elementary school.

For a substantial part of the 20th century the prevailing view was that computational skills constituted the “powerful mathematics” that was needed for effective citizenry and continuing mathematical growth beyond the elementary school. This emphasis on computational skills has sometimes been associated with an emphasis on meaningful mathematical learning (Brownell, 1935) and problem solving. During these periods of meaningful learning there have often been strong calls to produce a balance between skill and process, between instrumental and relational understanding (Skemp, 1971, p. 166), between procedural and conceptual knowledge (Hiebert & Lefevre, 1986, pp. 3–8), but such periods have been all too rare. Moreover, even periods of reform and enlightenment in elementary mathematics do not seem to have given most children access to the “deep ideas that nourish the growing branches of mathematics” (Steen, 1990, p. 3).¹

¹The reference to “the deep ideas that nourish the growing branches of mathematics” (Steen, 1990, p. 3) needs some explanation and contextualizing. Steen cautioned that there is much more to the deep ideas (root system) of mathematics than the traditional “layer-cake” sequence of arithmetic, measurement, algebra, and geometry that has characterized school mathematics. He goes on to identify some of these roots as specific mathematical structures (e.g., numbers, shapes), attributes (e.g., linear, symmetric), actions (e.g., representing, modeling), abstractions (e.g., symbols, change), attitudes (e.g., wonder), behaviors (e.g., iteration, stability), and dichotomies (discrete vs. continuous).
The emphasis on *all* students learning powerful mathematical ideas in elementary school is complex and did not come into sharp relief until the last 30 years. Even so, rhetoric on equitable access has been stronger than fact. For example, in the United States there is a plethora of research that documents the lack of achievement by disproportionate numbers of racial and ethnic groups, speakers of English as a second language, female students, and those from lower socioeconomic groups (e.g., Mitchell, Hawkins, Jakwerth, Stancavage, & Dossey, 1999; Secada, 1992; National Center for Education Statistics [NCES], 1995). This has been the reality despite efforts to provide an elementary mathematics curriculum for all students. For example, the National Council of Teachers of Mathematics (NCTM, 1990) asserts that “comprehensive mathematics education of every child is its most compelling goal” (p. 3). This same premise predicates national curriculum statements in most countries (e.g., Australian Education Council [AEC], 1990; Department of Education and Science and the Welsh Office [DES], 1991; Weber, 1990). Notwithstanding such ideals, Silver, Smith, and Nelson (1995) claimed that low levels of participation and performance in mathematics by these special groups is not due primarily to lack of ability, but to educational practices that deny access to high-quality learning experiences.

In the first part of this chapter we review and analyze the kinds of powerful mathematical ideas that should be accessible to all elementary school children in this new century. In carrying out this analysis, we will examine what research from the 20th century tells us about new domains and new technologies as well as extant mathematical domains that continue to be fundamental.

The second part of the chapter examines what research is saying about cognitive access to powerful mathematical ideas. In particular, we draw on research to uncover learning environments that will enable children to build new knowledge by enhancing existing knowledge structures.

In the final part of the chapter we examine curriculum access to powerful mathematical ideas. Although there has been no shortage of reform on curriculum issues that relate to mathematical access, the lesson of the past is that we lack a robust research base for evaluating the extent to which reform has been implemented. This lacuna in the research base generates questions about the kind of research methodology needed to link curriculum development and implementation and also raises concomitant issues about teacher enhancement programs.

**POWERFUL MATHEMATICAL IDEAS**

The issue of what constitutes powerful mathematical ideas raises questions that fall within the realm of historical and philosophical research. It is a discussion that will always be inextricably tied to cultural and political forces both within mathematics education and outside of it. For this reason we will examine powerful mathematical ideas in retrospect and also in prospect as we try to unfold research directions for the future. Moreover, given the increasing technological sophistication of elementary schools, we devote special attention to the role of technology in making powerful mathematical ideas accessible to elementary children.

We interpret Steen’s caveat as meaning that mathematics should not be viewed as “topics” to be layered with the curriculum indicating when to move to the next topic. Rather mathematics should be viewed as a meaningful interrelationship of deep ideas and patterns that can be revisited and strengthened from early childhood all the way through school and college. Moreover, we are claiming that these deep ideas and their linkages have not been the reality in mathematics teaching and learning during previous reform efforts.
A Retrospective View

What can we learn about the identification of powerful mathematical ideas for the elementary school from our endeavors in the 20th century? For most of the first half of the century, debate on what constituted powerful mathematical ideas for the elementary school was largely a nonissue. Guided by strong utilitarian and pragmatic needs, and fueled by waning support for mental discipline (Birkemeier, 1923/1973; Howson, 1982; Jones & Coxford, 1970; Niss, 1981), elementary mathematics was dominated by the need to train children to perform computational procedures. Even for those students who would progress beyond elementary school, a steady regimen of arithmetic skills was seen as the ideal diet for further manipulation of algebraic symbols in the secondary school.

The debate in the first half of the century was not on powerful mathematical ideas but rather on how arithmetical computation should be taught. Research was designed to compare and contrast computational approaches such as drill and practice, incidental learning, and meaningful learning (Brownell, 1935; Thorndike, 1924). It did not question the importance of or power attributed to standard algorithms for whole numbers and fractions. This emphasis on computation was complete and certainly understandable given the lack of computing technology and the needs of society during that first 50 years.

The period of the new math was another story. Mathematicians played a key role in arguing for revolutionary changes in mathematics per se (e.g., Howson, 1982; Jones & Coxford, 1970, pp. 68–77; Wooton, 1965). Their intent was to generate an elementary mathematics that encapsulated the structure of mathematics (Jones & Coxford, 1970, pp. 68–86; Page, 1959) and also better reflected the state of mathematics of the day. Although these changes were also accompanied by research on teaching and learning (Biggs & MacLean, 1969; Bruner, 1960; Dienes, 1965), this was a period of genuine change in the content of elementary mathematics.

The introduction of sets as a unifying idea for building concepts of number and space was a pervasive change in the quest for giving students access to powerful mathematical ideas. Through the use of sets, the reform groups of that time generated important representations for operations with whole numbers and fractions—even though the term representations appeared later in the century. For example, the addition of whole numbers was represented as the union of disjoint sets, and the intent was that connections between sets and numbers would provide a scaffolding for students’ learning of operations. Representations such as this were expected to not only support the learning of arithmetic but also to facilitate the transition from arithmetic to algebra. Computational algorithms were still a critical part of elementary mathematics in the new math, but underlying place-value representations and structural properties of the relevant operations were made more explicit to increase children’s understanding.

The power of sets also extended to the study of geometry and measurement. Sets were used to represent concepts such as points, segments, and angles and also to provide meaning for relationships such as intersection and parallelism. As it did in the case of number, the notion of correspondence was also implicit in capturing the fundamentals of measurement. Measurement was seen as a function that assigns a number to an object or, more specifically, to an attribute of the object such as length, area, or volume. Accordingly, function as a unifying idea played a subtle but key role in number and geometry, largely as a precursor to its more extensive role in algebra and calculus (De Vault & Weaver, 1970).

Much has been written on the outcomes of the new math and the differences between the intents of its architects and the realities of classroom implementation. It is not appropriate to reanalyze the outcomes of new math except to say that what happened in practice has been called “formalistic game-like plays in and with structures defined
in terms of sets and logic; often devoid of sense-making relations to matters outside the structures themselves” (Niss, 1996, p. 31). Our interest is focused on what we might learn from the kind of inquiry approaches and arguments that were used to identify powerful mathematical ideas.

The sources for most of the theoretical and philosophical arguments that generated the new mathematical content were mathematicians. They were in a unique position to make compelling arguments about the need for new content and for a new structural emphasis starting in the elementary grades. Although there were notable descriptions of collaboration among mathematicians, mathematics educators, and teachers (e.g., Wooton, 1965) in relation to the development of curriculum programs and experimental textbooks (e.g., School Mathematics Study Group [SMSG], School Mathematics Project [SMP]), it was the mathematicians’ arguments that determined what powerful ideas were to be included in the school curriculum. Mathematicians were also in a strong position to win external funding for school mathematics projects (Jones & Coxford, 1970) because this was an era of active political support for space exploration and scientific research.

Despite the development of large-scale and heavily funded curriculum projects in mathematics across the world, the developments did not produce the kind of systematic research methodologies that would have ongoing significance for the identification of powerful and accessible mathematical ideas. There were two reasons for this. First, it was early days in the paradigmatic shift from scientific-reductionist research in mathematics education to interpretivist research. Although there was some evidence of case-study approaches (De Vault & Weaver, 1970; Wooton, 1965), research at that time was more concerned with providing descriptions of the historical process than with analyzing and interpreting the argumentation used to identify key mathematical ideas for the curriculum. Had such research been undertaken, it might have revealed the “risks of following specialized mathematics too closely” and consequently selecting “subject matter and elements of mathematical language that do not make much sense outside of specialized mathematics” (Wittmann, 1998, p. 91). In fact, the research of the day was still focused to a great extent on the effect of modern mathematics on student performance (SMSG, 1972) and, as such, it largely followed statistical design models. Second, even if qualitative research had been carried out during this period, it is probable that the new math movement was simply too unique and too spectacular to provide a useful case study for the future.

In the wake of the new math there was a brief period of return to the traditional roots of elementary mathematics—that is, “back to the basics” of arithmetic (Schoenfeld, 1992). However, growth in technology and dissatisfaction with student mathematical performance especially in processes such as problem solving (e.g., Dossey, Mullis, Lindquist, & Chambers, 1988) soon led to a broadening of goals that were intended to “encompass the essential aspects of numeracy and ‘mathematical literacy’ in society” (Niss, 1996, p. 32). For the elementary school this resulted in greater emphasis on mathematical ideas associated with newer domains such as algebraic thinking, data exploration, and probability (AEC, 1990; DES, 1991; NCTM, 1989). Even in extant areas such as number there was a new focus that emphasized number sense, mental computation, and efficient use of technology in computation (Hembree & Dessart, 1986; Sowder & Schappelle, 1989).

Notwithstanding these changes in mathematical content, the most important shift during the last two decades was in the powerful ideas associated with mathematical processes. The NCTM Standards (1989) encapsulated this worldwide trend by giving preeminence to four process standards: problem solving, communication, reasoning, and connections. Social and utilitarian needs were still important, but mathematics was viewed as dynamic rather than static and constructive rather than prescriptive (Schoenfeld, 1992; Von Glasersfeld, 1984). In essence, elementary children
were expected to engage in mathematical problem solving, to collaborate with other students, and to build on their own conceptual thinking rather than rely totally on someone else’s standard procedures.

This strong focus on problem solving also led to a genuine emphasis on mathematical modeling in the elementary school. Mathematical modeling or applicable mathematics, as it was called, had been introduced into some secondary schools during the 1970s (Ornell, 1971). It gained a more extensive place in the secondary curriculum during the 1980s (Burkhardt, 1989; Usiskin, 1990), and this led to its introduction into the elementary school curriculum in recent years. Some researchers (Verschaffel & De Corte, 1997; Verschaffel, De Corte, & Vierstraete, 1999) have focused on modeling tasks that relate to the use of operations with whole numbers, fractions, and decimals. Such tasks not only provide rich experiences in mathematical modeling, they also reveal different aspects of number and operations and are generally supportive of aims that seek to enhance students’ number sense. Other researchers (Lehrer & Romberg, 1996; Lesh, Amit, & Schorr, 1997; Masingila & Doerr, 1998) have introduced model-eliciting problems that need greater mathematizing and also use conceptual knowledge from newer mathematical domains such as data exploration, probability, and discrete mathematics. These developments and others in the last 20 years set the stage for our prospective analysis of what might constitute powerful mathematical ideas for the 21st century.

A Prospective View

Our examination of elementary mathematics in the 20th century has revealed that powerful mathematical ideas were identified in response to a number of recurring goals: pragmatic and social needs of individuals and society and general formative goals that related to mathematics and applications outside mathematics. For the most part, educators of the day interpreted these goals as providing a mandate for computational skills in arithmetic and measurement. The approach to computation, at least for the first 80 years, varied with respect to level of formalism, degree of emphasis given to understanding, and the extent to which problem solving was incorporated in the learning of mathematics.

The enduring pragmatic goals of the 20th century still have core value for the coming century. However, there is already evidence that they will be embedded in broader goals and that this more complete set of goals will lead to powerful mathematical ideas and processes that are different from those emphasized for most of the 20th century. We are already seeing the emergence of “exterior and interior aims” (Niss, 1996, p. 32) that focus on the value of mathematics, the importance of the individual learner, the value of cooperation among learners of mathematics, and the need for autonomous mathematical thinking by individuals and groups. These aims will also incorporate process goals such as those identified by NCTM (1989). Process goals are likely to be even more expansive and might include problem solving, problem posing, modeling, exploration, conjecturing, reasoning, and the use of information technology. In essence, elementary mathematics will be the beginning of a process of “cultural initiation—one which might enable all members of a society to be in tune with the society to which they belong, to understand its most essential workings, and, as the case may be, to take an active part in scientific and technological development” (Chevallard, 1989, p. 57).

Given this emerging reorientation of the goals of elementary mathematics we might well ask what access students would need to extant mathematical domains such as number and measurement. In relation to this Ralston (1989) wrote, “Mathematics education must focus on the development of mathematical power not mathematical skills” (p. 35). He added that the single most important drag on any attempt to reform
the school mathematics curriculum is the emphasis in the first 6 to 8 years on manual arithmetic skills. Ralston’s powerful ideas include the kind of mathematical processes mentioned in the previous paragraph but he also adumbrates the need for elementary mathematics to be empowered by the growing calculator and computer technologies. Fey (1990) and Niss (1996) took a more balanced view with regard to number and computational skills. Fey wrote about the need for deep structural principles in number and noted, “For number systems a rather small collection of big and powerful ideas determine the structure of each system” (p. 81). Niss predicted that older mathematical ideas would simply be absorbed within new goals, and there is certainly a precedent for this in the research of recent years.

If we examine research on extant mathematical domains such as number (whole numbers, fractions, and decimals), proportional reasoning, geometry, and measurement, we observe that the research base is now very robust with regard to these areas. For example, in the domain of whole numbers, international research has not only classified semantic representations of addition and subtraction problems, it also has identified the kinds of hierarchical strategies that students develop in the early years of schooling (e.g., Carpenter & Moser, 1984; De Corte & Verschaffel, 1987). Interestingly, modeling and conceptual thinking play critical roles in this development. Similar representations and strategies have been generated for multiplication and division of whole numbers (e.g., Mulligan & Mitchelmore, 1997; Vergnaud, 1983) and also for the invention and understanding of multidigit addition and subtraction (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998).

Although theoretical knowledge is not as robust in areas such as fractions, decimals, ratio, and proportion (e.g., Hiebert & Wearne, 1986; Lamon, 1993; Mack, 1990, 1995; Moss & Case, 1999; Streefland, 1991) and geometry and measurement (Chiu, 1996; Lehrer & Chazan, 1998; Outhred & Mitchelmore, 1992; van Hiele, 1986), research in various countries has now generated valid and usable conceptual representations in these domains. Unlike the situation that prevailed in new math, we now have representations that are accessible to children; in fact, in many cases the representations are the constructed and validated models of children rather than of mathematicians. More will be said on the cognitive accessibility of these representations in the next section, but the key point here is that we now possess conceptually viable mathematical representations for a substantial part of what is powerful in extant fields of mathematics such as number, space, and measurement.

With respect to emerging but currently underrepresented mathematical domains such as algebraic thinking, data exploration, probability, combinatorics, and discrete mathematics, there are also promising developments in research for the new century. We should note in passing that inclusion of these underrepresented mathematical domains has been advocated worldwide, and the general consensus is that they must begin in a significant way in the elementary school (Borovcnik & Peard, 1996; Ralston, 1989). Moreover, in relating these new and powerful mathematical ideas to processes such as problem solving and modeling, Ralston wrote, “If instruction in these topics as well as in arithmetic is to achieve the larger goal of mathematical power, then problem solving needs to be emphasized throughout the elementary grades” (1989, p. 39).

There is also great potential for the kind of robust research carried out in extant fields of elementary mathematics to act as catalyst for underrepresented but emerging areas such as those identified in the previous paragraph. Methodologies used in studying extant domains may well carry over into emerging domains. In fact, in a number of these domains, models and frameworks have begun to emerge that identify representations that are and are not accessible to students: algebraic thinking (Bellisio & Maher, 1998); data exploration (e.g., Curcio, 1987; Jones, Thornton, Langrall, Mooney, Perry, & Putt, 2000) probability (e.g., Fischbein & Schnarch, 1997; Jones, Langrall, Thornton, & Mogill, 1997; Watson, Collis, & Moritz, 1997; Watson & Moritz, 1998) and
combinatorics (e.g., English, 1991). Even in discrete mathematics there is evidence that this domain offers students a new start in mathematics (Rosenstein, 1997), provides an alternative perspective on the power of algorithms, and is valuable in realizing the goals of the process standards (Casey & Fellows, 1997). Consequently, with respect to these emerging areas, we should be able to stand on the research infrastructure developed during the 20th century, especially the promising methodologies that have emerged during the last two decades. Moreover, emerging research will need to take cognizance of the increasing role that technology will play in revealing the powerful ideas of elementary mathematics and in giving children access to them.

**A Technological View**

Research over the last 30 years has begun to identify the potential of technology not only for generating powerful ideas in elementary school mathematics but also for giving elementary children curriculum and cognitive access to them. In this section we focus on the use of technology to generate powerful mathematical ideas and also on the mathematical connections that can be revealed through technology.

When examining the emerging role of technology in generating powerful mathematical ideas and potential areas for research, it would be naïve of us not to recognize that there is still reluctance, even resistance, to using technology in elementary school mathematics (Becker & Selter, 1996). Commenting on this, Balacheff and Kaput (1996) wrote, “For younger children, since it is widely felt that physical rather than cybernetic materials are more appropriate, relatively little software has been developed for targeting the learning of early number and arithmetic” (p. 473). Somewhat caustically they go on to add that even the more conceptually oriented arithmetic software is seen “to slow down the curriculum and the student—adding a flexibility and depth of understanding that does not seem to be valued as much as computational facility” (p. 473).

In the realm of arithmetic calculators and scientific calculators, where price is no longer an issue, Ruthven (1996) observes that there are still a number of factors inhibiting the development of calculator use in schools: public concerns about the effect of calculators on computational learning, testing policies that prohibit the use of calculators, and the treatment of calculators in some of the official curricula and textbooks of some countries. This resistance remains despite public policy documents (e.g., NCTM, 1987) and research showing that children’s number fact learning and their mental and written computational skills are not diminished by regular use of calculators (e.g., Groves, 1993, 1994; Hembree & Dessart, 1986, 1992; Office for Standards in Education, 1994). Moreover, these studies generally show significant positive effects on children’s reasoning in number sense and their problem-solving performance.

Despite a less than fully sanguine response from the public and educators to technology in elementary mathematics, there are signs as we commence the 21st century that research is producing the kind of technology that will give children access to powerful ideas in a number of areas of mathematics, including number and arithmetic. Concerning these latter two areas, recent research reveals that elementary children who use calculators identify new insights into modes of calculation, build earlier conceptions of large numbers, and develop different perspectives on checking arithmetical calculations (Groves, 1994; Ruthven, 1996; Shuard, Walsh, Goodwin, & Worcester, 1991). For example, in comparing the problem-solving processes of a class where students were expected to use calculators with one where students had no access to calculators (Wheatley, 1980), the calculator group exhibited more exploratory behaviors and spent more time attacking problems and less time computing. They also used different predominant processes for solving the problems and checking their solutions. For instance, with regard to checking, the calculator group used processes such as checking that the conditions of the problem had been met, retracing the steps,
and checking the reasonableness of their answers more often than the noncalculator group.

Groves (1994) also noted that primary children who had taken part in projects emphasizing the development of mental methods of calculation alongside use of the calculator did not in general make more use of calculators; rather they made more appropriate choices of methods of calculation. There is even evidence that children use calculators in unanticipated yet important ways for assisting their development of number. Stacey (1994) gave illustrations of children learning to write numerals such as 2 and 5 correctly by looking at the appropriate keys on their calculator. This finding appears to be consistent with Ruthven’s more general claim that pupils with less confidence in, or enjoyment of, number seem to experience through the calculator a means of matching the demands of schoolwork to their mathematical capabilities.

The outcomes of research using computer technology in number and arithmetic are more inchoate and problematic than those associated with calculator research. According to Balacheff and Kaput (1996), commercially available software is still aimed largely at teaching and automating computational skills in the form of algorithms, for example Math Blaster (Davidson & Associates, 1993) and Tenth Planet Explores Fractions (Sunburst, 1998). These authors went on to note that some more recent work (Kaput, Upchurch, & Burke, 1996; Steffe & Wiegel, 1994; Thompson, 1992; Tzur, 1999) has focused on the development of conceptual operations such as grouping, decomposition, and unitizing in topics such as whole numbers and fractions. In addition, within the Logo microworld environment (discussed later in this chapter), Noss and Hoyles (1996) claimed that when children work with the computer turtle they tend to build ideas of ratio and proportion naturally and as a consequence begin to think multiplicatively. More specifically, they claimed that once procedures for drawing figures had been built, students often posed for themselves the issue of enlarging and shrinking. Although growing and shrinking do not necessarily involve proportionality, Hoyles and Sutherland (1989) documented earlier how students used Logo input as a scale factor to change the size of a drawing in proportion and build procedures that reflected the internal relationship between figures. Clearly there is a need for further research into how interactive technology can foster students’ learning in important areas such as fraction, ratio, and proportion.

Interestingly, the power of technology in providing access to powerful mathematical ideas at the elementary school level has been more evident in content domains such as geometry, algebraic thinking, data exploration, probability, and mathematical modeling. Balacheff and Kaput (1996, p. 475) observed that geometry offers exciting developments based on new access to direct manipulation of geometrical drawings via software such as Geometric Supposer (Schwartz & Yerushalmi, 1984), Geometer’s Sketchpad (Klotz & Jackiw, 1988), Shape Makers (Battista, 1998), and Cabri-geometre (Laborde, 1985). Such access enables children to view conceptualization in geometry as the study of invariant properties of these “drawings” while dragging their components around the screen. That is, the statement of a geometrical property now becomes the description of a geometrical phenomenon accessible to observation in these new fields of experimentation (Boero, 1992; Laborde, 1992). In a real sense these invariant properties are the powerful ideas of elementary geometry and they provide the basis for describing geometrical objects and using such descriptions to build other geometrical properties. Notwithstanding these developments in geometrical environments, Kaput and Thompson (1994) note that research on these environments continues to be in short supply, and they advocate that mathematics educators pay greater attention to publishing research that examines children’s access to these kinds of powerful geometrical ideas. Certainly Kaput and Thompson’s call for action should not go unheeded in the early part of the new century.
Interactive technologies (graphics calculators and computers) in high school- and college-level algebra have centered on fostering students’ algebraic performance in using traditional formalism and graphics (Balacheff & Kaput, 1996). Out of these technological developments in higher levels of algebra, there has been an emergence of exploratory and interactive approaches that are applicable to fostering algebraic thinking in the elementary school. For example, Filloy and Sutherland (1996), Rojano (chapter 7, this volume), and Sutherland and Rojano (1993) used spreadsheets to focus children’s thinking on looking at numbers from the perspective of patterns and relationships. They suggested that this approach supports pupils’ thinking in making the key transition from arithmetic to algebra. Moreover, their research reveals that children can use spreadsheet language to build conceptions of functions and their different representations: rule (both written and symbolic forms), graph, and table. Rojano claims that spreadsheets provide access to the power of algebraic language thus removing one of the key obstacles associated with the development of algebraic thinking. She also maintains that the use of the computer frees children from the arithmetical activity of evaluating expressions, thus enabling them to focus on the structural aspects of algebraic thinking. The “list” facility of some graphics calculators (e.g., TI-73; Texas Instruments, 1998) could also be used as an alternative to spreadsheets in providing a more transparent learning environment for algebraic thinking.

In data exploration, probability, and mathematical modeling there is also evidence that elementary children can gain access to powerful mathematical ideas by using interactive technology. For example, Jones, Langrall, Thornton, Mooney, Wares, Jones, Perry, Putt, and Nisbet (2001) observed that Graphers (Sunburst, 1996b) computer software provided unanticipated benefits in helping Grade 2 children invent their own way of reorganizing and representing data. Rather than using the established software procedure to construct a graph, the children literally dragged data values across the desktop to reorganize the data and build their own graphs. This finding is important because data reorganizing is not only a powerful idea in statistical education; it is also a complex one for elementary children (e.g., Bright & Friel, 1998). Cobb (1999), Hancock, Kaput, and Goldsmith (1992), and Lesh et al. (1997) also provided evidence that technology may be a particularly effective instructional vehicle for helping students organize data and build different representations—the latter authors having made such an observation in relation to model-eliciting activities. Notwithstanding these supportive features of technology in relation to data exploration, Ben-Zvi and Friedlander (1997) offered the caveat that the computer’s graphic capabilities and the ease of obtaining a wide variety of representations may divert students’ attention away from the goals of a data investigation.

Most of the software in probability, for example, MathKeys: Unlocking Probability (MECC, 1995) has been designed with a single purpose: to generate data on probability simulations and ipso facto to provide experimental evidence on the probabilities of selected events. Although this provides valuable information for students, the software is restrictive from an interactive perspective and often requires the assistance of an adult. More recently Pratt (2000) reported impressive results in the development of 10- and 11-year-old children’s probabilistic thinking when they used the researcher-designed Chance-Maker microworld. Using this dynamic and interactive environment the children articulated their meanings for chance through their attempts to “mend” the computer tool so it would function as it was supposed to. The research documented the interplay between the children’s informal intuitions and the computer-based tool as the children constructed their own new internal resources for making sense of the probability tasks.

While the protestsions concerning the use of technology will continue into the 21st century, the research evidence accumulated over the last 30 years clearly demonstrates the potential of technology to make powerful mathematical ideas more
accessible. In particular, the expansion of more interactive software such as micro-
worlds is beginning to address the need for technology-supported constructivist envi-
ronments in the learning of elementary mathematics. As the computational paradigm
of elementary mathematics is hopefully laid to rest in this century, researchers will in-
creasingly face the challenge of how to build technology that will give students greater
access to the power of mathematical number sense and measurement and to newer ar-
eas of mathematics such as data exploration, probability, and mathematical modeling.

In summarizing this part of the chapter, we note that Wittmann (1998) made a
compelling case that “mathematics education is a systematic-evolutionary ‘design
science,’” the core activity of which is to concentrate on “constructing ‘artificial
objects,’ namely teaching units, sets of coherent teaching units and curricula as well as
the investigation of their possible effects in different educational ‘ecologies’” (p. 94).
This approach, which has parallels in educational development and developmental
research in The Netherlands (Gravemeijer, 1998) and teaching experiments in the
United States (e.g., Cobb, 1999; Steffe & Thompson, 2000) seeks to design instruc-
tional sequences or learning trajectories (Simon, 1995) that link up with the inform-
al knowledge and mathematical representations of children. Moreover, through
a process of reiteration and modification, this research seeks to enable children to
develop more sophisticated, abstract, formal knowledge while acknowledging chil-
dren’s intellectual autonomy (Gravemeijer, 1998, p. 279). In essence, research of this
kind has the potential not only to identify the powerful mathematical ideas that chil-
dren bring to school, but more importantly to find methods, including those supported
by technology, that will enable children to access even more powerful mathematics.
As Wittmann said, “There is no doubt that during the past 25 years a signi-
ficant progress, including the creation of theoretical frameworks, has been made within the
core [of mathematics education] and standards [of research] have been set which are
well suited as an orientation for the future” (Wittmann, 1998, p. 94).

COGNITIVE ACCESS TO POWERFUL
MATHEMATICAL IDEAS

It has been widely documented that children rely on informal, intuitive knowledge
when solving problems (e.g., Booth, 1981; Carraher, Carraher, & Schliemann, 1987;
Erlwanger, 1973). Moreover, research has shown that when children are given oppor-
tunities to build on their informal knowledge structures to make sense of problem
situations, they are capable of understanding significant mathematics that was once
reserved for older students or an elite minority (Romberg & Kaput, 1999). Because
understanding is not static, most complex mathematical ideas can be understood at a
variety of levels (Carpenter & Lehrer, 1999). Thus, when understanding is perceived
as emerging over time, we are able to broaden the range of powerful mathematical
ideas considered accessible to children.

Role of Teaching

Assuming that mathematical understanding is actively constructed over time does
not lessen the need for children’s learning to be influenced by teaching (Steffe, 1994).
New models for teaching mathematics have begun to investigate ways to develop
children’s informal knowledge structures (Simon, 1997). Dutch mathematics educa-
tors have developed an integrated model of mathematics teaching and learning based
on the perspective that children’s conceptual structures are developed through an
instructional process called guided reinvention (Freudenthal, 1991; Streefland, 1991).
In a similar way, the Japanese Open-Approach Method has been tailored to capture
a variety of students’ ways of thinking and learning (Nohda, 2000; Shimada, 1977). Although the teacher must map out a learning route for instructional tasks, both of these approaches provide children with opportunities to reinvent certain mathematical knowledge. Knowledge of how conceptual structures develop within a particular content domain and insights into children’s informal knowledge structures are vital elements in assisting the teacher to design the reinvention process.

Compatible with the Dutch and Japanese perspectives, Simon (1995, 1997) constructed a framework, called the Mathematics Teaching Cycle, that describes “the relationships among teacher’s knowledge, goals for students, anticipation of student learning, planning, and interaction with students” (1997, p. 76). A key component of this teaching cycle is the hypothetical learning trajectory or “the teacher’s prediction of the path by which learning might proceed” (p. 77). Simon’s hypothetical learning trajectory is essentially the same as the Dutch researchers’ learning route. It includes the teacher’s goal for student learning, plan for learning activities, and hypothesis of the student learning process. Teacher knowledge and interactions with students reflexively inform the generation and modification of hypothetical learning trajectories. More specifically, teachers draw on their knowledge of how children learn in general, and of how particular mathematical understandings are developed as they build up models of their students’ mathematical understandings.

All of these teaching models reflect the belief that instruction should be informed by a teacher’s knowledge of mathematics, of children’s thinking, and of the ways children learn mathematics (NCTM, 1991). According to Ball (1993), teachers need a bifocal perspective that involves “perceiving the mathematics through the mind of the learner while perceiving the mind of the learner through the mathematics” (p. 159). But how do teachers develop this perspective? We explore this question in the next section.

**Cognitive Models**

We claim that teachers need access to detailed models of children’s conceptual structures and how they evolve to design learning trajectories to foster the development of powerful mathematical ideas. Within the last decade, this has been a promising direction in research and one that has already begun to bear fruit in the elementary grades. Cognitive models incorporating key elements of a content domain and the processes by which students grow in their understanding of that content have been constructed for many of the extant mathematics domains (e.g., whole numbers, rational numbers, geometry) as well as some of the underrepresented domains (e.g., probability and statistics). These cognitive models have taken a variety of forms ranging from frameworks and taxonomies to detailed narrative descriptions. In some content domains the research is still emergent.

**Whole Number Concepts and Operations.** More than 20 years of research worldwide has yielded a knowledge base that describes children’s conceptual structures for whole number concepts and operations (Verschaffel & De Corte, 1996). There is evidence that children’s understandings in this domain progress toward “successively more complex, abstract, efficient, and general conceptual structures” (Fuson, 1992, p. 250). For example, detailed models of children’s concept of number (e.g., Fuson, 1988; Jones, Thornton, Putt, Hill, Mogill, Rich, & Van Zoest, 1996; Steffe, von Glasersfeld, Richards, & Cobb, 1983) outline a developmental progression from unitary to multiunit conceptual structures. Through early counting experiences children begin to develop concepts of unit and composite units. In turn, these conceptual structures provide a foundation for understanding mathematical topics that build on the concept of unit: place value, measurement, fractions, and proportional reasoning.
Perhaps the most robust body of research pertains to the development of children’s concepts of operations as reflected in the processes they use to solve different types of word problems. For reviews of this research see Carpenter (1985), Carpenter et al. (1998), English and Halford (1995), Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter, and Fennema (1997), and Greer (1992). The research provides a model of children’s mathematical thinking that includes a taxonomy of word problems, a detailed analysis of the strategies used to solve different problems, and a map of how these strategies evolve over time (Hiebert & Carpenter, 1992). As children solve different types of problems, they develop increasingly more abstract solution strategies that range from intuitively modeling the action or relationship to inventing multidigit algorithms (Carpenter et al., 1998). Moreover, this research identifies a number of powerful ideas or primitive constructs (Confrey, 1998) such as unitizing, part-whole, composing and decomposing number, and modeling. Further research is needed to determine how children build on and connect these powerful ideas within the whole number domain.

**Rational Numbers.** Although topics such as fractions, decimals, ratios, and proportions have been mainstays of the elementary mathematics curriculum, the research on children’s thinking processes in these areas is not as complete as for whole numbers. This is due, in part, to the complexity of the rational number domain, which is comprised of several related subconstructs: part-whole, quotient, ratio number, operator, and measure (Behr, Harel, Post, & Lesh, 1993) and is itself but one component of a more intricate multiplicative conceptual field (Vergnaud, 1994). Hence, rational number understanding involves the conceptual coordination of mathematical knowledge from many different domains (Lamon, 1996).

The research on rational numbers has followed two approaches: semantic analyses of rational number subconstructs (Behr et al., 1993) and studies of children’s conceptual understanding (e.g., Lamon, 1993; Mack, 1990, 1995). Although most of our research-based knowledge on rational numbers pertains to the analyses of subconstructs (Behr, Harel, Post, & Lesh, 1992), there is a growing body of research on children’s informal knowledge before instruction. This research investigates how children build on knowledge structures to develop more powerful ideas for rational number (e.g., Confrey, 1998; Lamon, 1993, 1996; Mack, 1990, 1995; Resnick & Singer, 1994; Streefland, 1991). It appears that children have informally developed understandings of some basic principles underlying rational number and that they are able to build on these understandings to construct meaning for formal symbols and procedures. The work of two researchers will be discussed to illustrate the nature of these findings.

Mack (1990) found that children could build on their informal partitioning strategies to solve a variety of fraction problems, including the more difficult ones such as subtraction problems with regrouping and converting mixed numerals and improper fractions. She noted that children “are able to relate fraction symbols to informal knowledge in meaningful ways, provided that the connection between the informal knowledge and the fraction symbols is reasonably clear” (p. 29).

Lamon’s (e.g., 1993, 1996) research has focused on children’s understanding of ratio and proportion. She found that children’s informal strategies, such as modeling and counting, were important in making sense of a problem and that, before instruction, children perceived some ratios as units and used them to reinterpret other ratios. Unitizing (constructing a reference unit and interpreting situations in terms of that unit) and norming (reinterpreting a situation in terms of a composite unit) have emerged from her work as a plausible framework for interpreting children’s thinking and building increasingly complex quantity structures.

Once again, the research has identified a number of powerful mathematical ideas such as splitting, partitioning, unitizing, part-whole, and modeling. Not surprisingly,
almost all of these ideas are linked to whole numbers reflecting their power to nourish several branches of mathematics. Forging these links throughout elementary mathematics remains a critical issue for research in the 21st century.

**Geometry.** Although research has examined how children develop knowledge about geometry and space, it is not as coherent as the research on whole numbers. It is generally agreed that children possess a great deal of informal geometry knowledge (Lehrer & Chazan, 1998) that can “serve as a launching point into formal mathematics” (Gravemeijer, 1998). In particular, children’s everyday experiences afford them rich intuitions about space and geometric constructs such as symmetry, similarity, and perspective (Lehrer, Jenkins, & Osana, 1998).

For almost two decades, van Hiele theory (van Hiele, 1986) has served as the leading cognitive model for describing the progression of children’s thinking in geometry (Clements & Battista, 1992). Recently, mathematics educators (see Lehrer & Chazan, 1998) have begun to question the adequacy of the van Hiele model, asserting that although it provides a broad framework for describing learning, it does not account for an individual child’s progression. According to Pegg and Davey (1998), the van Hiele theory may be more accurately described as pedagogical rather than psychological. They have suggested a synthesis of the van Hiele theory with the Structure of the Observed Learning Outcome (SOLO) taxonomy of Biggs and Collis (1991), merging the two complementary perspectives, one focusing on global thinking levels and the other on more micro levels of student responses. Pegg and Davey believed this synthesized model moves away from a single dimensional learning path toward a “true understanding of the nature of individual cognitive growth in geometry” (p. 133).

Recently, two research projects have reported detailed analyses of children’s reasoning about geometry and space. In the first study, Clements, Battista, and Sarama (1998) investigated third-grade children’s development of linear-measure during an instructional unit conducted in both computer and noncomputer environments. They provided descriptions of children’s thinking on tasks involving segmenting and partitioning length, composing and decomposing lengths, connecting number and spatial schemes, and conceptualizing turns. In the second study, Lehrer, Jenkins, and Osana (1998) studied the development of primary-grade children’s conceptions of two- and three-dimensional shape, angle, length and area measure, and drawing and spatial visualization. They reported rich descriptions of children’s problem-solving strategies and reasoning for each of these topics.

These descriptions may be the beginning of the kind of models of children’s thinking that have been generated in other mathematical domains. Certainly this research on geometry appears to be revealing new knowledge about children’s cognitive access to some of the same powerful mathematical ideas that were identified in research on whole and rational numbers: partitioning, unitizing, part-whole, and modeling.

**Probability.** Although probability is an underrepresented mathematical domain in most elementary school curricula, a considerable amount of research has been conducted on young children’s probabilistic thinking (Shaughnessy, 1992). Based on a synthesis of this research and observations of young children over two years, Jones and his colleagues (1997; Jones, Thornton, Langrall, & Tarr, 1999) developed a cognitive framework that systematically describes how children’s thinking in probability grows over time. The Probabilistic Thinking Framework incorporates six probability constructs—sample space, experimental probability of an event, theoretical probability of an event, probability comparisons, conditional probability, and independence—and encompasses four levels of thinking that range from subjective to quantitative reasoning. For each of the six constructs, the framework includes specific descriptors that characterize each thinking level.
Jones et al. (1999) also used the framework for informing an instructional program in probability with Grade 3 children. Two of their conclusions were interesting because they again revealed the power of part–part and part–whole thinking. In particular, part–part reasoning gave children some access to probability situations beyond subjective thinking. However, the integration of part–part and part–whole thinking provided more extensive access to probability including constructs such as probability comparisons and conditional probability.

**Data Exploration.** International calls for reform have advocated a more pervasive approach to statistics instruction at all grade levels. Generally, the treatment of statistics in most elementary mathematics curricula has focused narrowly on constructing and reading graphs rather than on broader topics of data handling (Shaunessy, Garfield, & Greer, 1996). Although research in this domain is still emerging, some aspects of children’s statistical thinking and learning have been investigated (Bright & Friel, 1998; Cobb, 1999; Curcio, 1987; Lehrer & Romberg, 1996; Mokros & Russell, 1995; Watson & Moritz, 2000). Based on the findings of this research and their work with elementary grade children over an entire year, Jones et al. (2000) have developed a framework for describing and predicting children’s statistical thinking. The Statistical Thinking Framework, modeled after their work in probability, incorporates four key constructs: describing data, organizing and reducing data, representing data, and analyzing and interpreting data. For each of these constructs, the framework includes specific descriptors that characterize four levels of children’s statistical thinking ranging from idiosyncratic to analytical reasoning. Although it is premature to make definitive statements about research on children’s statistical thinking, there is evidence that processes such as sorting, grouping, modeling, and sharing may provide access to powerful statistical ideas.

**Algebraic Thinking and Other Underrepresented Domains.** Although it is generally acknowledged that algebraic thinking should be developed across all grades levels (NCTM, 1998), there are few cognitive models to characterize children’s growth in algebraic thinking before and during instruction. There is exploratory evidence that children prefer to express generalizations in ordinary language; however, they can express generalizations algebraically provided that carefully designed activities support their thinking (Bellisio & Maher, 1998; Swafford & Langrall, 2000). There is also evidence that spreadsheets enhance children’s algebraic thinking and enable them to meet algebraic ideas in new ways (Ainley, 1999). As with rational number, algebraic reasoning is a complex domain comprising a wide variety of related subconstructs. Research on young children’s understanding of many of these subconstructs is emerging (Bednarz, Kieran, & Lee, 1996; Falkner, Levi, & Carpenter, 1999), and future studies are needed to build a general model that describes growth of algebraic reasoning over time.

Other domains that are currently underrepresented in elementary mathematics include combinatorics, discrete mathematics, and mathematical modeling. Although research on children’s thinking in these areas has begun to emerge, most of it has been isolated (e.g., Casey & Fellows, 1997; English, 1991). Research will need to develop cognitive models that can be used by teachers to inform instruction.

**Cognitive Access Through Technology**

As we enter the new millennium, technology serves a dual role in providing children access to powerful mathematical ideas. First, technology has the power to provide concrete embodiments of mathematical domains (Groen & Kieran, 1983, p. 372) and as such can enhance the salience and connectedness of mathematical ideas. We have
examined some instances of this in the first part of this chapter. Second, technology is enabling educators to develop more effective learning models as a result of research that uses technology to provide a window for viewing children’s constructions of meaning (Noss & Hoyles, 1996). Moreover, these learning models have the potential to produce mathematical learning environments that are more accessible to and flexible for children. To gain some perspective for this second role of technology, we will trace the ways in which computers and calculators have been applied in mathematics learning.

In presenting a framework for describing the use of computers in education, Taylor (1980) claimed that the computer can act as a tutor, tool, and tutee in providing children with cognitive access to domains such as mathematics. As tutor the computer can perform a continuum of tasks from drilling students in number facts to taking the learner step by step through computational algorithms, asking the appropriate questions at each stage, and checking students’ understandings before going on to more complex problems. Brown and VanLehn’s (1982) use of the computer to tutor subtraction and to diagnose and classify subtraction bugs is a well-known example of the computer’s power to facilitate the acquisition of mathematical skills. As tool, the computer can serve as a means for performing symbolic manipulations in arithmetic and algebra, generating graphical representations, and producing experimental data for probability. For example, Data Explorer (Sunburst, 1996a) provides students with the tools to carry out a data exploration by creating questionnaires, constructing and customizing a variety of graphs, and preparing reports. Finally, as tutee, the computer can present a problem-rich environment in which children solve challenging problems by programming the computer to exhibit arithmetic, geometric, and algebraic relationships. In the process of learning to program the computer, students develop new insights into their own thinking (Taylor, 1980) and develop an understanding of mathematical relationships. This use of the computer as tutee was pioneered by Seymour Papert (1980b) through the development of Logo and its accompanying philosophy of learning.

The concept of computer (and even graphics calculator) as tutee continues to provide the greatest potential for giving elementary children technological access to powerful mathematical ideas. According to Taylor (1980), when the computer functions as tutee, the focus of instruction shifts from product to process, “from acquiring facts to manipulating and understanding them” (p. 4). Papert (1980a) referred to this as teaching children to be mathematicians rather than teaching about mathematics. More specifically, using the language of Logo, Papert created intellectual environments that fostered learning through interactions involved with programming the computer. These environments, or turtle microworlds, were “constructed realities” (p. 204) structured to allow children to connect their intuitive understandings with formal mathematical knowledge. Papert’s microworlds were “sufficiently bounded and transparent for constructive exploration and yet sufficiently rich for significant discovery” (p. 208). In this way, he believed that the computer added “new degrees of freedom” (p. 209) to what children learned and how they learned it.

More recently, the goals of microworlds have shifted from having children program computers to having children devise their own tasks and subtasks for constructing and reconstructing mathematical objects and relationships (Noss & Hoyles, 1996). For example, in Tzur’s (1999) study, children used the objects and operations of a microworld (linear segments called sticks and operations on them such as partitioning and joining) to generate and abstract mathematical objects and relationships; that is, to build conceptions of unit and nonunit fractions as invariant relations. According to Noss and Hoyles, each object of a microworld is a conceptual building block that provides a means for connecting intuitions and existing knowledge with mathematical objects and relationships. They also maintain that the computer produces a
language through which meanings can be externalized and emerging knowledge can be expressed, changed, and explored (Noss & Hoyles, 1996). For example, after one of the children in Tzur’s study changed the color of the first two parts of a six-part stick and said, “this is two sixths, two out of the whole,” Tzur commented that the child’s language indicated that he had anticipated the structure of $\frac{2}{6}$ even before constructing it in the microworld.

In an even more poignant example, Olive (1998) reported on the effectiveness of the Geometer’s Sketchpad microworld in evoking the interest of his 7-year-old son in exploring the invariant properties of a triangle by dragging one of the vertices around the screen. According to Olive, his son “constructed for himself during that 5 minutes of exploration with Sketchpad a fuller concept of ‘triangle’ than most high-school students ever achieve” (p. 397). A surprising result of the child’s interaction with the computer was when he moved a vertex to the opposite side of the triangle, creating the appearance of a single line segment and concluded that the figure was still a triangle—“a triangle lying on its side.” Olive interpreted this comment as indicating an intuition about plane figures that “few adults ever acquire: that such figures have no thickness and that they may be oriented perpendicular to the viewing plane” (p. 397). This example highlights the power of dynamic microworld environments in providing children access to robust mathematical ideas. Some other examples of elementary children gaining access to powerful mathematical ideas through the use of microworlds can be found in studies on ratio and proportion (Hoyles & Sutherland, 1989), measurement (Clements et al., 1998), and probability (Pratt, 2000).

Contemporary microworld environments have generally retained the Piagetian learning model as espoused by Papert (1980b). According to this model, learning occurs as a result of breakdowns or incidents where predicted outcomes are not experienced. Thus, in designing microworlds, the developer must rely on a model of the relevant knowledge domain to predict where these cognitive breakdowns might occur (Noss & Hoyles, 1996). Similarly, Biddlecomb (1994) described the design of microworld environments as being guided by assumptions about the ways children learn. He reported that models of children’s conceptual structures provide an orienting framework for determining “what possible actions [are] to be included in the computer environment and how these actions [are] to be instantiated” (p. 97). Geo-Logo (Clements & Sarama, 1996) is an example of a microworld with a design guided by a model of the geometric structures that children constructed using turtle graphics. Although research has indicated that experiences with regular Logo were effective in helping children understand geometry, it was also found that children continued to rely on visually based, nonanalytical strategies (Clements et al., 1998). Geo-Logo was constructed to maintain a dynamic link between the commands entered by the student and the corresponding representations on the computer screen, thus “helping children encode contrasts between commands” (Clements et al., 1998, p. 220).

Clements et al. found that Geo-Logo was highly motivating to third-grade students and more particularly assisted their “constructions of mental connections between symbolic and graphic representations of geometric figures and between these representations and number and arithmetic ideas” (p. 221).

In effect, the Geo-Logo environment provided a window through which to study children as they continued to develop their understandings of geometry and measurement. Thus, while cognitive models inform the design of microworlds, these models are themselves informed by children’s interactions with the microworld. According to Noss and Hoyles (1996), technologies inevitably alter how knowledge is constructed and what it means to any individual. This is as true for the computer as it is for the pencil but the newness of the computer forces our recognition of the fact. There is no such thing as unmediated description: knowledge
acquired through new tools is new knowledge... Researching how students exploit autoexpressive computational settings to communicate, (re-) present and explain, not only provides descriptions of how individual students can express mathematical ideas, but can provide more general clues to the processes involved in learning, how knowledge is modified in the direction of mathematisation. (p. 106)

The challenge of future research will be to build on these new constructions of meanings through the development of mathematically rich experiences in both computer and noncomputer environments. Research in this century will need to continue to explore ways for technology to play the multiple and increasingly unified roles of tutor, tool, and tutee.

In summarizing this part of the chapter on cognitive access to powerful mathematical ideas, we note that research has revealed that children's informal knowledge structures accommodate powerful conceptual ideas such as partitioning, part–whole, and unitizing. Whether they constitute "the deep ideas that nourish the growing branches of mathematics" (Steen, 1990, p. 3) will be something that mathematics education research needs to investigate in the 21st century. Technology is certainly providing an effective setting for such research. For example, microworld environments have created a window through which to study children's constructions of powerful mathematical ideas and to analyze the development of these constructions in a fine-grained way not previously possible. Finally, research will need to reveal how these powerful ideas can inform curriculum and instruction because there is already evidence that teachers who are knowledgeable about cognitive models of children's thinking are effective in designing and implementing instruction that enhances children's mathematical understanding (e.g., Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier, & Human, 1997; Jeher, Jacobson, Thoyre, Kemeny, Strom, Horvath, Gance, & Koehler, 1998).

ACCESS TO POWERFUL MATHEMATICAL IDEAS: THE CURRICULUM GAP

At critical junctures during the 20th century, mathematics education leaders throughout the world called for reform in the school mathematics curriculum, in classroom implementation of that curriculum, and in related assessments (e.g., AEC, 1990; Cockroft, 1982; College Entrance Examinations Board, 1959; Commision on Post-War Plans, 1944; Council for Cultural Cooperation, 1988; Report of the Mathematical Association: The Teaching of Mathematics in Public and Secondary Schools, 1919, cited in Howson, 1982; NCTM, 1989, 2000). All of these reform endeavors have aimed at improving elementary students' access to powerful mathematical ideas. As noted earlier in this chapter, calls for restructuring the elementary mathematics curriculum have reflected ongoing societal needs, growth in the discipline of mathematics, changes in our understanding of students' mathematical learning, and increased availability and use of technology.

Mathematics educators today, enlightened by the experiences of the 20th century and large-scale assessments (e.g., U.S. Department of Education, 1996) recognize clear discrepancies among the desired curriculum as it exists in a national goal statement or a ministry of education syllabus, the implemented curriculum as it plays out in classrooms, and the achieved curriculum in terms of what children learn. These discrepancies raise two critical issues: (a) What kind of research is needed to address the discrepancy problem? and (b) What kind of curriculum development and teacher enhancement is needed to narrow the discrepancy? Ultimately, although these discrepancies remain, and the issues raised in (a) and (b) are still unresolved, we cannot guarantee that all elementary students will have access to powerful mathematical ideas.
What Kind of Research Is Needed?

The discrepancy among the desired curriculum, the implemented curriculum, and the achieved curriculum is not a new problem in mathematics education, but it is an intractable one. For example, when the results of the first international mathematics study were announced, critics of the new math blamed the comparatively poor performance of U.S. students on the new math curricula. However, the U.S. National Advisory Committee on Mathematical Education (NACOME, 1975) declared that, despite formal changes in school syllabi and curriculum texts of the new math era, the actual mathematical experiences of elementary school students during the 1960s reflected little of the reformers’ intended curricula. Consistent with this comment, Cooney (1988) later claimed that criticisms of the new math were inappropriate because “studies that carefully detail what happened in classrooms during the modern mathematics movement are virtually nonexistent” (p. 352). In essence, what NACOME and Cooney were saying is that although research revealed differences between the desired and the achieved curriculum, there was virtually no research that examined differences between the desired and the implemented curriculum. This lack of research focusing on linkages between the desired and implemented curriculum has engendered an ongoing sense of frustration, if not futility, in the curriculum development enterprise.

Even within the NACOME Report (1975), there was a call for descriptive studies that focused on the curricular and instructional activities of representative classes. The report also identified a number of pertinent questions that related to the implemented curriculum: How much class time is devoted to different mathematical topics? What is the relative emphasis on different levels of cognitive activity—factual recall, comprehension, or problem solving and critical thinking? Do textbooks dictate the curriculum? What is the influence of external exams? Who is involved in curriculum planning, and what value orientations do they bring to the task of preparing syllabi and selecting textbooks and tests? These questions are complex but are clearly just as crucial for this new century as they have been in the preceding one.

Fey (1980) responded to this challenge and in some sense established a framework for curriculum implementation research. He stated, “The effectiveness of future efforts to improve school mathematics programs depends on [research providing] a comprehensive picture of where we are and how public and professional influences act to shape school curricula” (p. 417). Fey also questioned the validity of existing methodologies that used questionnaire data to build up teacher-reported profiles of classroom activity. He added that studies in which researchers went directly to classrooms to observe how teaching time was used offered more fruitful directions for studying and analyzing curriculum implementation.

Consonant with the growth of qualitative and interpretivist research in mathematics education during the last two decades, methodologies are beginning to emerge that have the power to address Fey’s (1980) vision. These methodologies include educational development and developmental research (Gravemeijer, 1994, 1998), classroom teaching experiments (Cobb, 1999; Confrey & Lachance, in press), teacher development experiments and accounts of practice (Simon, in press; Simon & Tzur, 1999), and models focusing on teacher knowledge (e.g., Ball, 1991; Fennema & Franke, 1992). Even though these methodologies have different theoretical perspectives, they all meet Fey’s criteria of actually observing teaching and learning in classrooms. Indeed, they go beyond what Fey envisaged because they incorporate both instructional development and analyses of teaching and learning. Finally, they address these elements within the social situation of the classroom.

Notwithstanding these developments in research methodology, there is a huge leap in adapting microclassroom methodologies, like those identified above, so that
they can be used in analyzing the implementation of curriculum reform at a national level. Promising large-scale practices based on these microclassroom methodologies are beginning to emerge at both national (e.g., Ferrini-Mundy & Schram, 1997) and international levels (Stigler & Hiebert, 1999; Stigler, Fernández, & Yoshida, 1996). In the Ferrini-Mundy and Schram study (Recognizing and Recording Reform in Mathematics Education [R³M project]) a team of more than 20 researchers visited 17 school sites where changes in mathematics classrooms were occurring. More specifically, the R³M project is attempting to investigate the implementation of the NCTM Standards (1989, 1991) in schoolwide, districtwide, and statewide settings. The jury is still out on this research and its methodologies. However, the direction of R³M clearly captures the spirit that Fey (1980) foreshadowed when he called for case studies of curricular innovation in particular school settings. That is, case studies that could provide useful, raw material from which a broad understanding of the larger process could be pieced together (p. 417).

At the international level, the Third International Mathematics and Science Study (TIMSS) researchers (e.g., Stigler & Hiebert, 1999) studied classroom practices of elementary and middle school teachers in Japan, Germany, and the United States. In this study, based on significant international collaboration and cooperation, teachers were randomly selected from half the teachers whose classes took the test. The teachers were subsequently videotaped teaching a typical lesson, and they also completed a questionnaire that asked them to describe the goals of their lesson. Although it is not appropriate to discuss specific conclusions of this research, it is worth noting that the research has identified cultural differences in traditions of practice—differences that might impact student achievement. The videos also have the potential to impact teacher education and teacher enhancement.

As well-intentioned and promising as these micro and macro research developments are, we still require a robust body of research to guide the complex, multidimensional decisions that are needed to close the teaching and learning gaps among the desired, the implemented, and the achieved curricula that each country values (Stevenson & Stigler, 1992; Stigler & Hiebert, 1999). Moreover, until such a body of research exists, we cannot guarantee that children will have access even to the powerful mathematical ideas that are currently part of the intended curricula of the various nations in the world.

What Kind of Curriculum Development and Teacher Enhancement Is Needed?

Curriculum development and teacher enhancement are key elements in narrowing the gaps among the desired curriculum, the implemented curriculum, and the achieved curriculum. Even though we are already progressing toward research methodologies that will monitor the level of curriculum implementation and provide feedback for curriculum development and teacher enhancement, what is needed is a process to incorporate this feedback as part of the curriculum development and teacher enhancement cycle. So as to build a picture of how this cyclic process might work in the 21st century, two promising case studies will be considered: Realistic Mathematics Education (RME; Gravemeijer, 1998; Streefland, 1991; Treffers, 1987, 1993); and Project IMPACT (Campbell, 1996). In describing and analyzing these large-scale projects we will examine how they incorporate research, curriculum development, and teacher enhancement.

The Netherlands Project: Realistic Mathematics Education. Since the 1960s, the research of Dutch mathematics educators (e.g., Freudenthal, 1968; Gravemeijer, 1994; Treffers, 1987; Streefland, 1991) has provided the theoretical basis for their
“realistic approach” to curriculum development and the teaching and learning of mathematics. The original Wiskobas project that served as the catalyst for the reform of elementary school mathematics set in train the shift from a mechanistic orientation to teaching and learning to an approach that emphasized learning through reconstructive activity grounded in reality and sociocultural contexts. The Dutch researchers developed a new curriculum, textbooks, and tests; designed large-scale programs of preservice and inservice teacher education related to that curriculum; trained counselors and instructors; and monitored this activity through ongoing research. In essence, there was a strong articulation between the key elements of their program with research feeding the curriculum development and professional enhancement cycle and ipso facto the implementation of the curriculum.

The work that began with the Wiskobas project has continued to this day, with the perspective that school mathematics should be embedded in rich problem contexts that allow instruction to proceed from the reality of students’ informal strategies. Teaching in such a learning environment involves globally guiding students to be reflective and to develop increasingly abstract levels of mathematical reasoning that eventually lead to formal mathematization (Gravemeijer, 1991; Streefland, 1991). Their approach is reflected in the kind of mathematical modeling research (Verschaffel & De Corte, 1997) we mentioned earlier.

RME’s strong theoretical base has been developed through a distinctive research process titled “developmental research” (Gravemeijer, 1994, 1998). For Gravemeijer, developmental research combines curriculum development and educational research in such a way that the development of instructional activities is used as a means of elaborating and testing instructional theory. This combination does not take the form of a symbiosis between development and research in which research provides a formative evaluation of curriculum development. Rather, developmental research is seen as a form of basic research that lays the foundations for the work of professional curriculum developers. Moreover, developmental research is an iterative process in the sense that the development of instructional theory is gradual and cumulative; theory is slowly emerging from a large set of individual research projects (Gravemeijer, 1998, pp. 277–279). Over time this theory is increasingly being used to refine and enhance the Dutch approach to preservice and inservice teacher education.

In RME, powerful mathematical ideas such as problem solving are central to the curriculum and, by design, to the experience of every student engaged in that curriculum. Treffers (1987) and Streefland (1991) provided both qualitative and quantitative research evidence documenting that students using RME are especially successful in higher level problem solving and reasoning when compared with students who receive more traditional instruction. These findings are consistent with Lester’s (1980) research in which he concluded that the more students are engaged in real problem solving, the better they become. Moreover, the large-scale nature of the Dutch evaluation validates their ongoing cycle of developmental research, curriculum development, and teacher enhancement. In essence, it demonstrates that RME is beginning to narrow the gap between the intended curriculum, the implemented curriculum, and the achieved curriculum.

The U.S. Project: Increasing the Mathematical Power of All Children and Teachers (IMPACT). This project addressed the key concern that conventional mathematics instruction has failed to provide equitable access to powerful mathematical ideas for many students. Its intent was to address schoolwide reform in elementary school mathematics in predominantly minority urban schools where large numbers of students have not succeeded with traditional teaching practices. Project IMPACT emphasized building on children’s existing knowledge, problem-solving,
and instructional practices that elicited a high degree of student engagement and discourse (Campbell & Robles, 1997; Campbell & White, 1997).

A major focus of this 5-year project was its thrust on teacher development. This occurred through summer workshops focusing on content and pedagogy enhancement, on-site support from a mathematics specialist in each school, manipulative materials for each classroom, and the scheduling of weekly grade-level collaboration meetings that were devoted to planning and instructional problem solving. Special attention was devoted to teachers’ selection of rich problem tasks and to improving their questioning strategies in ways that elicited and promoted high levels of student reasoning and communication (Campbell & White, 1997).

In terms of improved student performance on tasks requiring conceptual understanding, higher level mathematical reasoning, and problem solving among students considered “at risk,” Project IMPACT has developed highly successful models for opening student access to powerful mathematical ideas. Moreover, the program of professional enhancement has enabled teachers to close the gap between the intended curriculum, the implemented curriculum, and the achieved curriculum. That is, as a result of teacher enhancement, Project IMPACT teachers have been able to move beyond uncertain and ineffective practices (Campbell, 1996; Campbell, Rowan, & Cheng, 1995).

Project IMPACT has built on prior research and also created its own. On the one hand it followed the Fennema and Franke (1992) teacher knowledge model as the basis for its teacher enhancement program. On the other hand, it created its own research by documenting critical yet different features of successful school programs (Campbell & White, 1997). The project’s research design examined changes in teachers’ knowledge and beliefs and the impact these had on classroom practice. It also documented the project’s emphasis on using real-life mathematics problems aimed at a slightly higher level than usual for their students and the growth in student engagement and discourse that were evident across all project schools. In essence, Project IMPACT used research on teacher change (see Fennema & Franke, 1992) to design an infrastructure that supported curriculum implementation and teacher enhancement. It then used its own research process to identify practices and principles that could guide effective mathematics instruction in elementary schools with high minority and poverty levels.

The Realistic Mathematics Education and Project IMPACT offer insights for providing equitable access to a mathematics curriculum rich in powerful mathematical ideas. The vision these projects offer is one that is closely tied to curriculum development, teacher enhancement, and research support. In RME the research drove the curriculum development and teacher enhancement, while in Project IMPACT the research captured the critical features of the project and made them available for wider dissemination and utilization.

**SUMMARY AND CONCLUSIONS**

Our analysis in this chapter suggests that the direction for elementary school mathematics in the 21st century will be more reflective of the last two decades of the 20th century than of the first 80 years. There is increasing research evidence that elementary school children need to engage in a mathematical experience, a “cultural initiation” (Chevallard, 1989), that will enable them to mirror the kinds of experiences in which mathematicians engage. This means that process goals that focus on problem solving, mathematical discourse, reasoning, and connections with technology will take precedence over pragmatic goals that have less salience in a society where technology has packaged the computational skills needed for effective citizenry. Elementary
mathematics should be a reality experience in which all children use powerful mathematical ideas with competence, confidence, and enjoyment. The emphasizing of all preempts a need for continued research to ensure that equity permeates the teaching and learning of elementary mathematics.

Given this focus on elementary mathematics as a cultural initiation, research must continue to investigate what is vital in extant areas of mathematics such as number, geometry, and measurement. Distinctions such as “conceptual knowledge” and “procedural knowledge” are helpful in enabling mathematics educators to identify powerful mathematical ideas from these domains. Newer mathematical domains (e.g., algebraic thinking, probability, statistics, and discrete mathematics) and children’s access to them through technology have the potential to empower children with pervasive conceptual knowledge that gives them access to both their present and their future reality. Moreover, research on processes such as mathematical modeling with respect to both extant (Verschaffel et al., 1999) and newer mathematical domains (Lesh et al., 1997) shows considerable promise for giving reality to the learning, integration, and application of mathematical ideas.

The research literature on cognitive access to powerful mathematical ideas in the elementary school is robust when compared with other areas of school mathematics. Cognitive models of children’s thinking are well represented in the literature on extant areas such as number, geometry, and measurement (e.g., Carpenter & Moser, 1984) and are beginning to emerge in newer areas such as algebraic thinking, probability, and statistics (e.g., Bright & Friel, 1998; Jones et al., 1997). These cognitive models have the potential to inform instruction in both traditional and technological environments (Tzur, 1999). Moreover, in this new century, teaching-experiment methodologies, with their emphasis on both psychological and sociological aspects of learning, may well forge the link between cognitive representations of children’s mathematical thinking and learning trajectories (e.g., Cobb, 1999).

Discrepancies among the intended curriculum, the implemented curriculum, and the achieved curriculum have proved a barrier to elementary children’s access to powerful mathematical ideas. This barrier is especially apparent for children from minority groups and poverty areas. This curriculum hiatus will continue to challenge us in the 21st century, but there are hopeful directions emerging. For example, in RME, an integrated approach to developmental research and curriculum development shows promise for establishing instructional practice that is consonant with the ideals of the intended curriculum (Gravemeijer, 1998). Project IMPACT with its strong focus on enhancing elementary teachers’ knowledge and beliefs is also producing learning environments that offer new directions for equity provisions.

Access to powerful mathematical ideas must be the right of every elementary student whatever their cultural background. Although it is neither realistic nor desirable to search for a solution to cognitive and curriculum access that is unique to every culture, increased globalization and technology offer unprecedented opportunities for international collaboration on these critical and enduring issues.

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This chapter addresses the theme of secondary school pupils’ access to significant mathematical ideas from different perspectives. One of these perspectives is related to the cognitive processes that take place in important conceptualizations, as well as in children’s evolution alongside themes of school mathematics at this educational level. A second approach has to do with the concept of junior secondary school. This concept varies from one country to another, not only as to students’ age but also concerning the emphasis placed on the role that this school level plays either as students’ preparation to enter into pre-university education or as the concluding stage of basic education, which in many countries represents the final schooling stage for an important proportion of the population. In this sense, such an emphasis determines curricular contents as well as teaching approaches. A third approach addresses the strong influence of incorporating new technologies to the teaching of mathematics on mathematical contents and classroom organization. Finally, I consider the unavoidable perspective of the new millennium because it imposes important reformulations on what to teach, how to teach, and why to teach. According to some authors, the new millennium will demand new mathematical preparation for all children, and school systems will have to accomplish this for younger students then ever before.

These four perspectives give rise to corresponding research issues, some of which are discussed in this chapter, where special attention is given to those issues that, from a cognitive processes perspective and the influence of new learning tools on mathematics education, could be considered critical in advancing our knowledge of key factors that may favor (or obstruct) adolescents’ access to powerful mathematical
ideas. Thus, the content of the chapter is structured according to these perspectives and related inquiry work. The second and fourth perspectives are combined to present the content of the last section.

**TRANSITIONAL PROCESSES IN THE ADOLESCENT'S MATHEMATICAL THINKING**

When referring to students’ access to significant mathematical ideas, the word significant can be interpreted in various ways. For instance, in terms of the transitional processes that teenagers experience when they begin studying algebra or synthetic geometry, becoming conscious of the power of generalization, working with “the unknown” (quantities), and verifying conjectures are considered significant ideas because they promote these transitional processes and allow students to access levels of thought that surpass specific, numeric, and perceptual thinking. In this sense, significant ideas in mathematics are not necessarily advanced and powerful mathematical notions but instead are key notions that provide real access to the latter. In this way, at least in the transitional processes context, a mathematically significant idea acquires a relative character because it depends on its power to aid the evolution of the student’s mathematical thinking toward more abstract, formal, and complex levels. In the following sections, I analyze some significant mathematical ideas, in the sense described above. That is, in the context of the transitional processes that occur in the passage from primary to secondary school.

**FROM ARITHMETIC TO ALGEBRA**

For a long period of time, the progression to algebraic thinking was assumed as occurring for the majority of students between 11 and 16 years old. This assumption changed at the end of the 1970s with research findings from authors such as C. Kieran who investigated the interpretation of the equality sign, which became an essential research topic for the elaboration of plausible explanations about the difficulties that students face when learning symbolic algebra (Kieran, 1981). Kieran’s work, along with the work of other researchers (Matz, 1980; Booth, 1984) who analyzed recurrent errors and misunderstandings in the study of algebra, helped to establish how the meaning variation of mathematical symbols during the transition from arithmetic to algebra represents an obstacle in the subject’s evolution toward the acquisition of algebraic language. Table 7.1 summarizes this change in meaning of some of the symbols that appear in school mathematics at both primary and secondary school levels.

Table 7.1 shows in a schematic way how the operational symbols change significance when changing from one knowledge domain to another. The symbols + and −, which in arithmetic represent executable operations with the addition and subtraction algorithms and which lead to a numeric result, relate terms containing literals in the field of algebra. These symbols also represent suspended operations (in expressions such as $2x + 7$), when algorithms or execution rules are not necessarily implemented; they also represent operations executable with algebraic rules (such as in $3x + x - 7x$) through which a result is obtained ($-3x$). In algebra, the symbols + and − can also be unary, as in the case of the relative numbers $-7$, $+5$, $-32$.

On the other hand, because of its close relationship with the operational symbols, in arithmetic the symbol $=$ works as an operator that “transforms” the left member of an equality into a numeric result that appears in the right member (such as in $12 + 7 = 19$). Meanwhile in algebra, the symbol $=$ can represent equivalence between two expressions (such as in $2(a + b) = 2a + 2b$); or it can also represent a restricted
TABLE 7.1

Changes in Meaning of Common Mathematical Symbols

<table>
<thead>
<tr>
<th>Symbols</th>
<th>Arithmetic</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>+, −</td>
<td>Binary operations; executable operations with arithmetic algorithms:</td>
<td>Binary operations; suspended operations: 3 + x, 2x − 7y</td>
</tr>
<tr>
<td></td>
<td>3 + 4 = 7, 37 − 18 = 19</td>
<td>Executable operations with algebraic rules: 3x + x − 7x = −3x</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Double meaning binary operations, suspended operations: 8a − b</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Binary operations executable in algebra: 23n − 11n = 12n</td>
</tr>
<tr>
<td>=</td>
<td>Operator: operations = result</td>
<td>Equivalence, restricted equality, functional equality: 2(a + b) = 2a + 2b</td>
</tr>
<tr>
<td></td>
<td>12 + 7 = 19</td>
<td>7x − 4 = 28x + 15</td>
</tr>
<tr>
<td></td>
<td></td>
<td>y = 3x − 2</td>
</tr>
<tr>
<td>a, b, c, . . . n, . . . x, y</td>
<td>Area volume and physics formulae: b × h/2, v = π × r^3, v = d/t</td>
<td>Unknown quantities, variables, and general numbers</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Characters concatenation</td>
<td>Additive meaning: 324 = 3 hundreds, plus 2 tens, plus 4 units</td>
<td>Multiplying meaning: 3a; three times a</td>
</tr>
</tbody>
</table>

equality or equation (such as in 7x − 4 = 28x + 15); or it can also represent a functional relationship (such as in y = 3x + 2).

Letters are used in arithmetic above all as labels that evoke very specific references but are susceptible to numeric substitution, such as in geometric formulae (a = b × h, c = 2πr).

The concatenation of symbols also obeys different conventions within arithmetic. The juxtaposition of numbers such as in 324 corresponds to notation in a positional system and is additive: 3 hundreds, plus 2 tens, plus 4 units. Meanwhile 3a in algebra has a multiplicative interpretation: “3 times a.”

The research carried out in the 1980s on algebraic thinking shows that these differences in meaning of the same symbols and symbol chains present serious difficulties for secondary school children in the learning of algebra, strongly bringing into question the old idea that algebra could be conceived, for teaching purposes, as “an extension of arithmetic.”

For its part, at the end of the same decade, great importance in the field of research was given to the approach of the evolution of school mathematical knowledge based on overcoming didactic obstacles of epistemological origin. In the specific case of school algebra, this approach is linked to the changes in the significance assigned to the symbols and the actions taken with them (Filloy & Rojano, 1989).
difficulties that natural language interpretations and actions generate when students transfer them to algebra also have been studied (Freudenthal, 1983). A classic example is that of left-to-right writing, a feature of languages such as Spanish and English that permeates algebra writing in such a way that students tend, for instance, to write a chain of equalities, instead of expressing the reestablishment of the equality in each transformation step during equation solving tasks:

Problem: Solve the equation $2x + 7 = 18x - 9$

Reestablishment of the equality (vertical sequence):

\begin{align*}
2x + 7 &= 18x - 9 \\
7 + 9 &= 18x - 2x \\
16 &= 16x \\
x &= 16/16 \\
x &= 1
\end{align*}

Equalities chain (sequence from left to right; usually present in algebra novice students):

\begin{align*}
2x + 7 &= 18x - 9 &= 18x - 2x \ldots
\end{align*}

Following is an example of a 13 year-old girl (Matilde) who had just been taught to solve linear equations using a concrete (geometric) model. These are her first steps in the algebraic syntax domain during an interview (Filloy & Rojano, 1989):

Equation to be solved: $129X + 51 = 231X$

Matilde writes down: $129X + 51 = 231X - 129X = 102$

$M$: “Therefore $X$ equals two”

The equality sequence, written from left to right makes sense in tautological transformations (algebraic identities) but not in equation solving problems. This confuses students because they do not possess the criteria to discriminate mathematical situations where it is possible to proceed as in other knowledge domains (arithmetic or natural language). These types of difficulties with the rules of algebraic writing which are, in part, due to the linguistic conventions of natural language can also be explained in terms of temporal order that tends to govern the sequence of actions or, as in Matilde’s case, the order of actions carried out in a concrete teaching method, which are transferred to the actions of transformation of an equation.

There is extensive research on the nature of students’ difficulties on understanding and using algebraic language due to the use of everyday languages as well as previously acquired notions such as arithmetic and the mother tongue. These include a wide range of studies, from those of a clinical and historic epistemological nature (Filloy & Rojano, 1984, 1989; Rojano, 1996a), to theoretical dissertations with an emphasis on the cognitive (Sfard & Linchevski, 1994; Herscovics & Linchevski, 1994), linguistic, or semiotic planes (Kirshner, 1987; Drouhard, 1992; Arzarello, Bazzini & Chiapinni, 1995; Puig & Cerdán, 1990), and experimental work or pilot studies (Cortes, 1995; Bell, 1996; Bednarz & Janvier, 1996; Bednarz, Radford, Janvier, & Leparge, 1992; Stacey & MacGregor, 1995; Kieran, Boileau & Garançon, 1996; Rojano & Sutherland, 1993).
Some of these authors have pointed out the existence of conceptual jumps or gaps that show the frontiers between arithmetic and algebraic thinking and confer great importance to the study of teaching approaches that can help students to overcome the learning obstacles rooted on those gaps. Summarizing, these authors argue the following:

- Novice students have difficulty working with “the unknown,” in other words, with unknown quantities. Evidence exists on students’ inability to extend spontaneously the actions done over an equation of the type $Ax ± B = C$ ($A$, $B$, $C$, known numbers) to find the value of $x$ into equations of the type $Ax ± B = Cx ± D$ because in these cases it is necessary to operate “the unknown,” that is, the terms containing $x$ (Filloy & Rojano, 1989). Sfard and Linchevski explained this phenomenon in terms of the process-object duality and of the transition from the operational to the structural through reification, pointing out that the reification step constitutes a source of enormous difficulties for novice algebra students (Sfard & Linchevski, 1994).

- The resolution of word problems, which in algebra explicitly include the translation of the text into algebraic code, represents another difficulty students face in their transit to the algebraic domain. Work by MacGregor and Stacey (1993), as well as research from Bednarz and Janvier (1996), clearly illustrates this cognitive jump. On the other hand, Puig and Cerdán (1990) used classic methods (the analysis-synthesis and the Cartesian methods) to establish existing differences between arithmetic and algebraic problems and to characterize them.

- The majority of students in secondary school are not able to connect by themselves the knowledge domains that constitute manipulative algebra on the one hand and instrumental algebra for problem solving on the other. Rojano and Sutherland (1993) showed how students can manage to conciliate both aspects of algebra through the use of intermediate codes (between natural language and algebra) similar to algebraic codes, in which the referents coming from the problem context are present (see the next section for a detailed explanation of the spreadsheets method to solve word problems).

- The study of algebra as a language both from the perspectives of the semiotics and the pragmatics (Filloy, 1999; Puig & Cerdán, 1990) or the linguistics (Kirshner, 1987; Drouhard, 1992), reveals intrinsic characteristics that can become obstacles for users to achieve proficiency in this language.

- Early introduction to algebra reveals that an adequate development of the operational sense (addition) allows students in primary school to experience transitional processes toward an algebraic form of thinking, for instance, through the addition of unknown quantities or arbitrary numbers (Slavit, 1999).

In summary, research conducted up to the 1980s warns us about the difficulties that students face in their transit to algebraic thinking and suggests the need to study in depth the nature of the didactic, cognitive, and epistemological obstacles that lead to these difficulties. This research refers us to the enormous influence that tendencies based on the everyday use of natural language and an arithmetic way of thinking have on the students’ interpretation and production of algebraic symbols, as well as on how students learn algebraic problem-solving methods.

In contrast, subsequent research reveals a tendency to answer questions identified in studies that unravel the nature of the difficulties on the acquisition of algebraic language. For instance, research on the early introduction to algebra and on the use of intermediate forms of expression and operation (between arithmetic and algebraic) bring together manipulative algebra and problem solving (e.g., Brown, Eade & Wilson, 1999; Goodson-Espy, 1998; Herscovics & Linchevski, 1994; Hoyles & Sutherland, 1989).
In both cases, reported results suggest further research is needed. For example, there are still many unanswered questions about the algebraic language "in use" and about the transformation routes existing between the child's intuitive methods and the school methods for solving algebraic tasks.

FROM THINKING SPECIFICALLY TO THINKING GENERALLY

The transit from the specific to the general is present in different degrees in every mathematical school task because generality and thus generalization, is endemic in mathematical doing and learning (Mason, Graham, Pimm, & Gowar, 1995). This transit is specially emphasized in junior secondary school because at this education level students are able to access symbolic (algebraic) representations that allow them to reach a manipulative level of generality. Generalization processes (the passage from the specific to the general) in school mathematics can be illustrated using the "generalization cycle" (Mason et al., 1995), namely:

- Perception of generality (recognizing a pattern, for instance, in numeric sequences)
- Expression of generality (elucidating a general rule, verbal or numeric, to generate a sequence)
- Symbolic expression of generality (yielding a formula corresponding to the general rule)
- Manipulation of the generality (solving problems related to the sequence)

Some authors (Lee, 1996; Mason, 1996) have criticized intensely the haste to symbolization when using a cycle of this type during the completion of generalization tasks in the classroom. There is an apparent tendency in teaching to abbreviate the first two steps, and this on some occasions precludes students from producing an algebraically proper equation for the stated problem. Lee (1996) discussed an example concerning a task used in an experimental study with adults in which, because of perception problems on pattern recognition, it was impossible to yield an algebraic equation that could lead the participants to successfully complete the task. This example is reproduced as shown in the dot rectangle problem (see Figs. 7.1 and 7.2 taken from Initiation into algebraic culture generalization (Lee, 1996)).

In the case of Fig. 7.1, focusing on the borders patterns corresponding to the numeric sequence 2, 4, 6, 8, did not lead the participants to a general equation because they faced a conflict with the given rectangle and dots from the first to the fourth rectangle. One would expect that the existence of this equivalence table could prevent the participants from focusing on the wrong graphical pattern. However, as Lee’s study results reveal, this did not happen in all the cases.

The example in Fig. 7.2, in which the task design contemplates the role of algebraic representations on manipulating generality, clearly illustrates the existing gap between theory and practice. This gap can be explained through one of the various peculiarities of cognitive processes in mathematical thinking, consisting of its close relation to individual preferences for different ways of representation (in this case diagrammatic, verbal, numeric, and symbolic algebraic). The existence of these preferences is reported in detail in Molyneux, Rojano, Sutherland, and Ursini (1999). In their Anglo Mexican study, “School based mathematical practices in the science classroom,” the differences mentioned above (which in this case also included the preference for graphical representations) are partly attributed to school cultural differences detected between the student groups in Mexico and in England that participated in the study.

The existence of cognitive tendencies, such as the preference for a determined representation form, does not weaken the theoretical argument stating that algebraic representations enable the calculation of generality to the point that it is feasible to solve a wide range of problems related to the generalization situation that is posed.

\[ \begin{array}{|c|c|}
\hline
\text{"How many ... hundredth"} & - \text{There will be 30 dots in the fifth} \\
100 \times 101 = 10100 & - \text{100} \times 99 = 9900 \text{ in the hundredth} \\
\text{multiply number of dots horizontally} & - \text{There will be} \\
\text{by number of dots on vertical line} & \text{n} \times (n + 1) = n^2 + n \text{ dots} \\
\hline
30 & \text{Blank} \\
9702 \infty & \text{Can't conceptualize it} \\
\hline
\end{array} \]


---

1The Anglo Mexican collaborative work (funded by the Spencer Foundation, Grant No. B-1493) was conducted by two research teams, one in England and one in Mexico. The research drew on the fields of cultural psychology and activity theory as well as the fields of science and mathematics education. The research investigated the school mathematical practices of 16 to 18 year old science students and the cultural influences on these school-based mathematical practices in both Mexico and England.
Thus, for example, in a sequence of numbers or figures governed by a general pattern, the algebraic expression of the nth element can lead to determine the place of an element with a given numeric value in the sequence, to calculate the element’s specific value for a determined place (prediction possibility), to analyze sequence tendencies, forward and backward (global appreciation possibility). Taking this into account, the question of under which conditions it is possible to promote the student’s awareness and appreciation of the algebraic code’s value in generalization tasks is an issue that should be investigated. Specifically, it would be interesting to investigate if an adequate use of the generalization cycle in teaching can support this awareness.

At this grade level, in the area of solving word problems, the transition to the general occurs when an algebraic expression (it can be an equation or a functional expression) synthesizing the relations between data and unknown quantities (equation) or between variables (function) within a problem’s statement is produced. In this particular case, this means that the students have to face the difficulties concerning the translation process from the problem’s text to the algebraic code and the difficulties related to the process of solving the corresponding equation(s). This process is commonly known as the Cartesian method, in which expressing the problem’s elements in equation form is acknowledged as the mathematization of the problem. Several studies using spreadsheets to help students to solve problems that are typically solved using the Cartesian method show that in this computer environment, it is feasible to use an intermediate language (between natural and algebraic) to express in a general way existing relations between data and unknown quantities with the possibility of changing the unknown quantities’ value to find an answer (Rojano & Sutherland, 1993; and Sutherland & Rojano, 1993).

When students use the spreadsheet method to resolve word problems, they organize the information contained in the problem statement on a spreadsheet, labeling the columns with names relative to the elements of this statement and introducing formulas written in Excel, which express the relationships between data and the unknown. After varying the numeric value of one of the unknowns (whichever one ends up being an independent variable in the set of formulas), the solution is reached through trial and refinement (Fig. 7.3 shows an example of this method to solve “the theatre problem”). In this method the Excel formulas constitute an intermediary language between natural and algebraic languages, and its construction comes from a process of problem analysis (Rojano, 1996b). In the particular case of “the theatre problem,” the basic unknowns can be identified: the number sold of child tickets and of adult tickets. Because there are 100 more child tickets than adult tickets (number child tickets = number adult tickets + 100), it is recommended that in the spreadsheet method the number of adult tickets be chosen as the “free” unknown to be varied. In this way, one of the spreadsheet cells (A1) is labeled as number of adult tickets, and the tentative number for the value of this unknown is introduced in cell A2, for example, 10. The name of the second unknown, number of child tickets, is written in B1, and the corresponding formula is introduced in B2 (= A2 + 100), which represents the relationship this unknown maintains with the former (in A2). In cells C2 and D2 formulas for the total cost of adult tickets (B2 × 120) and child tickets (C2 × 80) are respectively inserted. A formula for the total earnings of the event is introduced in E2 (= C2 + D2). With a change in the numeric input of A2 (one of the unknowns) the numeric values of the cells containing the formulas also change automatically. In this way the input in A2 can be continually varied until the value 30,000 appears in E2, which is one restriction of the problem. It is not difficult to find out that when 110 is introduced in A2, this restriction is fulfilled, and therefore the value of the other unknown (B2) would be 210.

In this way, students are provided with a tool that allows them to move gradually from an arithmetic approach to problem solving (a numeric approach centered on the specificity of the data), to the algebraic method.
The theater problem.

Tickets for a theater performance cost $120 for adults and $80 for children. A hundred tickets more for children than for adults were sold. How many tickets for adults and for children were sold if the total collected amount was $30000?

Use the spreadsheet to solve this problem.

<table>
<thead>
<tr>
<th>A</th>
<th>Number of tickets for adults</th>
<th>B</th>
<th>Number of tickets sold for children</th>
<th>C</th>
<th>Ticket price per adult</th>
<th>D</th>
<th>Ticket price per child</th>
<th>E</th>
<th>Total cost of tickets</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>A2+10</td>
<td>3</td>
<td>A2+10</td>
<td>4</td>
<td>B2*120</td>
<td>5</td>
<td>B2*80</td>
<td>6</td>
<td>B2+120</td>
</tr>
</tbody>
</table>

Let's assume that 10 adults go to the theater;

How much money will be collected if 10 adults go to the theater? $ ____________

Change the number (tickets for adults) in cell A2.

How many tickets for adults were sold? ____________

How many tickets for children were sold? ____________

Nevertheless, research on the feasibility of coupling the spreadsheet method with the equation formation process, and therefore with the algebraic method, is still pending (Rojano, in press). One of the main advantages of mastering this algebraic method is the opportunity brought by the ability to identify problem sets that can be solved using the same equation or equation system or, moreover, using the same type of equation or equation system. This involves another type of generalization: the generalization of the method.

FROM INFORMAL TO FORMAL METHODS FOR SOLVING PROBLEMS

The predominant use of intuitive or “personal” methods by student populations between ages 11 and 16 years old is discussed in a wide range of research studies, from the first systematic inquiries on frequent pupils’ errors in algebra (Booth, 1984; Matz, 1990) to the most recent investigations on word problem solving (Bednarz, Kieran & Lee, 1996; Rojano & Sutherland, 1993). This use is attributed on one hand to the difficulty that secondary school methods entail (e.g., the algebraic method, geometric justification, logical argumentation, probabilistic thinking validation) and on the other to students’ experiences with their “own” methods as means that will eventually lead them to reach a correct answer. These results have emphasized the
need for researchers, educators, and curriculum designers to include children’s own methods as unavoidable antecedents when learning school methods. In this respect, two possible pedagogic intentions can be identified: the usual one, which tends to replace the children’s methods with school methods, and other, which can be considered revolutionary, that tends to gradually institutionalize some methods that are closer to the children’s own methods. An example of the latter is the “trial and refinement” method that students frequently use to solve equations. Based on research findings recommending the consideration of the children’s informal methods, the trial and refinement method has been incorporated as a systematized version—which includes the use of calculators—into school practice in some countries such as England (Sutherland, 1999; The Royal Society, 1997). Sutherland referred to the 1997 Royal Society report, which discusses the confusion resulting from the failure to identify as an algebraic activity the mere manipulation of symbols, and from the characterization of the trial and improvement method as “algebraic,” to the point that students have come to believe that this is the official method. Sutherland argued that apparently this well-intentioned reform, centered on the student, has come to obstruct the students’ access to the powerful cognitive mathematical tools that have been developed through centuries (Sutherland, 1999).

Regarding the issue of how to consider the children’s own methods, it is possible to say that the access to other types of methods for solving word problems using computer environments such as spreadsheets permits a more intermediate position, one between the two described above, which are clearly situated on opposite ends. In the spreadsheets method (described previously) a mathematical relationship can be encapsulated by moving the mouse (or the arrow keys) without explicit reference to spreadsheet symbolism (Sutherland & Rojano, 1993). Therefore, a spreadsheet helps pupils to represent and try mathematical relationships without having to deal with a symbolic language, but they can see this relationship represented symbolically in the spreadsheet (spreadsheet formulas). The algebraic relationships are likely to be closely related to the numeric domain, and in this sense a spreadsheet provides a context for generalizing from arithmetic and systematizing pupils’ informal strategies. The definitive step toward the algebraic or Cartesian method, which explicitly assumes a translation from the word problem’s content to the algebraic code, unavoidably must take into account the existing differences between this method and the spreadsheet method. In the Cartesian method, the process of putting something in equation form corresponds to the action of finding two equivalent algebraic expressions for the problem situation (this equivalence is a local one, emerged from the restrictions of the problem statement) and then linking these expressions through the equality sign. In contrast, with the spreadsheets method all the partial (or elementary) relationships between givens and unknowns are symbolized in separate but related cells and all these relationships are finally synthesized in one expression, which serves as control of the variation of one of the unknowns. On the other hand, the solution to the equation itself using the Cartesian method does not have an equivalence with any part of the spreadsheets method, because to find the numeric solution to the problem in the latter, one must vary the numeric input in the cell that represents the unknown quantity; this comprises a purely numeric method.

Thus a didactic project that intends to use the spreadsheet method but at the same time intends to introduce the students to the Cartesian method should include in its design a way of linking the representation of the variables’ relationships in a spreadsheet with the algebraic code. In this way, students would be able to capture such relationships in an equation that could be solved with manipulative algebra techniques. Some results from the Anglo Mexican research project “Mathematical Modeling with Spreadsheets” suggest that spreadsheets can play an important role
in taking into account children’s intuitive methods as a basis to teach them “more algebraic” school methods of solving problems (Rojano & Sutherland, 1997; Rojano, in press). Nevertheless, how to help students shift from their own strategies to the algebraic (Cartesian) method, properly speaking, remains being an unanswered research question.

FROM DRAWING TO THE (GEOMETRIC) FIGURE

Some mathematics curricula contemplate the initiation of students into synthetic geometry. This presupposes previous intensive work with geometric objects and their properties during experimentation and inductive reasoning tasks. Of course, the latter assumes that the difficulty of shifting from working with the perceptual (drawings) to working with the conceptual (the [geometric] figure) has been overcome. There is compelling evidence regarding the small number of students that achieve this transition, particularly if their geometric learning experiences include only pencil and paper tasks, because this entails greater cognitive demands than, for instance, exploring and experimenting in dynamic geometry environments. The main problem source during the transition from the perceptual to the conceptual is the student’s confusion generated by the (geometric) discourse referring to figures that teachers and textbooks use. This discourse does not necessarily correspond to the students’ interpretation of these figures. Furthermore, it doesn’t even correspond to the properties that the teachers themselves intend to assign to the figures, so that their pupils can isolate the particularities of drawings and distinguish the invariant properties. The students’ focus on the geometric figures’ invariant features has been favored lately by the rise of dynamic geometry developed with the support of computer media.

Today, considering the possibility of developing an experimental geometry in schools, there are computer environments that allow students to directly manipulate geometric objects (such as Cabri-Geometre) to perform formulation tasks and conjecture testing. Other types of computer environments (such as the Anderson Geometry tutorial) have been designed to facilitate the learning of proof in mathematics. According to Balacheff and Kaput (1996), the didactic task of connecting the experiences developed by the students in these types of environments to help them to reconcile deductive and inductive reasoning is still pending. The latter will be at the center of geometry teaching interests at secondary school, especially if the transition to synthetic geometry and to proof is included.

Other aspects concerning education at this level are that of spatial sense, and in general that of three-dimensional geometry, in which the role of visualization becomes crucial (see Hershkowitz, Parzysz, & Van Dormolen, 1996).

Another problem source for geometry students in their transition to the conceptual is the lack of previous visual education that can aid the systematization of their visual experiences, for instance, in the search for patterns or in the distinction between the role of drawings as geometric objects or as diagrammatic models of these objects—a distinction based on the double role of the figures (as in Laborde, 1993). This lack of visual training during kindergarten and primary school has a severe impact on another fundamental aspect of education in secondary school: spatial sense, in particular, three-dimensional geometry. Here, the role of visualization becomes essential. Research conducted by Razel and Eylon (1990) with student groups in preschool and grade school suggests that students who have access to, and experience with, visual didactic media develop an ability to identify visual concepts in complex contexts (for instance, they can reproduce patterns perceived in a certain representation as different types of representations) as well as to apply these concepts
in visually complex situations. These researchers also studied the way in which visual experiences influence the development of mathematical concepts such as ratio and proportion (Razel & Eylon, 1990). They showed the importance of visual training as an antecedent to geometry learning in secondary school. A probable pending research task is precisely to study the transitional processes toward mathematization in geometry (both inductive and deductive) at secondary level, beginning with visual experiences in school and their effect on the child’s abstract and logical thinking. In particular, it would be important to investigate to what extent visual experiences (even if necessary for the development of certain types of geometric abilities) can provoke attachment to visual aspects of geometric objects and eventually become an obstacle to progress toward geometric knowledge that requires more abstract and deductive thinking.

**TOWARD ABSTRACT THINKING**

There is a common tendency to initiate progress toward more abstract mathematical processes and notions in students from 11 years of age and over. For instance, practically every teaching approach to algebra presupposes mathematical abstraction processes that are not always made explicit to the instructors in concrete proposals. In contrast, there is a wide variety of theoretical studies dealing with abstraction in mathematics, going from those considering abstraction as a decontextualization process to those denying decontextualization as a means to achieve more abstract levels of thinking. Hershkowitz, Schwarz, & Dreyfus (2001) provided a detailed review of these studies, pointing out the features that characterize each of these approaches and that lead to these conceptual differences. For instance, they referred to the notion of reflective abstraction (used by Piaget as the foundation of his cognitive developmental theory), which applied to the mathematics domain corresponds to the transit from action to cognition (in Piaget’s theory, Piaget, 1970) to the transit from the problem’s situation to mathematics (mathematization process). This adaptation of Piaget’s reflexive abstraction notion to abstraction in mathematics was developed by Vergnaud’s work on mathematization. Vergnaud (1982) conceived the latter as a process of progressive decontextualization through which the mathematics are extracted from the problem’s situation.

Hershkowitz, Schwarz, and Dreyfus (2001) placed on the opposite side those studies denying abstraction as a detachment from the referents (e.g., Mason, 1989) or those criticizing the conception of abstraction as a mental activity in which the environment’s role, both regarding the social interactions and the interaction with the tools, is ignored (Greeno, 1997). Although one could agree (or disagree) with Hershkowitz, Schwarz, and Dreyfus (2001), it is not difficult to accept the great gap that these authors observe between empirical or experimental research and abstraction processes, particularly in the field of mathematics education. Among the studies developed in this direction, however, the work of Mason is relevant for practices in the classroom because it analyzes the role of generalization in the learning of algebra (Mason, 1996) and its relation to mathematical abstraction and symbolization. Mason claimed that “If teachers are unaware of its presence [generality presence], and are not in the habit of getting students to work at expressing their own generalizations, then mathematical thinking is not taking place” (Mason, 1996, p. 65). Another example is the work of Filloy, which deals with the topic of abstraction in the learning of algebra by analyzing several observations on “concrete modeling” processes in a moment of transition (from arithmetic to algebra). The central foci of this work include (a) the role that “more concrete” languages or expression media play on modeling “more abstract” situations and (b) the role of “concrete modeling” on the production of the algebraic code
necessary to develop problem solving skills (Filloy, 1999; Filloy & Sutherland, 1996, p. 149).

Filloy specifically referred to the processes that occur when the teaching of algebraic syntax is deliberately intervened upon using some “concrete model” such as the balance model. The aim is that students will eventually associate the actions done with this model with the actions performed on the elements of a given equation. In this way, actions toward finding the unknown value (the unknown quantity’s value) in the following model situation can be associated with the transposition of terms in the equation (see Fig. 7.4).

The observation of eighth-grade students working with the balance model led to the detection of individual cognitive tendencies. On one hand, some students showed a preference for the algebraic syntactic level: Once these children could set up a correspondence between the actions they previously had carried out with the elements of the model and actions that can be performed with elements of the equation, they prematurely abandoned their work with the model. Thus, they moved to operate on the terms of the equation, using a partially constructed algebraic syntax. In the other extreme, some students showed the opposite tendency. That is, they were unable to transfer their actions on the model to actions on the equation and continued working in the context of the model, even in cases in which it did not make sense to use the concrete model (Filloy & Rojano, 1989). The latter poses several research questions in which the consideration of the subjects’ cognitive tendencies must be included to advance our knowledge on the abstraction processes that take place during specific teaching situations.
The emergence of new computer-based learning environments as well as the use of graphing calculators have given students access to advanced mathematical ideas, which would not be accessible at early ages with traditional learning tools. Among the wide variety of research and educational development projects that are currently being proposed, the incorporation of information and communication technologies (ICT) into secondary school is especially significant. We can identify two main trends regarding how technology is being used: one focusing on helping students face the typical difficulties that the learning of specific teaching contents entails, thus promoting the achievement of the school system goals; the other centered on introducing students to mathematical notions and contents that usually transcend curricular limits and educational goals in secondary school. Normally, these contents pertain to advanced mathematics, usually included in high school or university curricula.

The second trend includes the use of modeling and simulation applications, as well as different types of databases (graphics, tables). According to Balacheff and Kaput (1996), this trend unveils a new stage, which, in contrast to previous stages (focused mainly on facilitating the use of traditional formalisms such as the manipulation of algebraic expressions and function graphing), aims to connect the student’s personal experiences with the physical world (through simulation models) and with the mathematical experience (through databases, graphics, tables). SimCalc MathWorlds is a computer environment that provides this link, allowing the design of tasks that can make accessible the mathematics of variation and change to students even if they have not been introduced to, or are novices in, the algebraic symbolization typically required as the basic language for calculus. In this environment, it is possible to help students progress from the manipulation of a simulated motion phenomenon to more abstract and schematic representations and still be working with these phenomena using intermediate abstraction models (Kaput & Nemirovsky, 1995).

Research conducted by the developers of SimCalc suggest that this environment can be used for an early introduction to the mathematics of change, in other words, it can be used with children at primary school level (Stroup, 1996). (It is important to note that this is a case of democratic access to the powerful concept of variation in mathematics but that this access presupposes the clarification of the transition between mathematical notions for example, from the notion of ratio [clearly situated within the arithmetic domain] to the notion of rate [pertaining to the calculus domain]).

SimCalc has been used recently in an educational development project in Mexico² carried out with junior secondary school pupils. Some experiences from this project report that the lack of understanding, on the part of the teacher, of the conceptual change that is required to have access to the powerful ideas of the mathematics of change, led the pupils to remain working out the analysis of motion phenomena at a mere numeric (arithmetic) level. To implement educational innovation proposals such as the one mentioned earlier, it is necessary to conduct research on the cognitive

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²“Incorporating the use of new technologies into school culture: The teaching of mathematics in secondary school” Mellor, Bliss, Boohan, Oqborn, and Tompsett is a five-year project funded by the Ministry of Education and the National Council for Science and Technology in Mexico (CONACYT, grant GS263385). This project is aimed at incorporating gradually various pieces of technology into the mathematics and science national curricula at the secondary level. Initially, it covered 15 states all across Mexico, and four pieces of software were selected for the mathematics part: Cabri Géomètre, Sim Calc Mathworlds, Stella, and Spreadsheets.
processes that occur during the progression toward these notions and on the role of
the teacher’s interventions in this transition.

In SimCalc as in other computer environments, students can experiment with in-
termediate models, that is, those between physical phenomena and formal mathe-
matical models. A similar example is that of spreadsheets, where the work with
numeric columns (in which functional variations can be described) introduces a nu-
umeric approach to variation that considers the referents pertaining to the physical
world through the labeling of the columns.

In a spreadsheet, as in other computer environments, the access to the graphical
representation allows students to visually analyze a function tendency and its global
behavior. On the other hand, a symbolic representation with the spreadsheet code
allows students to manipulate variation itself, for instance, by changing the param-
ters of a functional expression. This possibility of using different systems of represen-
tation makes feasible the placement of the students’ work in a level of intermediate
representations that brings together their direct experience with phenomena and the
corresponding mathematical model (that is, with the analytic expression of a function).

The above examples illustrate how the use of certain interactive computer envi-
ronments can transform profoundly the way in which mathematics is understood
and learned in secondary school (connecting mathematics with the physical world;
using different representational systems) as well as the curricular contents character-
istic of this school level (the mathematics of change and modeling). In the previous
section in this chapter, it was shown how the use of dynamic geometry tools (such
as Cabri-Geomètre) can also transform mathematical practices, changing static work
into exploratory and experimental tasks. Chapter 13 discusses in detail how dynamic
geometry has a cognitive level impact on the way in which students construct geo-
metrical notions.

Regarding form and content, secondary algebra can also undergo great transfor-
mations depending on the electronic tools (computers or graphic calculators) used to
elaborate: a functional approach, an approach using equations and problem solving,
or an approach involving generalization modeling (Bednarz, Kieran & Lee, 1996).

There are mathematical topics that until now remained underrepresented in the
secondary curriculum. This is the case of recursivity, the presentation and handling
of information (probability and statistics), and modeling. The arrival of ICT makes
feasible the instruction of these topics in mathematics at this school level. Environ-
ments such as LOGO and spreadsheets allow, for example, the development of the
recursive function notion. The handling of information can also be accelerated and
transformed with the use of databases and diverse representational systems. Elec-
tronic simulations of chance phenomena can transform work in the classroom into
experimentation, recording, prediction, and analysis tasks that serve as a foundation
for the construction of probability notions.

Both exploratory and expressive modeling\(^3\) enable secondary students to work with
open situations and to pose problems. Modeling, as working with “artificial worlds”\(^4\),
according to Ogborn (1994), permits mathematics to be taught within the context of

\(^3\)Mellar and colleagues (from the London Mental Models Group) consider modeling as a means and a
tool with which the students can create their own world and use it to express their own representations
and to explore others’ ideas. When students create a model to represent, to express a situation or a
phenomenon behavior the model in question is called expressive. When students are given a ready-
made model to carry out explorations about aspects or characteristics of phenomena that have been
modeled, the model in question is called exploratory. They refer to this idea of modeling as learning with
artificial worlds. In this perspective, the focus is the nature of the ideas about the world that the human
mind constructs (Mellar et al., 1994, pp. 2 and 3).

\(^4\)That is, working with simplified, idealized models of aspects of the real world, which we know
everything about there components, simply because we decided what they were to be (Ogborn, 1994).
science, because in these worlds, physical, biological, environmental, and geographical phenomena can be recreated. Results coming from the Anglo Mexican study “School-based mathematical practices” suggest that modeling from the perspective of the sciences is a rich, meaningful source for formal mathematical models and that the different representational systems and forms used for that aim to conform structural resources of mathematical knowledge (according to Lave’s [1988] theory). Stella is another modeling tool, generally used in schools with computer resources, that has proved to be an important contribution in the development of student skills that go beyond the school objectives. Nevertheless, new educational reform trends advocate the teaching of mathematics in context and particularly its explicit relation to other areas of knowledge. This opens a genuine possibility for incorporating modeling as a curricular piece in school mathematics in the near future.

Almost every study currently using ICT, refers to the role that these tools play in cooperative and collaborative learning. When an important part of the feedback comes from the working tool and from the interaction among partners (both students and teachers) the mathematics class organization is affected. Generally, the existence of these tools along with a collaborative learning model helps to overcome the students’ passivity and to stop the information flow from being unidirectional (from the teacher to the student group). Chapter 13 discusses the Mexican project “Incorporation of new technologies into the school culture,” in which the incorporation of this tool model combination led to a substantial change in the teacher’s role, as well as in the organization of the class sessions and, generally speaking, in the school community.

Apart from the computer-based learning environments mentioned in this section, it also can be said that the introduction of information technology to the mathematics classroom has brought with it the possibility of democratizing mathematical knowledge (for example, movement mathematics), which previously was only accessible to a minority of students wanting to follow a scientific university degree. On the other hand, such technological surroundings (which include graphic calculators and computing algebra systems) also allow the “average” student access to pieces of mathematical knowledge more recently developed, as is the case for discrete mathematics, graph theory, probability, statistics, and its applications in social and natural sciences. With the latter, it would seem possible to close the breach between school mathematics and the more current applications of mathematics, to more widely define the profile of the mathematically educated citizen. In the field of educational research, certain questions, such as the following, would still require research:

- What is the scope of the knowledge gained in a technological environment in relation to the daily lives of individuals and in relation to their understanding of the physical and social world surrounding them? Is this knowledge inward when individuals work within a technological environment, and is this knowledge transferred beyond the environment itself and beyond the school surroundings?
- What is the influence of the cultural component in the assimilation of technological environments within secondary schools? Is this assimilation process influenced by the way scientific knowledge is socially valued? Or, on the contrary, does the process occur because the presence of technology in schools is valued?
- What is the nature of the knowledge generated from working in a technological environment and from a related pedagogical model (interactive collaborative, interactive cooperative)?
- What are the new abilities and mathematical skills developed from learning in a technological environment and how can these be made more profound?
- What factors contribute to the shift from traditional contents and pedagogic models to new approaches and contents, which are accessible through the use of technological environments, and to new collaborative learning models, based on the
theory of social construction of knowledge? Is it feasible to isolate key factors of this shift?

- Can we delineate possible paths of institutionalization of knowledge in the classroom? This is particularly important when the pedagogic model departs from pupils’ work within technological environments, in which mathematical notions are often approached through informal or simplified versions.

### INTO THE SECONDARY SCHOOL MATHEMATICS OF THE NEW MILLENNIUM

The features of the modern society related to the ICT definitely have significant implications for the content and outlook of the school mathematics. In particular, the intensive use of the ICT in the workplace is progressively requiring new mathematical abilities to be developed at secondary school. Nevertheless, thorough discussion on what the mathematical background of an educated student completing secondary school should be would require systematic enquiry about mathematical levels, skills, and knowledge currently sought by the commercial, financial, and industrial sectors of countries with different development levels. This raises the need to define the secondary school mathematics of the new millennium in accordance with the standards of developed societies as well as of developing countries and in accordance with the role of secondary school in such societies as well. Business leaders point out a series of attitudes and skills the school should promote in students so that they become aware of the power and relevance of mathematics in modeling situations of the world outside and of the importance of using ICT in modeling work at school. Clayton (1999) identified the required skills of industry’s future employees as follows:

- A sense of symbols to build mathematics models and to manipulate them envisioning further understanding of what is being modeled
- Experience in solving problems
- Awareness of how mathematics and ICT may be synergetically used
- Awareness of the importance of validation and verification of modeling applications
- Handling uncertainty

So according to Clayton, secondary school and college should contribute to the formation of human resources of societies of the future, including in its syllabus:

- Principles and applications of mathematical modeling
- Mathematical techniques and analysis methods taught in contexts that help understand how they may be used
- The use of numeric methods
- The effects of uncertainty and how to measure these
- The use of ICT for exploration, transformation of data, and re-creation of mathematical concepts with the aid of visualization
- Problem solving across the different subject areas
- Verification, validation, and estimation principles (Clayton, 1999)

Reviews on mathematical skills similar to the previous but rather related to commercial, financial, and scientific activities would give a better account of the necessary modifications that must be done to teaching contents and training methods. Elucidating the role junior secondary school plays in different societies (developed and developing ones) will play a decisive role in establishing educational policies that
may point to or stress such contents at this school level. In the majority of countries, junior secondary school represents the end of basic education as well as students’ preparation for pre-university and university studies. In other countries, secondary school has recently been incorporated within the basic compulsory education and practically represents the final schooling stage for an important sector of the population. As discussed in chapter 13, an appropriate use of the ICT can help these students access mathematical contents and ideas such as those Clayton described, as well as contents and abilities that may transform them into mathematically educated citizens.

The direct relationship between the use of ICT in schools and the possibility of making a student a scientifically educated citizen who is efficient in the workplace places the institutional responsibility of the way such tools should be used in the middle of a current debate. This debate occurs between two extreme visions: the technocratic one, which links the use of ICT with efficiency, speed, and the service to the interests of industry and the business world, and the sociocultural vision, which emphasizes issues of equity and access.

When Clayton suggested that school mathematics be orientated toward the development of skills and knowledge that could be useful in the future industrial and commercial spheres, it may appear as if he was adopting a technocratic vision. When placing the re-creation of mathematical concepts through visualization and exploration, transformation, and interpretation of data as part of future syllabi, however, Clayton’s position seems to be closer to the other extreme. That is, his position is closer to that of the sociocultural vision, given that these skills and experiences promote the training of well-informed individuals, with access to the world of statistics (coming from the world of finances and politics) and with rich mathematical experience.

SOME UNSOLVED QUESTIONS RELATED TO DEMOCRATIC ACCESS TO MATHEMATICAL KNOWLEDGE

In this chapter, perspectives on students’ possibilities in accessing significant mathematical ideas in the junior secondary school offer a picture of the needs to be fulfilled in the near future as far as educational development is concerned. In particular, when analyzing the role of secondary school in the societies of the new millennium, it is clear that new contents and mathematical abilities are to be included in the curriculum and syllabuses of this school level. It is also clear that the incorporation of the ICT within the teaching and learning of mathematics will facilitate this task. Nevertheless, there are many questions that need to be answered within the mathematics education research domain, to provide the necessary academic foundations to support the educational innovations to be implemented. Some of these questions are related to new contents and abilities that have up until now remained underrepresented in the secondary school. In this respect, for instance, it would be necessary to respond to the following questions:

- What are the intrinsic learning difficulties of the new school mathematics themes that may hinder 11 to 16 year olds’ democratic access to them?
- How can innovative teaching and learning approaches (such as computer-based and collaborative learning approaches) cope with these types of obstacles?
- What are the transitional processes involved in adolescent students’ constructing notions and developing abilities such as: handling data; analyzing mathematical models; validating models, conjecturing, and predicting; and using recursive processes?
In relation to the use of ICT in the teaching of mathematics, some questions to be answered include the following:

- What is the cognitive impact of using computer environments and graphing calculators on students development of exploration, conjecturing, and mathematical estimation?
- How feasible is it to take a step toward formalization or institutionalization of the knowledge produced when using computer-based approaches?
- Can research progress at the same pace as ICT development does? How will Internet access to teaching materials influence school mathematics?
- What new classroom models and school organization styles will exist as the use of ICT increases?

In relation to the mathematics education research work that has been carried out over the last two decades, it is reasonable to expect answers to questions such as

- How do abstraction and generalization processes manifest themselves in the scenario of new mathematical contents, abilities, and tools?
- How will researchers assimilate new paradigms in the new millennium, notably when dealing with adolescents' learning processes on one hand and when considering requirements and influences from contexts other than school on the other?

In addition to the impact that such questions may have on 21st century research agendas, it also is important to recognize that the step toward research perspectives on democratic access to powerful mathematical ideas should contemplate both applied research and the arduous work of basic research. Indeed, faced with the emergence of new and diverse research paradigms in the field of mathematics education, which is marking the onset of the new millennium, we must keep in mind the nature of the mathematical knowledge that might be generated through the new curricular contents to be taught, the new abilities to be developed, and the new learning tools to be used.

REFERENCES


A chapter written to comment on and to summarize the current state of mathematics education at tertiary level is almost bound to be fragmented and incomplete. To start with, all major mathematics education research domains do have significance, be it varying, to advanced mathematical thinking (AMT). Then we are faced not only with a bewildering range of mathematical theory but also how the learner copes with, thinks with, and processes this mass of information. Finally, we have to take account of the rather special political and social situation that mathematics educators must address in this sector of education.

A degree of specialization clearly is required. Recently there has been an influx of papers on attitudes and beliefs of both university students and lecturers. Hence, it seems a timely occasion to devote a short section on presenting students’ problems in their own words, then to discuss the transition from secondary to tertiary level mathematics and the teaching practices currently found at university, and finally to address the uncomfortable relationship that often exists between the lecturer of mathematics and the mathematics educator. With this we start our chapter. The second section undertakes a commentary (rather than a review) about the status of some major research domains in mathematics education in reference to AMT. This, out of necessity, has a highly selective character.

The third and final section is somewhat more original and specialized than is customary in handbooks but is written (partially) as a complement to the first two sections. Its purpose is to introduce, illustrate, and discuss a theme that we name reflection on mathematical structure (RMS). We feel that this theme may constitute a new theoretical perspective that will supplement more established mathematics education theories, which (we claim) are more dependent on assumptions involving continuous conceptualization. Also, the RMS perspective might be palatable to mathematicians’
ways of thinking, so it may form a good medium between the mathematics educator and the lecturer.

A BROAD SCENARIO

One way to gauge the difficulties and affects that students have when dealing with mathematical courses at college or university is to ask them directly about their attitudes and beliefs concerning mathematics and how it is taught to them. There recently has been a great deal of research in this area, but, perhaps because the structure of tertiary level education varies significantly between countries, relatively few are published in international journals. Nonetheless, common themes seem to occur, such as the following:

- Understanding is lost because of the pace of presentation of theory and the style of teaching.
- Success in mathematics depends on memory and being able to perform routine manipulations.
- Mathematics is too abstract and of obscure use.
- Mathematics is hard work, or requires innate ability, or both.
- There are doubts about the relevance of mathematics for career purposes.
- Images of mathematics as being asocial are pervasive.
- There is a lack of effective support frameworks.
- Students feel anxiety, only do the exercises that are assigned to them, show a “unique correct answer” mentality, and are not persistent when faced with difficulties.

All of these trends are noted in at least one of the following: Yosof and Tall (1999), Ricks-Leitze (1996), and Linn and Kessel (1996). Of course, these are only trends; many students find their studies in mathematics stimulating and worthwhile.

The chapters by Ricks-Leitze and by Linn and Kessel are very much in the tradition of the reform programs in the United States occurring over the last 20 years, inspired by reports such as *Towards a Lean and Lively Calculus* (Douglas, 1987). With the U.S. system students enter university with a relatively low level of maturity in mathematics, there is a high flexibility in changing directions in study, and many students are required to take many mathematical courses. Taken together these cause a problem, because there is a temptation for mathematics departments to regard particular courses (quite often the first course in calculus) as “weeding” courses. The impression to the student is that these courses are deliberately designed to intimidate, and passing is a matter of the “survival of the fittest.” Another analogy used is that of a filter; the aim of the reform is to transform the filter more into a “pump,” see Steen (1988).

This problem really is becoming global. As Hillel (1996) noted,

> The problem of the mathematical preparation of incoming students, their different social-cultural background, age, and expectations is evidently a worldwide phenomenon. The traditional image of a mathematics student as well-prepared, selected, and highly motivated simply doesn’t fit present-day realities. Consequently, mathematics departments find themselves with a new set of challenges.

This call for more democratic rather than elite-based teaching raises many issues. Major ones include the following:

1. Students should learn more than is presently customary the “process of mathematical thinking” rather than the “product of mathematical thought” (terms borrowed from Skemp, 1971).
2. What can be done in teaching practices or support systems to help students manage the transition between school mathematics and tertiary level mathematics?

3. What distinguishes “advanced mathematics”? Why does it cause problems for students and how can we help (ultimately) to improve, or to inform, teaching practices or support systems?

4. Are lecturers willing or able to effect the necessary changes in their teaching practices, and are they prepared to take into account the advice of mathematics educators?

The rest of this section briefly looks at these four selected issues in turn.

Process of Mathematical Thinking versus Product of Mathematical Thought

Traditional methods of teaching tertiary mathematics stress content of mathematical theory rather than the motivations and thoughts that underlie this content. At least this is the strong impression given by the students interviewed in the studies mentioned in the previous subsection who did not understand or see the relevance of the mathematics they were learning. This lack of understanding clearly manifests itself when students are asked “nonstandard” but far from intricate tasks; quite often success rates are alarmingly low (see, for example, Eisenberg, 1992; Selden, Mason, & Selden, 1989; Stein, 1986). This phenomenon in turn militates against the students’ confidence. In particular, Stein’s paper includes an example by which it is argued that traditional values should be reexamined: Employers are more interested in graduates who can think independently rather than those who only possess mathematical facts. Serious considerations should be given to find ways to enhance the process of mathematical thinking, even if some sacrifice in content may be needed to achieve this. This paper was published in 1986; in the next subsection we shall describe some measures that have since been taken in this direction.

New Directions in Teaching Practices and Support Systems

Hillel (1996) summarized a study (conducted by P. Kahn) about teaching policy of 50 mathematics departments in the United Kingdom by listing the following trends: “reduction of content; introduction of new courses (e.g., geometry, problem solving, modeling), and changes in perspectives to existing courses (e.g., problem-based lectures, use of computers, project work, linkage to other disciplines).” We shall expand on some of these (and other allied) topics.

The character of the new courses being introduced are ones that allow a stress on process of mathematical thinking rather than on theoretical content. Quite often they are considered as “transition courses,” that is, their introduction is meant to make smoother the transition from school education to university education. Some of the issues involved in this transition will be considered in the next subsection.

Transition courses vary from institution to institution, but many are based on developing skills in problem solving or proof making. Many also are designed to meet some “remedial” material on basic conceptions, especially on variable and function. The ultimate aim of these courses is to introduce students to powerful patterns of mathematical thinking. There are some dangers, however; Hillel (1999) noted that transition courses may become standardized, counter to their spirit of attempting to make mathematics seem flexible and creative. Gray, Pinto, Pitta, and Tall (1999) pointed out that it might be necessary to teach weak students ways of thinking in a procedural way, thus burdening these students with more procedures than before.
the other hand, the study of Yosof and Tall (1999), shows how effective a transition course can be in positively changing students' attitudes. The same study, though, indicates how quickly these gains are canceled out when traditional methods of teaching are resumed.

This suggests that changes ought to be made throughout the syllabus. The U.S. reform movement particularly espouses projects; help (called scaffolding) is provided so students can adapt to this kind of work; project work is usually highly collaborative; projects provide opportunities to write about mathematics; see, for example, the paper by Linn and Kessel (1996).

The style of delivery of lectures and tutorials is a major factor in our subject, but we touch on this topic in another subsection. Information technology now influences ways of learning and teaching at all stages of mathematics education, as discussed in other chapters of this handbook. Presentation of some undergraduate textbooks are becoming more sensitive in meeting some of the cognitive needs of their readership. For example, the book by Jones, Morris, and Pearson (1991) provides within its proofs some of the thoughts underlying the proof, hence approaching the idea of structured proof as conceived by Leron (1983). Raman (1998), however, suggested that textbooks that are differentiated by level of abstraction seem to hold different "messages" about the character of mathematics that are difficult for students to decipher.

Another idea toward democratizing the process of the learning of mathematics is to devote some lecturing time for free debate about mathematical issues between students. One protocol for such debates is summarized in a chapter by Alibert and Thomas (1991). Such debates are particularly useful for students to realize the importance of hypothesis, the flexibility and uncertainty involved in finding arguments, and the possible value of even an aborted attempt at a problem. Debates are time-consuming, and this might account for their not being widely used.

What Is Advanced Mathematics?

Educational research on tertiary level mathematics may be regarded simply as a specialization of the general mathematical education research agenda. The specialization may be described in terms of the (special) concerns of students in tertiary education. The next section of this chapter, Mathematics Education Research at the Tertiary Level, categorizes such research and some directions it may take in the future.

As we have seen, students' comments clearly express a perceived difference in learning mathematics at school and at tertiary level education. Some of this difference may be explained by teaching practices at college and university that require much more independent learning than the students are accustomed to. What the students perceive seems to go much further than this: They feel that the very character of the material taught has radically changed, and they don't understand fully how, why, and how to cope with this change. Because of this, it is an important issue for AMT researchers to try to delineate advanced mathematics.

A clear picture does not emerge, however. It seems that many closely related aspects are involved, most of which may be claimed in varying degrees to be represented within school syllabi in any case. For example: (a) students studying tertiary level mathematics often complain that mathematics is "too abstract," as referred to by Yosof and Tall (1999). Of course, children are used to abstract objects from their primary school years' experience with arithmetic. What the students often lack at the advanced level is the sense of why the abstraction was made (see, for example, Dorier, 1995). (b) Somewhat related to abstraction is the issue of rigor, which can be regarded as a terminal point in a process of making arguments more precise within the medium of mathematical terminology. Recently, quite a controversy (in the United Kingdom at
least) has arisen about whether to reintroduce more rigorous proofs into school syllabi (see, for example, the collection of opinions in the discussion paper “Teaching Proof,” 1996). (c) A distinction of advanced mathematics is that conceptualization tends to take place after the presentation of a formal definition. This process is nicely described by Gray et al. (1999).

Many other components may be thought of to fill in the mosaic of advanced mathematics. These include an increase of the number of ideas that must be put together for any particular task; more stress placed on method rather than result (without this stress it is difficult for the student to appreciate the reason to study real analysis); conceptual and calculational orientations becoming blurred; encapsulation of processes as objects; and new habits of thinking. In the final section of this chapter, we discuss reflection on mathematical structure, which we claim acts as a fairly good “umbrella” for talking about some of these aspects.

Delivery of Lectures and Cooperation Between Lecturers of Mathematics and Mathematics Education Researchers

For university students studying mathematics, one of the most uniformly held negative attitudes is directed at the delivery of lectures (see Yosof & Tall, 1999). Some factors contributing to dull or inefficient teaching are obvious; lecturers rarely are instructed in teaching, and, for many, doing research takes precedence over lecturing. Raman (1998) made another worthwhile note: When lecturers teach first-year courses, which need special sensitivity, they are often teaching mathematics at a level of sophistication far below their full competence (as far as mathematical material goes). But perhaps also we should turn to lecturers’ beliefs about the nature of mathematics, for these surely must also influence their teaching practices.

There is a popular image of a mathematician’s beliefs, and perhaps this is well represented by the following quote from Davis and Hersh (1981): “the typical mathematician is both a Platonist and a formalist—a secret Platonist with a formalist mask that he puts on when the occasion calls for it.” This certainly would explain the tendency to lecture without respecting cognitive considerations. Some studies, however, such as the ones by Mura (1993) and Burton (1999a), surveying lecturers’ beliefs of mathematics and research practices indicate a far more heterogeneous picture. Burton’s study, together with another paper (Burton, 1999b) drawn from the same survey but focusing especially on intuition, shows that most mathematicians, when engaged with research, work in a highly collaborative way, they often see connections set within some global image of mathematics as an important part of ‘knowing’ mathematics, and they also see intuition and insight as being very important to their work. On the other hand, Burton also noted that the mathematicians were on the main not interested in analyzing their own intuitions and in communicating their enhanced understandings in their teaching. Why is this?

Burton (1999a) suggested that mathematicians believe in a global image of mathematics that “students must learn before they can begin to think of mathematizing.” This seems to reflect an attitude on the part of the lecturer that undergraduate learning should be mostly about acquiring theoretical knowledge, because sophisticated trains of thinking depend on this knowledge. Clearly this view is highly undemocratic in spirit. Traditional delivery of lecturers based on this perspective is only geared to the highly gifted and motivated students who are able to extract meanings, connections, and understanding more or less independently.

However, the recent trend of democratizing teaching practices in universities may reveal a difficulty for lecturers that they were more able to sidestep before. This difficulty is eloquently put by the following quote from Freudenthal (1983), p. 469:
I have observed, not only with other people but also with myself..., that sources of insight can be clogged by automatisms. One finally masters an activity so perfectly that the question of how and why [students don’t understand them] is not asked anymore, cannot be asked anymore and is not even understood anymore as a meaningful and relevant question.

This point may well epitomize the need for lecturers, even as experts in their academic fields, to have advice from educators about their teaching methods. Although this need now is acknowledged by some mathematicians (for example, Thurston, 1990), overall there seems to be a lingering mistrust about the worth of the educators’ work from the side of the lecturers. (In this discussion, it might appear that we consider educators and lecturers as nonintersecting populations; however, this is just a simplification for ease of discussion). Typical criticisms and attitudes shown by lecturers toward educational research addressing tertiary level mathematics include the following (many of which are expressed by S. A. Amitsur in an interview given to Sfard (1998b)):

- Researchers should be careful to respect expert knowledge.
- It is not the role of educators to decide what should be taught and how; their role rather should be as a kind of consultancy.
- Papers indulge too much in psychological theory, giving little practical suggestions in presenting particular themes or concepts. On the other side, much research fails to take account of the totality of the whole issue at hand.
- Methodology and scientific status are questioned.
- Too much attention is paid to “non-strictly mathematical” activities.

Sfard (1998a) explained the friction between the two communities in another way, in terms of mathematicians tending to have views consonant to Platonism, whereas mathematics educators usually keep some form of constructivism as their guiding philosophy.

Holding in mind the studies of Mura and Burton, we feel that if we replace Platonism with images of mathematical structure, we obtain a more accurate portrait of the mathematicians’ outlook. Also, we contend that most researchers in AMT adopt moderate versions of constructivism; their principle concern is that traditional presentations of mathematics do not connect with the students’ need to develop their own intuitions and ways of thinking. Put in this way, the two sets of views is far from being irreconcilable; at the end of the last section we say more on this issue.

It is not surprising that some lecturers feel defensive and perhaps even a bit threatened about the influence of educational theory. As described by Artigue (1998), this influence moves lecturers away from their areas of expertise, and furthermore the mass of mathematics education literature may seem overwhelming and not directly addressing their teaching problems. It is difficult for mathematicians to understand some aspects and aims of mathematics education, especially those that deal with cognitive theories without making explicit teaching recommendations. It is largely the responsibility of the community of AMT researchers to find a better forum to explain to lecturers their aims and to make more accessible the main issues, results, and controversies of their discipline.

A concrete way that a mathematics educator can advise lecturers, or, perhaps to put it better, induce lecturers to reflect on certain aspects of their teaching, is for the educator to observe some of the lecturers’ classes. Nardi (1999) noted that this has proved a useful exercise for tutors at Oxford. In another study, Morgan observed some lectures, but it should be pointed out here that the lecturer involved is himself deeply involved in mathematics education research (Barnard & Morgan, 1996). The present skepticism held by many mathematicians for the mathematics education researcher means that similar cooperations may remain rare.
MATHEMATICS EDUCATION RESEARCH
AT THE TERTIARY LEVEL

As mentioned in the previous section, no completely satisfactory description of advanced mathematics is available, so the identity of the research domain determined by the community of researchers in AMT must depend largely on the issues that it chooses to tackle. (In the previous section, we noted some issues dealing with attitude as well as social and environmental aspects relating to tertiary institutions; here we are concerned with factors relating more directly with mathematical cognition.)

Up to now the issues of AMT have mostly found expression within the contexts of concept acquisition and proof making (and reading). Even though these two mental activities are both of tremendous scope, together they represent a rather traditional and artificial divide. In this chapter we present two ways to remove, or at least lessen, this divide. In parts of this section we consider some domains of interest of mathematics education research that currently does not attract much attention from the AMT community but we feel has a potential to play a role in a broader understanding of AMT. In the third section, we consider a perspective we call reflection on mathematical structure, which entails a model of mathematical meaning that straddles concept acquisition and proof making (and indeed more).

Before starting, we wish to make several notes. The framework we use imparts a bias to favor research that identifies itself strongly with the broad educational traditions that we have chosen to include, and hence our discussion of AMT within this section will not be fully representative. There is such a wide scope in content and philosophy to be found in AMT research literature that some degree of specialty can hardly be avoided. The framework we take is unusually broad, and the reader can appreciate that the bibliography had to be selected frugally to limit it to a reasonable size. The framework works only as a loose organizational convenience and it is not meant to be taken theoretically. That is, we do not wish to try to relate the theories that we have introduced together to draw a unified picture of a certain aspect of mathematical education. Nonetheless, when we talk about the potential of AMT in each theory discussed (our main purpose), we do suggest some links between them within this limited context.

As in the previous paragraph, we use the word “theory” to refer to any area of interest of mathematics education research, even though this clearly is not strictly appropriate in some cases. Sometimes, though, it is convenient to refer to tools rather than theories in cases where the topic considered seems to largely deal with a facilitating focus and vocabulary within mathematics education. (Examples are concept images and epistemological obstacles).

Research Directions in Concept Acquisition

Although concept acquisition is relevant to all stages of mathematics education, it is particularly pertinent in advanced mathematics, where a significant amount of processing in terms of meaning is needed to interpret formal presentation. We start with the tools of concept image and epistemological obstacle.

The tool of concept image allows two factors: how interpretation of a concept may be accommodated in the mind and how the practitioner may fail to understand or may misunderstanding some aspect of the concept. The first factor allows individuality in how the concept is recovered mentally; the second explains phenomena in an individual’s behavior that are contradictory to the concept definition. Multiple concept images may be held by the same person for the same concept, and these also may have potential conflict factors between them. (There has been some debate about whether conflict-free teaching should be espoused. However, there is now quite a strong consensus
that, at least at tertiary education, cognitive conflicts in mathematics are inevitable, and conflict resolution is a major activity of the student.

In Moore (1994), a schema is made between concept image, concept definition, and concept usage. In this schema mathematical language plays an important part. Some of the mathematical language needed for concept usage is extracted from the statement of the definition. Hence, if students have strong concept images but without linking these well in their minds with the definition, they may be unable to write down a proof (for example) involving the concept even if they had succeeded in discovering an intuitive strategy to find it. Moore also noted how different equivalent forms of a definition in some cases may encourage concept imaging, whereas others may suggest some specific kinds of concept usage. Although the article is in the proof agenda, Moore acknowledges that the notion of concept usage is a rather limited (if important) tool in doing proof. The way in which the article is especially valuable is in showing that at AMT-level concept acquisition usually has a utilitarian aspect.

Informally, epistemological obstacles may be described in terms of “old” and “trusted” knowledge suddenly becoming inadequate in the face of new problems or as discontinuities between common thinking and scientific thinking. Already we have two descriptions that are not completely complementary, and indeed the notion of epistemological obstacle largely fell from grace with the community of researchers because it could not agree on a stricter definition (see Sierpinska, 1994, pp. 123–128).

Some general observations may be made, however. On the more mundane level, obstacles are situations that students may or may not enter (and even if a student does enter, the obstacle is not necessarily realized until an explicit conflict is encountered). In this case, there is some assumption that obstacles can be anticipated without necessarily observing them to evolve in fieldwork. This enables research (for instance) to develop material especially directed to particular potential conflict factors that might otherwise be overlooked. On the grander scale, when the tradition was at its peak, researchers held quite an expansive view of the role of epistemological obstacles. This view greatly extends the idea of cognitive conflict already mentioned for concept images; indeed conflicts were thought of as being more or less integral in the evolution of the concepts of number, function, infinity, and limits. Some researchers went as far as to suggest that the process of resolving conflicts is the only way to gain insight. Until a workable framework for epistemological obstacles is agreed on, these claims will remain plausible theory but lacking means of analyzing it.

The tool of epistemological obstacle, mentioned above, was developed within the milieu of epistemology. Here we confine ourselves to an epistemology of meaning, as it is identified by Sierpinska and Lerman (1996). Put in a simplistic way, epistemology in mathematics education involves identifying “targets” in terms of mathematical understandings endorsed by the professional community and investigates how and why students diverge from these targets. Typically the targets form a wide net around a particular concept, where not only direct aspects of the concept are addressed but related concepts are introduced so these can be contrasted with the original. Sierpinska (1990) called these targets “acts of understanding.” Epistemology has a predictive aspect, as shown by its strong interest in historical struggles in developing mathematics as being somewhat indicative of the problems of the modern student studying mathematics.

Epistemology clearly fell foul of the more assertive constructivism, and its influence is presently much reduced even in AMT, the area that might be expected to be affected the least. Some geographic regions seem to have retained some interest in the tradition, especially France and some South American countries. Another factor that no doubt contributed to epistemology falling into disfavor was its intellectual prerequisites, which raise some legitimate criticisms. For example, the collection of acts of understanding associated with a concept needs a painstaking analysis of the
mathematical interface, which could be an exercise needing the collating of ideas gathered at different times over several years of education. Not even professional mathematicians, let alone students, usually take such explicit attention to this kind of collation. Second, evaluating historical developments is, if taken seriously, a rigorous discipline. Mathematicians are likely to find epistemology too “fussy.” The weakened status of epistemology, however, was accompanied with a distinct feeling that AMT had lost an overall direction. A recovery in epistemology in some form would be welcome, and recently there are signs that such a movement is gaining momentum (see for example Sierpinska, Trgalova, Hillel, & Dreyfus, 1999).

The resultant vacuum in research direction was partially filled by Dubinsky’s (and his group’s) ideas and methods. Dubinsky’s earlier work in mathematics education is summarized in his chapter in Advanced Mathematical Thinking (1991), and it is centered on the notion of reflective abstraction; more recently Dubinsky was largely responsible for introducing APOS theory (see for example Cottrill et al., 1996). The kernel of both ideas is underlined by the importance in mathematics of processes “becoming” objects. In this respect, there are other related theories. To mention only some of the more well known, we have Skemp’s relational and instrumental understanding (Skemp, 1978); Douady’s dialectique outil-objet (Douady, 1986); Sfard’s dual nature of mathematical conceptions (Sfard, 1991); and Gray and Tall’s procept (Gray & Tall, 1994). Despite substantial differences in theory, what most evidently distinguishes Dubinsky’s work from all the others is how it incorporates methodology, starting with the conception of a research program through to pedagogical implications and implementation. His theory has been applied to an impressive range of mathematical fields, including induction, predicate calculus, function, limit, and group theory. It now has a sizeable and well-organized group of adherents in research circles and presently is arguably the most influential group in AMT research. It sits fairly easily with constructivism: “One can think of reflective abstraction as trying to tell us what needs to happen whereas the other notions attempt to explain why it does not” (Dubinsky, 1991). Dubinsky argued its globality in the same paper: “The goal of our study of reflective abstraction is a general framework that can be used, in principle, to describe any mathematical concept together with its acquisition.” The methodology is flexible, and Dubinsky often stresses his “democratic” view that his theories constitute only one out of many other possible explanations for phenomena found in students’ work (Cottrill et al., 1996).

The assumption that all conceptual acquisition can be somehow acquired within the schema formed around an objectification seems doubtful, though. Indeed, what is noticeable in the APOS papers are the simple forms of the schemata. The schemata clearly show the processes, objects, and their relations in the given context but little else. The interest of the paper depends on the richness of this kind of structure. The perspective of APOS is an important one and should have a continued future, but it has its limitations. In particular, Tall (1999) criticized APOS as suggesting too much of a linear progression from Act, then Process, then Object to Schema. He noted that a major portion of the brain is devoted to vision and perception of objects. He therefore suggested that a theory based only on processes becoming encapsulated as objects cannot tell the whole story. For example, cognitive development can involve refining a mental image that is conceived from the start as an object and modified into a formal structure.

Furthermore, Tall, in a paper coauthored with E. Gray, brings in a wider context for the role of process than that found in APOS by introducing the notion of procept (an amalgamation of the words process and concept) (Gray & Tall, 1994). A recent expository paper dealing with procepts is Tall et al. (1998). A procept involves a symbol that may be regarded as being a pivot between a process to compute or manipulate and a concept that may be thought of as a manipulable entity. Even though in the theory the concept is sometimes described as the output of the process, the relationship between the concept and process is far more subtle than simply a matter of encapsulation. This
is not only because of the influence of the symbol, but also because of the multiple procedures by which the same process may be achieved, each one contributing at different times to the total cognitive understanding of the output. Hence, procept does not emphasize objectification, and concept acquisition is portrayed as a discontinuous progression. (A similar principle of breaks into the continuity of conceptually based thought from a more general viewpoint will be important in the exposition of our theory of reflection on mathematical structure in the final section of this chapter). In terms of students’ problems in doing mathematics, the theory largely dwells on the so-called proceptual divide; students who can think in terms of process and concept can maneuver their thoughts more efficiently and compactly (as well as flexibly) compared with students depending more on procedures. Although the role of procept becomes less with formal mathematics, the notion has proven to be a fruitful one in mathematics education research from the most elementary to the AMT level. The theory might profit in the future from a more concrete identity, however, in laying out more explicitly both what kind of mathematics can be treated and specific directions for conducting cognitive research.

We mention here briefly a new theme in mathematics education: neural networks. This field is impinging from two sides; the first from physiology and psychology in studies how the human mind works, the second from the work of artificial intelligence. Recently significant progress has been made in examining brain function, resulting in the publishing of important nontechnical books in the topic, such as the one by Dehaene (1997). We are far from being able to link these results directly with mathematical thinking of any complexity, but some researchers have started to think that it may be worthwhile to experiment with models of doing mathematics based on neural theories, which may have the potential to be applied even to advanced mathematics (for example, the inventors of procept were influenced in this way to some degree). The idea is intriguing, but even if it succeeds, the models will probably fail to be as powerful as is hoped in the same way this is true for APOS.

**Proof, Problem Solving, and Problem Posing**

There are two fields that seem to cover the majority of the concerns of AMT researchers: concept acquisition and proof. The research on proof is the more recent discipline, but the literature now is large and growing rapidly.

Proof, as an educational realm, is somewhat difficult to put a finger on. It involves questions about formality, rigor, and logic and at the same time about persuasion, meaning, generalization, generating ideas to solve a given problem, plus the handling of complicated structures. Working on such a broad front, it is not surprising that the research in this field is diffuse.

There seems to be some controversy about the status proof should have in secondary school mathematics syllabi. Most researchers would claim with Hersh (1993) that proof at this level should only involve convincing and explaining, yet a sizeable minority would agree with Gardiner (1995) that a development of mathematical precision and language leading toward formal proofs and methods is also an essential component of school mathematics. Somewhat surprisingly, this controversy is paralleled by another found in the mathematics research community. The advent of powerful technology has shaken the traditional outlook mathematicians have about what constitutes proof. The increased acceptance of experimental mathematics, a changing perspective about conjectures, new ideas such as zero-knowledge proof, and recent studies indicating that proofs in papers accepted in mathematical journals are not so rigorous as might be expected have all contributed to make some mathematicians to view proof in a somewhat looser context than before. Some current views of mathematicians about proof can be found in Kanamori (1996) and in Hanna and
Jahnke (1996). Although this issue is one mostly touching attitudes of mathematics researchers, it does suggest that the position that students must justify all proofs strictly deductively has been significantly weakened. Davis (1993) gave an example in this direction in which the author argued that visual evidence in cases can provide “something deeper than formulaic–deductive mathematics and hence can contribute to a wider view of mathematics”; with this perspective Davis pursued a notion of “visual theorems.”

If the professional mathematician now is unsure about what constitutes a proof, we can be certain that students will be more so. Unlike secondary school, there is no real question at the tertiary level that proofs should be largely treated formally. Nonetheless, it should be made clear to students that the introduction of abstraction and formality is not an arbitrary imposition, but a necessity to allow more precise argument. To argue, however, you cannot afford to annihilate understanding even when working abstractly; you need an ability to form some kind of comprehension of mathematical structure (our final section expands on this). This message and understanding is difficult to impart, especially in syllabi typically based on mathematical theory building. In practice, university students often see formal proofs as games of unmeaning manipulation of symbolism, and as a result students find difficulties when tackling even simple proofs (see, for example, Moore, 1994). Hanna and Jahnke (1996) exhorted the use of proofs that explain (if they are available) over proofs that simply prove; however, sometimes proofs that explain depend on intuitive representations that are counter to the trend of precision in explanation. We are back to the dilemma found at secondary school level. On the other side, tactics often used in deductive reasoning, such as mathematical induction, argument by contrapositive, argument by contradiction, and argument by counterexample, can seem eccentric and rather insubstantial ways to ascertain mathematical “truths” for many students. Hence, it is often difficult for students to appreciate the enhancement of precision afforded by formal proofs. They generally feel more comfortable when they can judge mathematical arguments by empirical or intuitive evidence rather than strictly logical considerations (see, for example, Finlow-Bates, Lerman, and Morgan (1993). Douady (1986) gave an account of the respective roles of explanation, understanding, argument, and proof in an educational and an historical context).

Students’ problems in particular broad types of approaches to proof, as just mentioned in the previous paragraph, have attracted much attention by the literature. Apart from these types of overarching logical structuring, there is often at AMT level very complicated structures involving mathematical ideas and concepts within a proof. How do students cope with these? It is perhaps useful to discriminate between two types of substructure of the structure of a proof. The argumental block is a section of a proof that accommodates a more or less integral argument used in the proof, whereas line-to-line connection means the cognitive input needed to accept the validity of one line from the previous one.

Typically in complicated proofs, line-to-line connections tend to obscure argumental blocks, and as a result an overall understanding of the proof is lost. One attempt to help students to assimilate the total structure of the argumental blocks is by what are termed structural proofs, (see Leron, 1983, 1985). These basically allow a tight but nonformal description of each argumental block, explaining their form in terms of their cause of introduction. A structural proof is not part of a formal proof but acts as a kind of running commentary of it.

Line-to-line connections depend both on local structure and on direction provided by the argumental blocks. This is a complicated and wide-ranging situation. Perhaps the most realistic educational research tool currently available to cover line-to-line connections is given by the notion of cognitive units of Barnard and Tall (1997). A cognitive unit is a piece of cognitive structure that can be held integrally in the
mind, but its character is both a compression of information and an enabler to make connections.

Evidently, proof can be regarded to be subsumed under problem solving. In practice, however, what is called problem solving is given a somewhat limited range, so that proof making and problem solving have some distinguishing characteristics. Chief among these is that problem solving typically involves a transparent task, whereas for proof even the proposition to be proved may be difficult to understand and hence a clear view of what to do (let alone to achieve it) also may be obscure. Proof usually is the more formal and has the more sophisticated logical structure. For this reason, the research perspective toward problem solving is distinct from that of proof.

With problem solving, we are starting to tackle research areas in which AMT does not dominate. Hence we shall from now on be more concise. A leading figure in research in problem solving is Schoenfeld, and the article mostly cited by AMT researchers in the field is his state-of-the-art exposition, “Learning to Think Mathematically” (Schoenfeld, 1992) in which five broad aspects in problem solving are identified and discussed: the knowledge base, problem-solving strategies, monitoring and control, beliefs and affects, and finally practices. Knowledge base is about how knowledge is accessed from the memory, strategies are closely identified with heuristics as described in Polya’s How to Solve It, monitoring is about “on-line” self-evaluation of your work and practices are usage of standard known methods. This picture seems too clean, however. Solving problems at the advanced level, in common with proof, tends to produce sequences of cognitive clashes that the student must overcome, and the dynamics of thought that this causes do not seem well represented in the present theory.

Although we disagree with the view that exercises at tertiary level mathematics tend to be overtly procedural, they often are set on purpose to give opportunities to practice and explore the mathematical methods currently being taught. This means course exercises, regarded as problem solving, may be somewhat dominated by practice. Researchers in problem solving usually do not want to have the “interference” of a strong influence from the development of a mathematical field, and they are probably justified in not wanting it. Being able to solve problems is such a basic skill that we must be fairly confident students have some proficiency in it before burdening them with too much theoretical knowledge. Thus, we now are seeing in some universities first-year courses where the stress is on general methodology and problem solving rather than the mere building up of theory (see the first section of our chapter).

With the introduction of such university courses, one would expect a concurrent interest within AMT research about the traditions of the researchers studying problem solving. Surprisingly, this seems to be slow in happening. Up to now, most AMT literature referring to problem solving is concerned with issues revolving about the courses themselves (syllabi, affective factors, etcetera) rather than exploiting or developing ideas from existing problem-solving literature from a more theoretical perspective. Moves in this direction surely will take place, however, and some work has already been done (e.g., Hegedus, 1998).

Proof and problem solving share a certain artificiality; they represent activities for which some of the structure is given, which would not be expected to exist in a purely creative picture of mathematics making. To address this concern, some researchers have turned their attention to problem posing. Problem posing has potential to be highly significant to AMT research, but as yet the approaches taken in this area are really only satisfactory for lower stages of mathematical education. The reason for this is that posing tasks conducted up to now have been done largely independent of development of mathematical ideas. In the previous paragraph we stated that problem solving without reference to theory was defensible at AMT level; in the case of posing, we believe it is not. Posing that is not placed within some sort of mathematical evolutionary context lacks motivation and direction. These features are noticeable in
most extant fieldwork in the area; in particular, usually restricted physically based contexts are used, which in practice are never transgressed in the posing activity, and the posing often leads to problems that are not in a suitable form for mathematical treatment (see Silver, Mamona-Downs, Leung, & Kenney, 1996).

Small-scale project work has for long been a feature of tertiary level mathematics educational, although rather underemployed in many institutions. These invariably involve a certain degree of posing while maintaining a tight mathematical context. The level of imagination and sophistication needed in posing may be regulated by the openness of the task given. Students’ results from project work (at AMT level) have never been analyzed closely in this respect; perhaps some of the existing frameworks for posing at earlier stages of schooling may be adapted for this purpose.

**Representations, Visualization, Analogies, and Metaphor**

A theory in mathematics education, which has shown some strength over the last 15 years, is that of representation. Two fairly recent collections of papers in this area are Goldin and Janvier (1998), and Steffe, Nesher, Cobb, Goldin, and Greer (1996). As the titles of these works suggest, representation theory has been developed so that it encompasses a more or less general perspective of learning, teaching, and doing mathematics. Indeed, according to the introduction of Goldin and Janvier (1998), the various interpretations of representation include associations with physical situations (p. 1), with “linguistic embodiment” (p. 3), and with “internal, individual cognitive configurations” (p. 3). The last association in particular is very encompassing, but at the same time endows the whole theory with a somewhat indistinct identity.

For our purposes, we shall choose to severely restrict the meaning of representation by considering only “mathematical constructs that may represent aspects of other mathematical constructs,” another interpretation given in the introduction of Goldin and Janvier (1998, p. 3). We believe that the other interpretations are adequately covered by the topics of mental images, visualization, analogies, and metaphor. (The term mental image is used rather carelessly; in spirit, however, it is similar to concept image because it has to do with mental accommodation of meaning; however, a mental image may refer to meaning pertaining to any mathematical situation rather than only to an isolated concept. With some modification, much that was discussed previously about concept images transfers to mental images. The topics of visualization, analogies, and metaphor will be briefly dealt with later in this subsection.)

Even within our restricted meaning of representation, there is much room for subjectivity. For instance, sometimes tables giving certain function values are considered representations of the function even though the values provided by any such table would coincide with those of many other functions. Also any function always has innumerable situations from which the same function can be extracted; in what circumstances should these be considered representations? It may be useful to remember that there is always a construct to be represented and one that represents. Hence, there is an association between the two constructs, but not a free one; the one construct has a particular role to inform in some way or another about the other, and in the context of the representation it has no other role. Hence, for example, the function \( f: [0, \infty) \rightarrow \mathbb{R}, f(x) = x \) has an obvious embedding in the spiral of Archimedes \((r = \vartheta, \text{in polar coordinates})\), but the spiral is unlikely to be considered as a representation of \( f \). (Conversely, why cannot \( f \) represent the spiral? Here we come to another subjective issue. The construct that represents should either have a concrete character, or it is placed within a richer environment than that of the construct represented.)

In educational terms, the identity of representations is further described by ascertaining how representations are useful. We suggest the following characterization. The construct representing should provide the construct represented with one or more of
the following: (a) better understanding, (b) better way of analysis, (c) better facility in manipulation, and (d) new mental images. Examples might be graph of function and the Argand diagram for (a) and (b), symbolism for (c), and tangent notion from graph of function for (d).

The usual point of view in the literature is that representation has more of an intuitive aspect than a structural one. The literature is particularly concerned with “multiple” representations and the danger of “compartmentalization” of knowledge about a concept due to different representations (see, for example, Leinhardt, Zaslavsky, & Stein, 1990, which reviews the sizeable literature on representations of functions). Another point is that the theory tends to take a much more expansive view of representations than is taken in practice, where most representations considered are pedagogically imposed. (This reflects a problem; if you want to introduce a concept, you are almost bound to introduce representations of it before the concept itself.)

Perhaps because of various aspects that imbue it with some degree of subjectivity, representation as a theory has not significantly influenced AMT research. Indeed, when one takes an intuitive approach to representations, it is sometimes difficult to distinguish them from mental images. Hence, it is possible that some ideas in the literature on representations could transfer into this area of AMT interest and might help to lead to a revival of interest in analyzing mental imaging. Even if this linking may be tentative, the usefulness of representation as a tool is surely indisputable, especially if we in AMT take a more structural approach. In this case the representation becomes more than a mental image because of the existence of an explicit description of the association between the construct to be represented and the construct that represents. The significance of this may be illustrated by the Argand diagram. As noted in Tall (1992), “complex numbers—where the process of taking the square root of a negative number was carried out without giving a meaning to $\sqrt{-1}$ for a century and a half—were given meaning through representation as points in the plane.” (For a historical narrative, see pp. 626–632 of Kline, 1972). But would students attain this meaning? Perhaps they must think of complex numbers as having some concrete identity rather than being abstractly characterized by a certain structure that can be captured completely in a representation.

Visualization is a large branch in educational research but suffers from a lack of a uniform definition; interpretations can be in terms of the pictorial, the geometrical or graphical, internal versus external stimuli, or of intuition. A categorization of visual imagery is offered in Presmeg (1986). Clements and Battista (1992) also gave an overview on visualization and imagery. Researchers frequently remark that the educational system seems to deter students from visualizing, and this must be considered a serious inhibition even at advanced mathematics. (The problem is particularly evident in calculus teaching; see, for example, Eisenberg, 1992). Researchers have argued against a simplistic characterization of mathematicians either as being geometers or as being analysts; in Zazkis, Dubinsky, and Dautermann (1996), a model is made in which visualization and analysis interact so that they ultimately become so intimately connected in the mind that they can hardly be distinguished.

The mathematician’s view is often contrary to this model; although today most would say visualization is an influence, it is little more than that. For example, Amitsur stated in Sfard (1998b, p. 455);

“But in mathematics proper, in mathematics itself, what I really have to know is something different: it is how to draw conclusions from things I know about things I don’t know yet. This cannot be done with pictures and other visual representations.”

At the opposite extreme, Davis (1993) endorsed so-called visual theorems.

At tertiary level, it is almost traditional to regard the word analogy as a cue for attempting generalization. This may be as true for the researcher in mathematics education as for the lecturer. Notwithstanding the importance of generalization, there
are also many general ways of thinking that do not lend themselves to tight definitions (c.f., our description of decentralised notions in our last section). But the main point is that analogy can be a powerful cognitive tool for students (without worrying about forming new structure), and this still can be significant at AMT level. Burton (1999a) found, however, that experienced mathematicians think in terms of “connectives” and the “big picture” of mathematics (terms coined by Burton). The meaning of connective seems to be somewhat stronger than analogy and depends more on compatibility in mathematical structure than on intuitive identification of similarities between different constructs.

Lakoff and Nunez (1997) suggested that “metaphor” forms a basis on which all mathematical ideas can be explained. According to this philosophical essay, metaphor is the vehicle by which mathematics can be thought of as “essences.” “Neither formalism, nor constructivism, nor Platonism has any room for an account of mathematical ideas” (Lakoff & Nunez, 1997, p. 31). An application of the theory is made, extending from sets, functions, elementary algebra right up to space-filling curves. The ideas of Lakoff and Nunez certainly are provoking and ultimately may have implications for mathematics education research.

**REFLECTION ON MATHEMATICAL STRUCTURE**

In the previous section, we considered how AMT is currently represented in some major theories contemporary mathematical education research has adopted. In this section, we take a specialized look at the question, what are the important factors of AMT that cannot be adequately covered by the existing main-line mathematical education theories? We select one such factor we feel is important and broad, yet fairly tangible (as opposed to, for example, topics such as mathematical creation and inspiration). This factor concerns how practitioners of mathematics (both the learner and the expert) mentally interact with mathematical structure. We invent our own terminology for this: reflection on mathematical structure (RMS). Although some of the ideas within this section are not new, the drawing together of these ideas may have some claim to a certain level of originality. In particular, our aim is to make a case that the mathematics education community at AMT level should find a way to incorporate RMS in their research and pedagogical agendas. Although some suggestions are made, how to achieve this is largely left open.

**A Characterization of Reflection on Mathematical Structure**

We describe Reflection on Mathematical Structure (RMS) as: conscious mental response to the form in which constructs (objects, expressions, procedures, proofs, etc.) are presented mathematically. Perhaps this statement communicates more by its ramifications rather than by its expression, so this section will be devoted to discussing a broader characterization of RMS, which will highlight the spheres of influence of RMS in terms of certain ways the mind interacts with mathematics. In this way, we aim to convey to the reader the spirit of RMS.

Within the discussion we also involve the theme of discontinuities in maintaining meaning while doing mathematics. We assume that all such discontinuities occur because the practitioner has been engaged temporarily in RMS, where a new shift of meaning has been extracted from mathematical forms. We will stress this theme almost as much as our description of RMS because of its more direct connection with cognition. Indeed, if we were to propose an educational field studying RMS, its focus might well be that part of cognition dealing with discontinuities.

In the following, we refer to our description and to the discontinuities as the two identifying traits of RMS. The argument is divided into three parts. The first concerns
definitions; the second, understanding of mathematical constructs; and the third, on-going mathematical work. This division is reflected in the final statement of our characterization. Notationally, we do not distinguish RMS from the practice of employing it or any theory built around it; all indiscriminately may be referred to as RMS. Context should indicate the sense intended.

RMS in Creating Definitions and Understanding Given Definitions

Definitions often mark the start of a line of study of mathematics. Hence the way that a definition is understood can be influential in setting a tone for an entire mathematical topic. Indeed, the very identity of a topic can depend on a single central definition, and in this case much conceptual focus must be closely related to this central definition. Thus, the role of definition is important, and this has been acknowledged from early on in the history of AMT research with identifying differences between concept images and formal definitions as described in Tall and Vinner (1981). We also shall give definitions a special place in our consideration of RMS.

The reason why definitions should involve RMS significantly is that in passing to advanced mathematical thinking, there is an increasing tendency for definitions to determine concepts rather than vice versa (e.g., Gray et al., 1999). We extend on this theme.

There are several issues underlying a definition; it must address the purpose for which it was invented, it must be unequivocal structurally (i.e., it must be well defined), and it must be framed in a form that can be used in practical terms. For a student presented with a formal definition, as much as the mathematician who created it, these are the first-line considerations for comprehension of the definition.

The creating of a new definition and the comprehending of a presented definition can be very different. The creator always has a motive in mind. As she crafts her purpose in practical and familiar mathematical ideas and language, the purpose may have to be obscured or even compromised. Purpose always involves meaning, and this meaning may be transformed; hence, we may have discontinuities in maintaining meaning while we seek a workable expression in terms of previously known constructs presented mathematically.

Definition creation is an activity rarely required for undergraduate students (see Vollrath, 1994). However, the restrictions caused by the process of creating the definition strongly influence comprehension of the presented definition. The student has to face formal definitions that do not necessarily communicate meaning immediately. The student is not aware of the reflection that led the creator to develop the definition (and this may be just as well). Often students must be informed informally of an intention in introducing a definition that would be broadly open to the intuition; the task of the student is then to try to evoke from the definition a mental image with the original intuition as a guide, but to allow the new image to refine or even alter this initial intuition. A good example of this would be the standard definition of the limit of a real sequence, see Mamona-Downs (2000). Again, we are greatly involved with our two identifying traits for RMS.

Sometimes definitions only respect an original intuition up to the point that they seem consistent. This seems to be particularly true for definitions of properties. Let us take an example from elementary probability theory. The definition of independence of two events \( E, F \) is given by

\[
P(E \cap F) = P(E) \cdot P(F).
\]

In this case, the original intuition (which may be on the grounds, roughly, that \( E \) and \( F \) are independent if they do not affect each other) is likely to be too vague to be
transferable into mathematics. What we are left with by the definition is a decontextualized rule that allows for all the cases recognized by intuition, but as a price of being precise and workable mathematically, it will also admit cases beyond the intuition. Hence, we sacrifice meaning once more in forming mathematical structure.

**Mathematical Constructs Understood Independently From Their Definition**

Definitions dealt with above implicitly are ones for which the identity of the construct being defined is largely accessed by reflection on the form of the definition itself. There are many definitions that act in other ways, however.

For example, many definitions simply name constructs that already have been identified through the analysis or exploration of an established mathematical system. The naming of the construct helps maintain the construct as a focus. Institutional naming can be confusing for the student, however, because such naming can hint to references to associated properties to which the student might not have access. As an illustration, the common intersection point of the medians of a triangle is often named the centroid of the triangle, where the name centroid (in general) refers to a much more esoteric and broad invariance principle of geometric figures/vector spaces (see, for example, section 13.6 of Coxeter, 1989). (A problem in such premature naming is that we may have a situation in which we ask a student to prove that a point already named the centroid is the centroid.) This note is somewhat of a digression. What we want to stress is that mathematical constructs may attain identity independently of definition through the understanding of other mathematical constructs relating to it. Hence, in our simple example, the centroid of the triangle (even unhelpfully named) will in usual circumstances convey the triangle, its three medians, and the fact that they intersect at a single point. All other relevant facts subsequently discovered are likely to be thought of as consequences of this basic identity rather than as adding to it. This situation involves a special case of RMS in that mental response to the form of existing constructs acts toward conceiving another. We might say that we are initializing a new aspect in the system that allows an outlet for assigning meaning.

Some definitions do not communicate the identity of the construct being defined, nor do they act as simple naming roles. A class of such definitions is one in which the motives of making the definition are largely utilitarian (for example, definitions made to facilitate manipulation of technical notation). Such definitions obtain meaning (if any) through the properties that become evident in using the definition. A more tangible category of definitions that shares the tendency of being understood by seeking out properties is made by definitions of subclass. Here we suppose a whole class of objects is comprehended, and a certain subclass is picked out. The whole class may have its own strong identity, but as soon as we pass to a subclass, that strong identity may no longer be of much use as we then are trying to distinguish the subclass. Often an analysis must be made to find a valuable interpretation of the subclass. (For instance, symmetric real matrices are matrices with entries arranged to satisfy some symmetry conditions, but they are understood as those matrices that, when considered as linear transformations, have eigen-vectors that span the whole space and are orthogonal to each other.)

Clearly, the understanding of constructs through properties that can be deduced from their mathematical form falls into the first identifying trait of RMS.

**Working With Mathematics**

In the two previous subsections, we talked about the identity of those mathematical constructs that largely act as prompts to first undertake mathematical working. Now we consider the role of RMS when working with mathematics.
When we are working with mathematics, we suppose that the furtherance of a given mathematical argumentation depends on one or both of two things: a continuous flow in conceptual meaning paralleling the argumentation and an operational sense. To explain what we mean by operational sense, we describe its two components. Operational usage is the known maneuvers and procedures available in a particular system (as represented by the present state of mathematical argumentation) and the instinct of the practitioner about which ones to apply. By idea engineering (on the operational level), we mean the ideas that are generated while reflecting directly on the present state of the mathematical system, together with accessing previous knowledge, and combining these ideas. This is done to identify targets for the argumentation and to determine how to achieve them. For a modest example, suppose that a student has to solve an indefinite integral of the following type:

\[ \int \frac{dx}{ax^2 + bx + c} \quad a, b, c \in \mathbb{R}. \]

He hasn’t learned how to solve this general type of integral, but he knows the solution of

\[ \int \frac{dx}{x^2 + 1}. \]

The student’s main problem now is to access the technique of completing the square. If the student does not know the technique, perhaps we could not expect him to invent it by himself. If the student does know the technique, however, it may be reasonable for him to discover the solution on his own. Returning to the general, clearly the creation of constructs to fulfill particular mathematical needs is important within idea engineering, and because of this many students may be limited in this particular kind of thinking.

Idea engineering is highly consonant with the first of our identifying traits of RMS. Operational usage differs, however, from idea engineering in that it works without much conscious control of the practitioner, but application of operational usage can significantly transform the character of the system supporting the contemporary state of the argumentation. If this changed character is recognized and internalized, we move in a discontinuous way from one point of understanding to another. Recovering understanding in an argument, after it was temporarily suspended, by extracting meaning from the present form of the mathematical system is important because it induces a new meaning intimately drawn on structure and thus highly consonant with idea engineering. This involves both of the identifying traits of RMS. Symbolic manipulation revealing new forms that can be separately interpreted constitutes a common and important type of example of this aspect of operational usage. For a simple illustration, we consider a situation that, although not placed within an ongoing argument, shows how powerful interpretation of even slightly different forms of symbolism can be.

We consider the polynomial \((1 + x)^n\) where \(n \in \mathbb{Z}^+\). First we express the polynomial as \((1 + x) \cdot (1 + x) \cdots (1 + x) [n\text{-times}]\). This expression could lead someone to identify the coefficient of \(x^r\) (i.e., the binomial coefficient \(\binom{n}{r}\)) with the number of ways that one can choose \(r\) things out of \(n\).

Second we express the polynomial as \((1 + x)^{n-1} (1 + x)\) from which we can read the relationship:

\[ r-1 \binom{n-1}{r-1} + \binom{n-1}{r-1} = \binom{n}{r}, \text{ for } n \geq 2, r \geq 1. \]
The above should not be just thought of a property of binomial coefficients. Given obvious initial conditions \( \binom{1}{1} = 1, \binom{1}{0} = 1 \) and adopting a convention that \( \binom{n}{-1} = \binom{n+1}{n} = 0 \) (for all \( n \in \mathbb{Z}^+ \)) the relationship above suitably extended would (inductively) reveal the whole structure of a self-supporting system. This structure can be abstracted (as suggested by Pascal’s triangle). Another way, then, of looking at this situation is that we can use the polynomial \((1 + x)^n\) as a catalyst to find a transparent way to explain why any entry in the Pascal’s triangle expresses the number of ways of choosing a certain number of things out of another certain number.

(The operational usage above is minimal and only involves the equation \((1 + x) \cdot (1 + x) \cdots (1 + x) = (1 + x)^{n-1} \cdot (1 + x)\). Everything else refers to finding interpretation for which there are two levels. The first level is to give a special meaning to the coefficients and to find a relationship for these coefficients. The second level is to further reflect on the form of the relationship to obtain a new meaning of it as a self-contained abstract system that “forgets” its original referents. The human mind retains both levels, however, and the association is mentally kept.)

As a summary of the last three subsections, we propose a characterization of RMS. RMS is always involved whenever at least one of the following becomes a concern:

1. The need to define mathematical constructs with an eye toward clarifying any intuitive basis and hidden assumptions, toward brevity, and toward a form that is practical and productive; to understand given definitions in the same terms; and to define properties in a decontextualized way.

2. The need to understand a mathematical construct through (a) its relation with other mathematical constructs, (b) properties deduced from the mathematical form of the construct (in contrast to properties that are perceived as being intrinsic), or both.

3. The need (a) to identify targets for mathematical argumentation to reach and to deliberate on the operational possibilities how to achieve them (in particular, an important part of this is the creation or introduction of new constructs into the system to fulfill particular mathematical needs, or to inspire strategy); (b) to extract meaning from a mathematical system that has been evolving for some time only operationally, including (re)interpretation of evolving symbolism.

An Extended Illustration

The previous subsection should have communicated to the reader that RMS is a mental undertaking, yet the terms in which we have couched for RMS may have seemed impersonal. To give the reader an impression of how RMS might act in terms of more personal thinking habits, we present the following illustration. Nonetheless, the illustration, because its purpose is to highlight the role of RMS in a particular mathematical task, still concerns an imaginary “practitioner,” who thinks in an unrealistically clean way and whose thoughts are deliberately directed in ways consonant to RMS. As such, the practitioner should not be considered as a typical student. However, each idea comprising the solving procedure will be neither unnatural nor particularly esoteric, and thus the approaches should not be considered outside the potential of a student to achieve at least in outline. Hence, if wished, fieldwork could be conducted to identify psychological factors that may obstruct or help students engage with the presented arguments or, on the contrary, to see whether the students will try to deal with the situation in a way not so closely allied with RMS.

The practitioner wants to show that if \( f, g \) are any two differentiable real functions, then the product function \( f \cdot g \) is differentiable. She then wants to find a simplification for \((f \cdot g)'\). She wishes to employ the formal definition for differentiation. This, of
course, is an unrealistic situation in that the task constitutes a standard theorem, but here we suppose the practitioner is meeting the task for the first time.

**First Argument.** She first writes down

\[ \lim_{h \to 0} \frac{f(x + h) \cdot g(x + h) - f(x) \cdot g(x)}{h}. \]  

(1)

She appreciates that \( f \cdot g \) is differentiable if and only if the above limit is defined for each \( x \in \mathbb{R} \). The form suggests that any argument would have to rest on previously known results for limits involving general functions. This reflection though does not help her progress further. The practitioner now thinks rather indirectly and supposes that \( (f \cdot g)' \) exists and admits a simplification. What would be the natural features that might occur? The functions \( f', g', f, g \) may seem likely candidates to be involved, and because of the terms \( f(x + h) \) and \( g(x + h) \) appear, it is plausible to believe that both \( f', g' \) should figure. Through the practitioner’s knowledge of rules of limits and her knowledge that

\[ g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h}, \]

she starts to wonder how she might introduce the term \( (g(x + h) - g(x)) \) into \( f(x + h) \cdot g(x + h) - f(x) \cdot g(x) \). This cannot be done with usual manipulation of symbolism. The expression has to be somehow operated on without affecting its value. The practitioner realizes that one option is that she can add any term if she also adds as another term its negative; there is a purpose in doing this if one term is absorbed into one construct, and the other in another. Hence she can easily isolate \( (g(x + h) - g(x)) \) by operating as follows:

\[
\begin{align*}
&f(x + h) \cdot g(x + h) - f(x) \cdot g(x) = f(x + h) \cdot [g(x + h) - g(x)] \\
&\quad + f(x + h) \cdot g(x) - f(x) \cdot g(x),
\end{align*}
\]

where by factoring she also naturally obtains the term \( (f(x + h) - f(x)) \); the result now follows by applying simple rules of limits.

**Second Argument.** The practitioner focuses on the nominator \( f(x + h) \cdot g(x + h) - f(x) \cdot g(x) \). The natural interpretation of the difference of a function’s values at different variable values here does not seem to offer much help in how to progress. However the practitioner, otherwise at a loss, tries to access richer systems within which this expression may be imbedded, the interpretation of which in such a new system may afford a natural useful reexpression. The practitioner has a tentative idea: The expression is simply the difference of two products of numbers \((x \text{ and } h \text{ being held constant for now})\). From experience (indeed from primary school), she is used to the idea of representing the product of two numbers by a rectangle with the same dimensions. Given the difference of two products, this could be represented by the “area” of the union of two rectangles less their intersection (where it is understood that some of the “area” might have to be taken in the negative sense).

The practitioner reflects that the limit process involved in the definition means that \( h \) should be thought of as a variable. Even if she draws only two rectangles, how she draws them also should reflect how any other rectangle from the relevant infinite family would be accommodated in the diagram. This concern in control in representation makes her create as many common features to the rectangles as possible. Another concern that influences her to move in the same direction is a wish to make
the region representing the difference as simple as possible. Hence, she arranges for the rectangles to share a corner and axes, and for ease also assumes that \( g(x + h) > g(x) \) and \( f(x + h) > f(x) \); Fig. 8.1.

The diagram now catches the practitioner’s attention, and she notices that the shaded region representing the difference has two perpendicular “legs” of width \( (g(x + h) - g(x)) \) and \( (f(x + h) - f(x)) \). She even might have had this aspect in mind when she was deciding how to form the diagram. These features are highly significant because when transferred back to the formal expression of \( (f \cdot g)' \) they will yield \( g'(x) \) and \( f'(x) \). What started as a tentative idea now seems highly propitious.

The practitioner now needs to operate on the region so that she is both preserving and isolating the thickness of the legs. An obvious way to do this is to divide the region into two rectangles in one of two natural ways. This allows her to write down (for example):

\[
\forall x, h \in \mathbb{R}, \quad f(x + h) \cdot g(x + h) - f(x) \cdot g(x) = f(x + h) \cdot [g(x + h) - g(x)] + g(x) \cdot [f(x + h) - f(x)].
\]

She divides by \( h \) and takes the limit to obtain the result.

**Notes.**

1. Both arguments most closely fit with that part of our characterization of RMS dealing with identifying and resolving targets. Both depend on constructing new objects. Nonetheless, the role of the construction in the two cases are very different. In the first method, the introduction of the two canceling terms fulfills already formed needs, whereas the diagram of rectangles in the second method is created to inspire strategy.

2. Though the two methods are clearly very different cognitively, in a written presentation the ideas how each argument was thought of may be obscured. For example, if the practitioner had conceived of the diagram of rectangles only in her mind, there would be no reason to refer to it explicitly in her exposition. The diagram in the end simply helps her transform the algebra into a more convenient form, and because of this she does not even have to be careful to consider different cases for the diagram.
In this circumstance, the written presentation would not inform the reader of which method was employed.

3. Of course, there would be other ways of approaching the task and different colors to the two methods we have presented here. For example, if the practitioner formed the diagram of rectangles, she might be able to use it as a mental aid to “see” the result via a mature understanding of differentiation interpreted as instantaneous rate of change. In this case, she is operating far more conceptually. (Although papers such as Thompson (1994) would strongly suggest that this kind of intuition about differentiation in geometric or physical contexts is extremely poor for the majority of students). On the other hand, the practitioner may simply look at the form of the region given by the diagram between the rectangles and instinctively partitions it into two rectangles because it simplifies the form. This in itself could be regarded as a rather trivial local act of RMS in relating one construct with others, as in the second part of the characterization of RMS. However, it would weaken the case that the whole of the second approach met that part of the characterization dealing with identifying and resolving targets because some of the sense of deliberation is lost in the overall argument.

The Reflection in RMS

*Reflection* is a word used repeatedly in many research papers of mathematical education, but often only with an intuitive and somewhat indistinctive meaning. Indeed further precision is frequently not needed, because the word is used simply to indicate that a certain mental functioning is operating without having to analyze it. In RMS we are treating reflection in a particular context, and we are interested in analysis of reflection. In this subsection we extend this theme and compare RMS with other educational traditions that involve analysis of reflection (i.e., reflective abstraction and metacognition).

We do not define reflection beyond some basic “principles” that it should always satisfy. (Even these principles would not be shared in other authors’ views). Mathematical reflection involves thoughts that are (a) conscious, (b) not spontaneous, (c) personal, (d) reactional from a particular situation, (e) meta-mathematical (i.e., addressed toward handling mathematical issues). Principle (d) comes from the following consideration: Reflection must have a subject on which to reflect, and there must be something perceived within that subject to provoke that reflection. Note also that (a) and (b) exclude unconscious “incubation” that can lead to sudden inspirations, such as related in the famous anecdotes of Poincare (discussed at length in Hadamard, 1945), so the reflection in RMS does not cover all mental interaction with mathematical structure.

We have suggested before that RMS is closely concerned with discontinuities in maintaining meaning, that is there are places in mathematization where we sacrifice some flow in our intuitions to allow a formatting, which provides us with a new starting point for our thoughts. The reflection in RMS is a reaction to this situation and has two sources of generating ideas; one is structural, where we consider which aspects of mathematical expression are to be extracted and used; the second is more cognitive, concerning our understandings, motivations, and our expectations in handling structure. Typically these two sources develop mutually, but of course conflicts are also possible.

The identification of these sources locally would then become the basis of the analysis of the reflection in particular circumstances. For example, we refer back to the illustration of the last subsection. For method 1, we can nominate as the cognitive source the expectation of $f', g'$ to appear in the final expression of $(f \cdot g)'$ and for the structural source how to accommodate such terms into the expression of its formal
### TABLE 8.1
Domains of Mathematical Activity vis-a-vis Reflection on Mathematical Structure

<table>
<thead>
<tr>
<th>Domains</th>
<th>First Source (Structural)</th>
<th>Second Source (Cognitive)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Modeling</td>
<td>Mathematics used in model</td>
<td>Physical question</td>
</tr>
<tr>
<td>2. Thinking aside</td>
<td>Mathematical working</td>
<td>Insight of what “looks right”</td>
</tr>
<tr>
<td>3. Postreflection</td>
<td>Structural reexamination of completed piece of work</td>
<td>Seeking for significance, meaning, and transparency</td>
</tr>
<tr>
<td>4. Posing problems</td>
<td>Checking for relevance and tractability</td>
<td>Different ways the mind follows for posing</td>
</tr>
<tr>
<td>5. Analysis of</td>
<td>Examination of “local structure” (or “partial structure”)</td>
<td>To assimilate local structure back to whole structure</td>
</tr>
<tr>
<td>mathematical object</td>
<td></td>
<td>To study the properties of a construct in terms of another with parallel structure</td>
</tr>
<tr>
<td>6. Embedding</td>
<td>Isomorphism (homomorphism)</td>
<td></td>
</tr>
<tr>
<td>7. Generalization</td>
<td>Abstraction of a property commonly found in several contexts and axiomatized to form a single construct</td>
<td>Why and how to abstract</td>
</tr>
<tr>
<td>8. Forming decentralized</td>
<td>Identification of analytic tools commonly used in several contexts usually lacking general abstract definition</td>
<td>To allow similar arguments in one context to be used in another</td>
</tr>
<tr>
<td>notions</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Definition. For method 2, we have two stages. For the first stage, the structural source of the definition expression is examined for a visual representation (the cognitive source); once this representation is obtained, this becomes a new cognitive source to guide manipulation of the expression.

This notion of investigating sources of generating ideas for the analysis of RMS also has the potential to be taken theoretically in more global terms. For example, in Table 8.1, we indicate some wide domains of mathematical activity, each of which can be described broadly as having overall a certain structural identity as well as a cognitive one. The linking of these two identities should have a strong interest for AMT researchers.

We do not have the space to expand on the separate domains listed in Table 8.1, except for decentralized notions that will be discussed in the next two subsections. Some domains are familiar ones of general level mathematical education (for example 1 and 4), but the interaction of the two sources suggests a more mature approach than we would expect from school students. In AMT literature, domains 2, 3, 7 are commonly tackled, but again the full potential of the interplay of the structural and the cognitive often is not fully realized. Finally, we have identified areas (domains 5, 6, 8) that we feel are underrepresented in the literature.

While talking about analysis of reflection in RMS, perhaps it is as well to mention the problem that reflection, like all thought, cannot be observed and may only be gleaned in indirect ways. We do not wish to dwell on this theme here, but the problem is acute in RMS (as in metacognition). In more conceptually based theories, the coherence of the concepts themselves gives the researcher many clues of how a student is diverging from or adhering to the concepts. In RMS we do not (necessarily) have such a steering framework. For one practitioner to communicate his reflection on mathematical structure to someone else, he typically has to make statements of intention as well as describing the mathematical steps he has taken.
We conclude this subsection by comparing RMS with other mathematics education perspectives and theories in which reflection has a role. Let us consider first reflective abstraction. Reflective abstraction was a term coined by Piaget, and the notion behind it was an important and recurring one in his work. Put broadly, the notion could be considered as the interiorization and coordination of actions. It constitutes an extensive view of abstraction evolving with meaning, with a strong construction aspect as well as the more traditional way of looking at abstraction as decontextualization. Although Piaget tested reflective abstraction only for young children, he also regarded the notion highly significant for AMT:

the whole of mathematics may therefore be thought of in terms of the construction of structures... mathematical entities move from one level to another; an operation on such "entities" becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternatively structuring or being structured by "stronger" structures. (Piaget, 1972, p. 70)

Here we see mathematical structure (looked at in a particular way, stressing a global hierarchy) and objectification playing essential roles.

Dubinsky’s version of reflective abstraction for the special use of research in AMT (Dubinsky, 1991) stresses encapsulation (interiorization of a dynamic process as a static object). RMS does not share this focus (for example, in RMS the study of an object in the form of a formal expression may be the starting point for understanding that object). Dubinsky’s theory of reflective abstraction ultimately led him and his colleagues to develop APOS theory (see the second section of this chapter). This latter theory suggests a standardized hierarchy of levels in understanding for any particular concept (the levels being represented by action, process, object, and schema). Although the attainment of a higher level of understanding in this hierarchy may be thought of as a concrete step that constitutes a substantial jump in mathematical thinking, we do not consider this as a discontinuity in maintaining meaning such as those that characterize RMS. In APOS, each step in fact maintains past meaning because the new meaning is obtained by an act of assimilating the past meaning. On the other hand, RMS suggests that some meaning is (temporarily) suspended for the sake of direct examination of the structure. In spirit, then, the two perspectives are not complementary in character.

We now consider metacognition. Metacognition could be described as self-awareness of how one’s mind is interacting with a subject matter (perhaps including a socially driven awareness also). Clearly, metacognition must be the result of considerable reflective processes. This description is too wide to be of much use. The term metacognition was adopted mostly by researchers interested in problem solving. They categorized metacognition for their purposes into two main categories: individuals’ knowledge of their cognitive processes and self-regulatory procedures (later adding a third category dealing with beliefs and affects; see Schoenfeld, 1992). Nonetheless, doubt that personal applicable mathematical knowledge always fits in with self-knowledge of cognitive processes has been expressed by some researchers, such as Garofalo and Lester (1985). This has contributed to metacognition literature to mainly stress regulation.

Self-regulation has aspects that are not strictly structural; for example, a decision to abandon one approach to a problem may be made on the grounds that “I cannot see how to progress,” “The last expression seems too complicated to expect to simplify,” and so forth. Conversely, RMS can act outside the sphere of problem solving. Also the reflection in RMS operates more on the cognitive level rather than the metacognitive, which allows it (for example) to be involved more directly than self-regulation with strategy making and heuristics (which are not necessarily activities
with metacognitive weight). Self-regulation and RMS do seem to be complementary to a degree, with self-regulation considered naively as a decision-making mechanism in doing mathematics and RMS available to provide a basis to allow this mechanism to work effectively. Whenever RMS does constitute such a basis, we include the decision-making mechanism itself within RMS; in this way self-regulation and RMS are intersecting.

Despite this, RMS and metacognition have strongly different identities because that RMS stresses structure and metacognition literature stresses more general aspects of problem solving. The difference in identity is clear when the two perspectives are decomposed into subthemes. For example, we could contrast the different style of the categorization of self regulation into “reading, analyzing, exploring, planning, implementing, and verifying,” found in Schoenfeld (1992), with the domains in the table given in this subsection suggesting natural arenas for the application of RMS that refer much more explicitly to mathematical issues.

The Significance of RMS in AMT and Decentralized Notions

In part of the first section of this chapter, we considered the identity of AMT. Although RMS is not a characterization of AMT, there is a strong correlation. We say this even while acknowledging that RMS of significant level does occur in school mathematics; for example, the ability to use the range of methods given at school to find simple integrals often needs a deliberation beyond straightforward appliance of procedure. Two general traits of school mathematics severely limit RMS at this educational level, however. The first is that mathematical presentation at this stage is mostly intuitively based, and it is usually only in cases of cognitive conflict when serious attention may be placed on points of structure. The second is that school mathematics is largely result oriented, whereas tertiary level mathematics is more oriented toward analyzing methodology. Although the operational skill in applying a method may involve RMS, the understanding of why the form of the method has to be as it is generally involves a much higher level of sophistication in RMS. At the same time, we do not claim that all work done at AMT will substantially involve RMS; sudden inspirations and more or less continuous lines of thought based on highly developed insight of particular complicated systems may take place, and these do not fit well with our description of RMS. However, the vast majority of AMT depends on mathematical structure, the role of which could be thought of (in cognitive terms) as a focus for the mind to concentrate on in partial or temporary isolation of other (semantic) details in the system.

The major significance of RMS is in its allowing a mathematical understanding that may be independent of continuous conceptual thought. In this way, we assert that RMS (as a theory) can reveal areas of cognition that are important for success in AMT but that have received little attention in mathematics education literature. One indication of this may be gleaned by inspecting the list of broad mathematical activities given in the previous subsection. Later, we illustrate this by discussing one of these, decentralized notions.

RMS also is important because it is relevant to most of the main theories we mentioned in our second section. In this role, RMS may be thought of as a perspective that can complement these theories at the AMT level. We briefly give some examples. The idea of concept image has long ago been extended to the notion of mental image that can accommodate meaning in a broader context than just concept acquisition. Because RMS may act to evince new meaning from structure, it is highly relevant to mental images, and because RMS always stresses an interaction between mathematical structure and some more cognitive source of thought, some of the ideas involved in the schema of concept image, concept definition, and concept usage should
transfer. Educational studies depending heavily on epistemology often target on particular highly important notions found in mathematics; RMS can contribute by identifying some such notions through the idea of decentralized notions. We have mentioned before that APOS theory runs counter to the spirit of RMS; however, the more flexible theory of procept seems more consonant, with the structural feature of symbol appearing explicitly within its basic cognitive framework. The reflective aspect of RMS also should help students to conceive indirect lines of thinking, facilitating them to do proof; the stress that RMS gives to structure should enrich the already highly reflective theory of problem solving. Attention to structure as well as context will contribute to making students pose more plausible problems. Finally, representations with RMS are more likely to be constructed systems rather than givens from teaching practices, hence how students conceive representations rather than interpreting them becomes more pertinent. We now discuss decentralized notions, after putting their importance into context.

There is an inherent dilemma within mathematical education theories that becomes progressively problematic as the target fields involve more and more advanced mathematics. To put it simply, as the mathematics becomes more diverse and sophisticated, any cognitive framework drawing the mental processes used in a unified way seems to be more and more remote from the specifics of the mathematics being done. On the other hand, if more local perspectives are introduced to cover special mathematical issues, it may be difficult to draw all the resultant information together in an integrated way.

Currently, the trend is for unifying cognitive frameworks to dominate the literature, which may constitute somewhat of an imbalance. One way to address this imbalance may be for some research to specialize in decentralized notions. Examples are decomposition, symmetry, order (in the sense of arrangement), similarity, projection, equivalence, inverse, invariance, dual, canonical forms, (to mention just a few). Decentralized notions are distinguished by having roles cutting through mathematical theories, but at the same time preserving the resultant plurality of context, not being of the character of being usefully or readily described by abstract generalization. They constitute standard ways of thinking in mathematics rather than representing parts of the mathematical output. We contend that for the AMT level, the acquisition of decentralized notions is essential for students’ progress and that this should be distinguished from straightforward concept acquisition or generalization. Decentralized notions provide topics that are broad and accessible enough to make cognitive analysis and to allow specialized frameworks to be designed for each. The coordination of decentralized notions would then form an umbrella research agenda and would represent a significant part of advanced mathematical thinking. In particular, we hypothesize that contextualizing decentralized notions often plays an important part in heuristics.

Concluding Remarks on RMS

Let $g$ be a continuous real function and $a, b, k \in \mathbb{R}$. In Eisenberg (1992), the question whether

$$\int_a^b g(x)dx = \int_{a+k}^{b+k} g(x-k)dx$$

is true is included in a test that the author claimed that the majority of graduates would fail even after taking introductory calculus courses. The author attributed this largely on the fact that students avoid visualizing. In Dreyfus (1991), the same question is mentioned. Dreyfus remarked that this question might seem straightforward to an
expert practitioner, but the mental processing involved may not be available to the student. Perhaps, though, the RMS perspective provides another angle to the issue of why it is likely that many students would have problems with the question. The symbolic form of the question conceivably may act as a barrier to the initialization of the mental processes to which Dreyfus refers. How this could happen has many possible levels. For instance, interpretation of symbolism may be an unfamiliar practice or even regarded as regressive by the students, hence prompting a preference to some sort of “algebra” of integrals, for which there are, in this case, no obvious “handles.” Even if an interpretation is attempted, there may be various stumbling blocks. There is an institutionally imposed image of

$$\int_{a}^{b} g(x) \, dx,$$

meaning the area under the curve of $g(x)$ over the interval $[a, b]$. For the student, the interpretation of this symbolism may be highly sensitive to what might seem to be an unconventional form, such as

$$\int_{a+k}^{b+k} g(x - k) \, dx.$$

First, the mental imaging might be regarded as too much of a whole, with the result that the role of each component of the symbolism is not properly realized on its own. Second, inspecting the second integral, students might be distracted from the basic simplicity of the “reading” of the symbolism by inventing for themselves other factors; for instance, they may be worried about not finding a “consistency” between the integrand variable $(x - k)$ and the symbol $dx$. They might also have a problem in even appreciating the spirit of the question, in which case the role and the character of $k$ may be mysterious. But perhaps the most vital factor is whether the relationship between $g(x - k)$ and $g(x)$ is understood; the issue here is if the conception of the relationship is inherently difficult, or the symbolism $g(x - k)$ in itself contributes in obstructing the forming of this conception. (For completeness sake, we note that another image, that of the antiderivative of $g$, is available through theory and by adopting a special designation, say $F$ for such an antiderivative, the problem becomes trivial, by symbolically substituting $F$ in both integrals.)

The significance of this discussion lies in the following: In Dreyfus’s interpretation the suggestion is that the students lack the resources to “think through” the given problem (and by extrapolation, many other “nonstandard” mathematical tasks); from the RMS perspective, students may simply be prevented from doing mental processes that they are able to perform because they have difficulties in negotiating the given mathematical structure. An important difference between the two perspectives is that Dreyfus’s message does not easily allow remedial measures, but these are not a priori ruled out if the students’ problems lie in reassigning meaning to mathematical notation. Also the first perspective stresses problems in facing complication (not appreciated by the expert), the second more in discovering simplicity in apparent complication.

In this paper we have discussed RMS without placing it within clear-cut agendas. Our aim is to make a case that RMS may fill a substantial gap that may exist in present mathematical education theories. Should subsequent work adopt RMS as a perspective, we would expect it to refine the character of RMS and to identify distinct roles that it might take within mathematics education. As an illustration of a research agenda that could be envisaged, let us return to the discussion of the example above. The example given is such that a mathematics lecturer might include it as a line in a proof, say, without any further comment. To the lecturer, there is no ‘trick’ to explain, it
simply needs the ‘right’ interpretation. (In the terminology we introduced in section 2, it could be called a line-to-line connection). Our agenda then might be: (a) to define more closely the type of tasks we are dealing with when we are taking our example as a prototype; (b) to identify through research fieldwork the problems the students have with such tasks. (Does the consideration of lack of conceptual maturity versus conceptual blockage due to mathematical presentation provide a good framework? If so, is there a dominance of one component over the other?); and (c) if there are student problems in reading mathematical presentation, to research ways students may become more proficient in this skill. (How much is attitude, such as unwillingness to visualize, a factor in this?).

We hypothesize that drawing meaning from symbolism can be difficult for students, and perhaps this merits some consideration in bridging courses. Nonetheless, we stress that we are only considering one strand of RMS here. Another important strand is decentralized notions. We contend that these form powerful ways of thinking in mathematics but do depend on context provided by mathematical theories for their initialization and development. The more sophisticated the decentralized notion, the more likely the notion is cultivated by a sophisticated theory. Calls for sacrifice in content (if necessary) to allow time for activities encouraging students to think independently and to express their thoughts (see for example Stein, 1986) have recently been materialized, with many mathematics departments forced to make concessions in the face of a more and more heterogeneous clientele (e.g., Hillel, 1999). A sacrifice of content however, is always accompanied with some sacrifice in potential enrichment of general ways of thought in mathematics, which may well include some decentralized notions. The trade-off of better understanding for ‘less’ mathematics would seem justified overall, but still the trade-off should be taken in the spirit of compromise.

Dreyfus (1991) stated that “our goal should be to bring our students’ mathematical thinking as close as possible to that of a working mathematician’s.” Probably many researchers in AMT would concur with this sentiment (notwithstanding a possible doubt about the uniformity of experts’ thinking habits). The argument in the previous paragraph suggests that because an expert’s power of thought partially relies on experience in doing mathematical research on abstruse theory and in forming new concepts (an activity rarely done in undergraduate education; see Vollrath, 1994), we are almost bound to fall well below that ideal. Brown (1997) even questions the ideal itself. Regarding the rather structurally based standpoint of mathematics learning, found in a perspective of “mathematization” developed over many years by Wheeler (e.g., Wheeler, 1982), as a model for “Thinking like a Mathematician,” Brown (p. 37) states that “the act of understanding ourselves and becoming educated is fundamentally at odds with the qualities we associate with ‘mathematization.’” With our stress on RMS dealing with how to proceed in mathematical working when there is a discontinuity in lines of thought, we can be fairly certain that Brown would feel much the same way about RMS as about Wheeler’s mathematization. Brown’s may represent an extreme point of view, but perhaps it does reflect an even subconscious reservation to the wish of mathematics educators to bring students to think as much as possible like an experienced mathematician. This reservation is that much of a mathematician’s thought is done at the structural level, and even though this does not necessarily contradict constructivist principles, there is remoteness in it from constructivist sensibilities. (Especially in the possibility of not maintaining a direct chain of thought starting from a physically or psychologically based image). Maybe it is because of these sensibilities that no major mathematics education theory up to now has considered incorporating a perspective similar to RMS. Even though university lecturers hold a fairly broad outlook of what mathematics is, there still is a strong tendency toward an internal structural viewpoint (see Mura, 1993, and Burton, 1999a). In Tall’s (1992, p. 2) words;
we, as educators, most reconcile any cognitive approach with that development pursued by the wider community of mathematicians of which we are part. This must be done either by meeting the community beliefs part way, or offering a viable alternative.

Perhaps the perspective of RMS could fulfil both roles.

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Because mathematics is the systematic description and study of pattern, it is not surprising that the world of mathematics opens onto so many other worlds. Most people know generally that mathematics incorporates logical and precise reasoning and appreciate at least some of its concrete, everyday uses. Many see that it offers a cross-cultural language and set of tools vital for the natural sciences, engineering and technology, economics, business, and finance and that its methods also find application in psychology, the social sciences, law, medicine, and a host of other professions and activities. It is common wisdom that the level of mathematics achieved during school years has enduring consequences, facilitating or impeding lifelong ways of understanding, learning, and communicating. Thus the idea that broad, democratic access to mathematical knowledge serves a valuable social purpose—that of opening the doors of experience and economic opportunity—is not very controversial.

Nonetheless the nature of mathematical power, and the extent to which it is widely achievable, are not so generally agreed on, nor are the processes through which mathematical understanding develops. In my work as an educator and researcher I have been committed to rendering accessible the abstract ideas, language, and algebraic and geometric reasoning methods of mathematics, as well as everyday skills, at least to the large majority of learners. Some would see this goal as fundamentally impossible, citing wide ability differences among individuals or populations. They might argue that what most students—80% or more—are capable of learning is computational arithmetic and consumer mathematics, when these are simply taught and well drilled; with only a small subset—20% or fewer—able to attain real understanding of algebra and geometry at, let us say, the traditional high school level. Some would further characterize the objective of broad accessibility as diametrically opposed to the aim of enabling talented, high-achieving students to accomplish the maximum possible for them.

Many educators whose values lead them to embrace the ideal of near-universal access remain at a loss as to how to achieve it. At the least it is an ambitious and difficult undertaking, one that requires sound models at mathematical, psychological,
sociocultural and political levels and a set of working tools based on them. If the quest
is not futile, some clearer expressions of vision, theory, and method are needed.

Lately this issue has been joined with others in an increasingly vigorous, sometimes
rancorous, conflict surrounding public policy in mathematics education in the United
States and some other countries.¹ Perhaps because it has an egalitarian, antielitist
ring to it, the statement of a “universal access” goal is easily subsumed in belief
systems and rhetorical frameworks that employ popular catchwords but obscure the
nature and central importance of the mathematical concepts, methods, and reasoning
capabilities that constitute the very substance of the goal. The language that is used
to describe mathematical learning and teaching itself entails assumptions that are
increasingly treated as ideological rather than scientific. Then we confront the political
debate that currently surrounds “reform,” without the objective research base that
might resolve it.²

There is a pressing need for a shared, scientific, nonideological framework for
empirical and theoretical research in mathematical learning and problem solving.
The present chapter reflects my view that the constructs of representation, systems of
representation, and the development of representational structures during mathematical
learning and problem solving are important components of such a framework.

To discuss representation, we must be able to consider at a minimum configurations
of symbols or objects external to the individual learner or problem solver, configura-
tions internal to the individual, and relations between them. I regard these basic
notions as essential to characterizing the nature of the patterns that mathematics is
about. They are likewise essential to a psychologically adequate formulation of what
mathematical understanding consists of, and how individuals acquire it. Research
on representation thus involves some external and/or behavioral variables that are
straightforwardly accessible to observation, together with other, internal constructs
that require careful, often context-dependent inference. It can and should draw on
both quantitative and qualitative research methods and assessment instruments, ac-
cording to the desired purpose. The study of representation in mathematical learning
allows us—at least potentially—to describe in some detail students’ mathematical
development in interaction with school environments and to create teaching methods
capable of developing mathematical power. It is thus an important tool in achieving
wide access through public education.

For these reasons, I am greatly encouraged that “Representation” is one of the five
broad “Process Standards” included and elaborated in the National Council of Teach-

Here I would like to put the above ideas into a scientific and philosophical con-
text, relate them to some other perspectives on the nature of mathematical learning
and teaching, and use them in suggesting a meaningful alternative to the current

(1998, p. 119): “Who’s to blame for the math crisis? The answer to this question is very simple: The
National Council of Teachers of Mathematics (NCTM), to whom teachers, curriculum developers, and
administrators have always looked for expert advice, has betrayed us.”

²A pointed and wryly humorous column by Diane Ravitch, Assistant Secretary for Educational
Research and Improvement and Counselor to the Secretary, U.S. Department of Education during the
G.H.W. Bush administration from 1991–1993, contrasts the research-based medical treatment she re-
ceived with the state of educational research. Ravitch writes, “Medicine, too, has its quacks and char-
latans. But unlike educators, physicians have canons of scientific validity…. Why don’t we insist with
equal vehemence on well-tested, validated education research? Lives are at risk here, too.” (Ravitch,
1999). The column formed the basis of a plenary panel in July 2000 at the 24th Conference of the In-
ternational Group for the Psychology of Mathematics Education, where discussants took a variety of
positions on the feasibility and value of achieving validity and reliability in mathematics education
research.
ideolesical debate. The chapter is organized as follows. In the first part I characterize briefly some of the issues in that debate and highlight their roots in earlier or recent theory. My intent is to clarify a few of the nonscientific reasons why some mathematics educators have resisted or considered inadmissible the notion of representation and related constructs—and hopefully, to dispose of these issues in the mind of the reader. In the second part, I explain some of the key ideas based on representation, providing a brief summary of concepts for this Handbook and highlighting how they contribute to a unified perspective in the study of mathematical development. In the third part of the chapter, I use the notion of representation to address one controversial educational issue—the curricular question of abstract mathematics versus mathematics in context—and suggest an alternate point of view.

IDEOLOGICAL DEBATE AND DISMISSIVE EPISTEMOLOGIES

This section, although not addressing representation per se, outlines some of the problems exacerbated by the absence of a suitable, shared theoretical framework in mathematics education research. Behind these problems lie tacit or explicit belief systems, based on epistemologies I have termed dismissive. We shall see why these systems have downplayed, skewed, or disallowed entirely the notion of representation.

Mathematics Education Ideologies

To say that two camps have formed, and to call them traditional and reform, risks great oversimplification and may evoke emotional responses by advocates or opponents. But I see no other way to provide the needed overview.

Let me first describe briefly the two sets of ideas, in language I think many of their adherents would accept, using reasonably well-defined terms. Because the descriptions are idealized, I have not sought to attribute them to particular individuals.

**Traditional Views.** The traditional camp, which includes some leading mathematicians, advocates curriculum standards that stress specific, clearly identified mathematical skills at each grade level. Ideally these are to be developed step-by-step, and then abstracted or generalized in higher level mathematics. This recognizes that much of mathematics is structured hierarchically, with more advanced techniques presupposing mastery and a certain automaticity of use of more basic ones. Arithmetic operations with whole numbers, fractions, and decimals are fundamental at the elementary level, forming the basis of most of the mathematics that follows. Abstract or formal mathematical methods are valued for their power.

Principal attention should be given at all levels to the strength of the curricular content, the correctness of students’ responses, and the mathematical validity of their methods. Standards should be measurable, and standardized achievement tests based on explicit goals should provide the main objective measures of standards attainment. Expository teaching methods are valued, including considerable individual drill and practice to ensure not only the correct use of efficient mathematical rules and algorithms, but also students’ ability to interpret and apply them appropriately. Based on this mastery, more complex mathematical ideas can be successfully introduced.

In this spectrum of opinion, calculator-based work should be deemphasized until computational skills are well established. Children are recognized as differing greatly in mathematical ability, so that some significant numbers of them may not have the capacity to succeed in higher mathematics; for these children, achieving the basics is
especially important. Class groupings should tend to be homogeneous by ability, at least after a certain grade level, to permit advanced work with high-ability students and attention to the basics with slower learners.

Reform Views. The reform camp, including many leading mathematics educators, advocates curriculum standards in which high-level mathematical reasoning processes are central and universally expected. It values students’ finding patterns, making connections, communicating mathematically, and engaging in real-life, contextualized, and open-ended problem solving from the earliest grades, with correspondingly reduced emphasis on routine arithmetic computation. Such learnings are best assessed through open-ended, “authentic,” or alternative assessment methods, and assessed least well through short-answer, standardized skills tests.

Hands-on, guided discovery teaching methods are encouraged that involve exploration and modeling with concrete materials. In this spectrum of opinion, teachers should have children solve problems cooperatively in groups as well as individually, encouraging them to invent, compare, and discuss mathematical techniques as they construct their own, viable mathematical meanings. Contextualized mathematics is valued for its meaningfulness and relevance.

Extensive, early use of calculators and computer technology is seen as desirable, with the goal of pursuing more advanced mathematical explorations and projects unhindered by the limitations of pencil-and-paper computation. It is recognized that children have different learning styles; for example, those who seem to learn routine arithmetic or algebra operations slowly or imperfectly sometimes show surprisingly strong visual, spatial, or logical reasoning ability in less routine mathematics. Thus low expectations may be self-fulfilling, and should be raised. Most often it is thought that children should be grouped heterogeneously to allow interaction among those with different learning styles and characteristics and to achieve greater equity.

Discussion. Which is right? Without the distorting lenses of ideology, it is evident that most of the stated ideas are not contradictory at all, but complementary. In particular, skills and reasoning are not opposites; each involves the other. As a mathematical scientist, mathematics education researcher, university teacher, and organizer of New Jersey’s Statewide Systemic Initiative, I see much of value in both sets of views—and of course would introduce a few essential qualifications. Either set alone is, in my judgment, wholly insufficient.

Some of the statements are open to empirical study. The methods advocated (such as expository or guided discovery teaching, individual or group work, homogeneous or heterogeneous grouping) are likely to be appropriate under the right conditions, and optimized in a reasonable balance that takes into account many variable factors—characteristics of the teacher, the students, the community, the mathematical knowledge to be developed, the problem-solving tasks, school organizational constraints, available resources, and so forth. Good research makes the effects and interplay of such factors explicit, provides useful empirical information, improves on our theoretical constructs, and leads ultimately to generalizable results.

Unfortunately, as so often happens in the political and social arena, the most pure or most radical exponents of a belief system receive on balance the most attention. Rational advocacy of complex solutions to complex problems is drowned out by the noise of sound bites. Thus each camp counts among its most powerful and vocal spokespersons advocates of extreme positions. Each accommodates itself to the willful disregard of contravening evidence and tacitly adopts negative, value-laden terminology for characterizing the views of others. We then move from thoughtful, research-based consideration of difficult problems, with possibly complex ideas for solving them, to a state of ideological and political conflict.

Consider, for a moment, some of the extremes.
9. REPRESENTATION IN MATHEMATICAL LEARNING

**Ideological Poles.** On the one hand some traditionalists, at least tacitly, define mathematical knowledge to be that which is measured by the standardized tests they favor and mathematical ability to consist exclusively in students’ accuracy of response under timed test conditions. With these definitions, the only acceptable interpretation of meaningful understanding, “real” achievement, talent, or educational merit is to be found in high test scores—speed and accuracy become the outcome observables. Quantitative measures are admissible, whereas qualitative ones are not. The main way for less talented students to achieve speed and accuracy on traditional tests is through the discipline of systematic drill in the skills to be tested. The admissible empirical evidence demonstrates, then, that drill focused on testable skills raises scores. Inclusion of the long division algorithm in the core of any proposed curriculum becomes a quick litmus test for its mathematical soundness, whereas calculators are to be banned entirely from the lower grades.

Opponents are stereotyped, in their ideas as well as personally. Open-ended exploration of any but the most directed kind is called “fuzzy mathematics.” Those who favor guided discovery learning are accused of valuing all children’s responses equally and of devaluing or negatively valuing correct answers. Those who use calculators are said to want to “dumb down” the curriculum. Heterogeneous grouping of any kind is regarded as denial of ability differences, and equity concerns are dismissed as “political correctness.” The term constructivist is generalized to label, without distinction—but with considerable stigma attached—all who might think any of these things. Advocates of reform or educational equity are characterized as mathematically unqualified, ignorant people, holding positions in schools of education that do not value mathematical knowledge or objective research, and caring more about equality of outcome by race and gender than about mathematical achievement.

On the other hand, some reformists define the teaching of mathematical rules and algorithms, with accompanying student drill and practice, as exemplifying—by its very nature—meaningless or rote learning. The placing of value on correct responses, or on the objective validity of mathematical reasoning, is labeled rigid, absolutist, or destructive of children’s natural creativity or inventiveness. Indeed, the very terms correct, objective, or valid are taken as highly objectionable. Quantitative measures are negatively valued and qualitative ones esteemed. The problem of basic skills prerequisites for higher mathematical learning is denied and circumvented rather than addressed. In particular, calculators and computers are seen as having rendered computational skills obsolete.

And opponents are again stereotyped. Exponents of expository teaching methods are characterized as advocates of authoritarianism. Those who seek the highest levels of achievement by the most capable students must be elitists—spokespersons for class, race, or gender privilege. Those who question calculator use at the expense of learning fundamental mathematical operations are considered Luddites, who place oppressive classroom rituals ahead of modern technology. Abstract mathematics, standardized testing of any kind, formal logical reasoning, or homogeneous class grouping, are deemed per se racist, sexist, or both. Professional mathematicians are stereotyped as an arrogant group of men claiming special access to “truth,” ignorant of schools and their problems, and expressing the narrow values of a white, Western, masculinist culture—one that values abstract rules and theorems at the expense of human beings.

As with most stereotypes, there are (unfortunately) individuals who seem to fit the caricatures drawn by each camp, although the majority of those who care about educational issues do not.

**A Quick Historical Look.** The current “math wars” are, of course, not a wholly new phenomenon. I was educated in the traditional mathematics of the 1940s and 1950s in the United States, with a great deal of memorization, rule learning, and
training in routine problem solving. The pendulum swung. In the late 1950s and 1960s, my younger siblings began to study the “new mathematics,” the product of a mathematician-led movement funded by the U.S. National Science Foundation, to teach concepts and structures rather than procedures (see Sharp, 1964). Topics such as operations with sets, systems of numeration other than base ten, structural properties of number systems, probability, and transformational geometry supplanted the flash cards, the tables of arithmetic facts, and the memorization of rules and algorithms. Pattern-finding and mathematical discovery became valued over rule learning.

This was called a revolution—and there followed, inevitably, the counterrevolution (Kline, 1973; NCTM, 1968, 1970). The “back to basics” movement of the 1970s, intensely critical of the leadership by academic mathematicians, refocused attention on computational skills and rule learning with emphasis on measurable, behavioral outcomes (Mager, 1962; Sund & Picard, 1972). Most of the earlier innovations were discarded, or at most reserved for select student populations, and the mathematics community seemed to withdraw, licking its wounds, from its former leadership involvement in public education.

But the pendulum swung again. Nonroutine problem solving came into fashion in the 1980s, and by the 1990s there had developed in the United States—at least at the level of rhetoric, although not as frequently in practice—a renewed emphasis on mathematical exploration and discovery, group activities, open-ended questions, alternate solution methods, contextualized understandings, and uses of technology (NCTM, 1989). There was a corresponding deemphasis on computational algorithms and on uniform curriculum standards based on them.

Now, in 2001, the restoring force of a second back to basics movement has overtaken the trend. This time, in one of those ironic twists of history, academic mathematicians are at the helm of the traditionalist movement, acclaimed as heroes or denounced as villains according to the ideology of the “true believer.”

**The Role of Dismissive Epistemologies**

Opposing forces over the years have found supporting intellectual bases in the academic research arena. Extreme educational ideologies often draw, tacitly or overtly, on radical theoretical or epistemological “paradigms,” the exponents of which have achieved prominence in part by dismissing—often on a priori grounds—the most important constructs of other frameworks.

To be clear, the frameworks I am terming ideological or dismissive are those for which the system is closed to falsification either by empirical evidence or by rational inquiry or where the fundamental tenets exclude by fiat consideration of the theoretical or empirical constructs of nonadherents.

In the psychology of mathematics education, such schools of thought have come in and gone out of fashion like clothing styles, dependent more on the cultural climate and marketing than on their rational coherence or the empirical evidence for them. This process may be explained partly by the simplistic appeal of all-encompassing constructs, especially those that are sufficiently vague or general as to lend themselves to the popular jargon. The sociology of university-based research in the “soft sciences” appears to favor—with fame, or at least with wide attention—“isms” that distinguish themselves by branding as illegitimate the conceptual entities of rival perspectives. Rarely does the new movement build on or acknowledge what went before. In succession the dismissive theories arise, gain adherents, educate graduate students in their tenets, and after some decades are discarded—not because they are wrong (as a theory in the physical sciences might be abandoned in the face of contravening evidence), but because newer fashions have rendered them no longer in vogue.
This pattern repeats itself, despite the fact that those who study mathematical learning and problem solving from the different perspectives of such theories do want ultimately to understand and explain similar observable phenomena.

**Behaviorism.** One such fashion that provided theoretical support for back-to-basics advocates across roughly four decades has been the psychological school of behaviorism and its subsequent elaboration as neobehaviorism. Founding their movement on the radical empiricist epistemology called positivism (Ayer, 1946), behaviorist psychologists rejected on first principles any incorporation into theory of internal mental states, mental representations or cognitive models, thoughts or feelings, understanding, or information gained through introspection. Because none of these are susceptible to direct, empirical observation, behaviorists claimed that according to the verifiability criterion of meaning asserted by the positivists, none could possibly have meanings beyond the observable behaviors from which they might be inferred. Therefore, they were simply ruled out—the words were forbidden.

Reinforcement of observable stimulus-response (S-R) connections through timely reward, an empirically verifiable phenomenon, was adopted as a nearly all-encompassing mechanism to which learning, including mathematical learning, could be and should be reduced (Skinner, 1953, 1974). Neobehaviorists were somewhat more flexible, accepting constructs built up from “internal responses” to previous responses that could serve as stimuli (allowing chains of S-R bonds, rules, and so forth) and focusing more directly on structures in external environments while continuing to reject all “mentalistic” explanations on first principles.

The statistical methods of psychometrics were compatible with the behaviorists’ insistence on predefined, observable outcomes. Together these provided an academic rationale in the United States for the behavioral objectives approach to mathematics education, combined powerfully with performance-based accountability measures. Legions of mathematics teachers rewrote their schools’ curricular objectives during the 1970s to accord with the approved terminology. Qualitative research was devalued to the extent that it became unacceptable in some journals. Today it is difficult to appreciate how dominant behaviorism became in American mathematics education in this period and how unacceptable were other points of view.

Although the behaviorists claimed to be scientific, their epistemology was not. It is true that earlier in the 20th century, positivism had gained credibility from the need to address through operational definitions the modified concepts of space and time associated with Einsteinian relativity and the problems of measurement raised by quantum mechanics. Successful scientific theories have always relied not only on observable data, however, but also on constructs that are not themselves directly observable but that help to unify empirical observations and provide explanatory or predictive power. This aspect of physics did not change with the advent of relativity or quantum theory; a modern example is the theory of quarks in fundamental particle physics. Furthermore, qualitative and exploratory research have continued to play well-established, essential scientific roles—most apparent in the biological sciences, astronomy, and emerging disciplines of chaos and complexity theory.

The behaviorists’ ban on internal, mental states and related ideas as legitimate constructs was, from the standpoint of sound philosophy of science, a wholly arbitrary one, but it greatly energized back-to-basics advocates. Without the admissibility of internal or mental phenomena, mathematics educators could focus easily on discrete, testable skills but were forbidden to discuss cognitive structures or conceptual understanding. Without the possibility of complex, explanatory models for students’ cognitions, psychometrics—claiming statistical rigor—lent support to the reification of some behavioral patterns as aptitudes, abilities, traits, or general intelligence, and the neglect of other, perhaps more important, indicators.
Challenged by Piagetian developmental psychology, unable to resist the appeal of the information-processing sciences generally or cognitive science in particular, and never able to account for the complexities of mathematical or language learning, radical behaviorism went into decline. There is no opprobrium today in criticizing it within most mathematics education research circles. Rather, it seems trite to do so because few students spend time learning about behaviorism and it is so widely discredited.³

But ideologies rarely become influential without some grains of truth. The important and valid reasons that fueled the ascendance of behaviorism were, in its rejection, also largely forgotten. One of these was a prior reliance on inadequate or overly simplified mentalistic constructs as psychological explanations, where the process of inferring these had no scientific reliability or validity. A related reason, perhaps more important for us today in mathematics education, was the tendency of psychology to lose touch with its scientific, empirical foundations, to mistake values for evidence, and to overgeneralize from anecdotal reports and clinical interviews.

**Radical Constructivism and Social Constructivism.** A second fashion, one that has fueled the reform movement in mathematics education since the 1980s and remains current in mathematics education research circles, is radical constructivist epistemology and its offshoot, radical social constructivism (cf. Confrey, 2000; Ernest, 1991; von Glasersfeld, 1990, 1996). In contrast to the behaviorists, who barred internal or mentalistic constructs, radical constructivists rejected on a priori grounds all that is external to the “worlds of experience” of human individuals. Excluding the very possibility of knowledge about the real world, they dismissed unknowable “objective reality” to focus instead on “experiential reality.” Mathematical structures, as abstractions apart from individual knowers and problem solvers, were likewise to be rejected. In advocating the (wholly subjective) idea of viability they dismissed its counterpart, the notion of (objective) validity. Thus cognition and learning were seen exclusively as adaptive to the individual’s experiential world, and never in principle as reaching “truths” about the real world. Those who paid close attention to the processes of constructing knowledge during learning and problem solving, but did not accept the radical constructivists’ fundamental denial of the notions of objectivity and truth, were labeled “trivial” or “weak” constructivists.

Radical social constructivists saw mathematical (and scientific) truth itself as merely social consensus and dismissed the possibility of any “objective” sense in which reasoning could be correct or incorrect. This perspective was consonant with the fashionable trend toward ultrarelativism. Because each cognizing individual constructs his or her own knowledge, population studies or empirical investigative methods in education based on controlled experimentation were to be effectively replaced by in-depth case studies—research on human beings could never be replicated because no two individuals or populations could (in principle) ever be shown to be the same.

Radical constructivist epistemology, unlike positivist epistemology, aimed more at challenging the supposed objectivity of science than it did at claiming scientific validity for itself. But it was deeply flawed (Goldin, 1990, 2000a). It offered no explanation of the extraordinary degree to which science and mathematics succeed in permitting

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³However, the dismissal on first principles of notions such as understanding, based on their unobservability, recurs in mathematics education research. For instance Lerman, adopting a sociological/postmodernist perspective, writes, “First, it is high time we abandoned words and phrases such as ‘understanding,’ ‘misconceptions,’ and ‘acquisition of concepts’ in mathematics education. They are useless from a teacher’s and a researcher’s point of view, since they are in essence totally unobservable, and are effectively tools of regulation, since we take it upon ourselves to be the only ones qualified to identify when understanding has taken place” (In Sfard, Nesher, Lerman, & Forman, 1999, p. 85). In this view, the terms are not only epistemologically unsound but morally offensive.
accurate prediction, control, and design, whereas superstitious belief systems do not. If I apply its initial assertions—that cognizing individuals have access only to their worlds of experience and can never have knowledge about the real world—directly to myself, I arrive at a well-known and not very useful solipsism. If I apply it to others as well as to myself, as radical constructivism intended, I simply bypass the problem of how I (having access only to my own experiential world) can validly infer cognition in others. If I can do that, am I not assuming knowledge about a real world in which other human beings and their experiential worlds exist? If I cannot have such knowledge, how can I consistently make assertions about other cognizing individuals and what they may or may not have access to? The response to such objections, asserting not the validity but the viability of knowledge, created a system impervious to argument or evidence. Each belief system was viable for its adherents—and there one had to stop.

The radical constructivists’ ban on objective knowledge begged important questions in the philosophy of science. But in challenging scientific hegemony, it proved a powerful energizing force for reform advocates intent on overthrowing behaviorist ideology in mathematics education. No longer were there right answers in mathematics, only more viable or less viable constructions, and this appeared to strengthen the legitimacy of researchers’ wanting to study and interpret students’ spontaneous, nonstandard ways of reasoning. Complex, explanatory discussions of cognition, cognitive structures, and conceptual understanding became not just admissible but highly desirable, as long as no objective validity was claimed for them. Mathematics educators could now devalue the objectivity of discrete, testable skills—not based on empirical evidence but on the a priori basis of a philosophical movement.

Although sharp criticism of radical constructivism still invites powerful disapproval in some academic circles, it is becoming clear that the movement as a whole is entering the realm of the passé. And, as in the case of behaviorism, the most important reasons for its ascendancy are also being forgotten—the inadequacy of behavioral measures alone in describing meaningful learning and understanding, the need for complex, cognitive-developmental models to describe and account for mathematical learning and development, the value in complex domains of qualitative as well as quantitative research investigations, the importance of social and cultural variables in understanding learning in classroom contexts, and so forth.

Other Dismissive Theories. These are, of course, not the only examples of dismissive theorizing. For instance, insisting that all thinking must be information processing, some artificial-intelligence-oriented cognitive scientists maintained in effect that theoretical models are impermissibly vague unless they are written as computer code. This lent great legitimacy to descriptions of cognition by readily programmed constructs such as problem-solving search algorithms, whereas thought processes more difficult to simulate were downplayed. Some cognitive theorists maintained for a while that all cognitive encoding should be represented propositionally on a priori grounds of parsimony, thus rejecting any kind of internal imagistic representation (Pylyshyn, 1973). At another extreme, some language theorists seem now to claim that all mathematics is metaphor, attributing the fact that theorems “remain proven” to the stability of metaphor and devaluing the study of formal foundations (Lakoff & Nuñez, 1997).

4Ultrarelativism with regard to the notion of right or wrong in mathematics is not the exclusive province of radical constructivists. From the perspective of embodied cognitive science, Nuñez (2000, p. 19) suggests, “The so-called ‘misconceptions’ are not really misconceptions. This term as it is implies that there is a ‘wrong’ conception, wrong relative to some ‘truth.’ But Mathematical Idea Analysis shows that there are no wrong conceptions as such, but rather variations of ideas and conceptual systems with different inferential structures…”
The theories I have mentioned make their most valuable contributions by focusing attention and study on particular domains of empirical phenomena or particular sets of theoretical constructs—structures of observable behavioral patterns, and their reinforcement (behaviorism); cognitive-developmental processes and subjective experience in the construction of knowledge (radical constructivism); the role of social and cultural processes in knowledge development (social constructivism); the importance and ubiquity of metaphor, especially bodily metaphor, in human language (embodied cognitive science); and so forth. But a single-minded insistence on excluding other phenomena and other constructs, even to the point of the words that describe them being forbidden, is intellectually insupportable. It leads to built-in, unnecessary limitations.

Consilience and Unification

We should learn from the history of progress in the natural sciences that the denial on first principles of the admissibility of one or another kind of construct is rarely fruitful. The need is for a theoretical framework that is not ideological or fashion driven but scientific—in which complex models are permissible, constructs are subject to validation, claims are open to objective evaluation, and conjectures can be confirmed or falsified through empirical evidence.

The idea of the coherence and compatibility of knowledge in different domains, termed consilience and discussed interestingly by Edward O. Wilson (1998), is perhaps useful here. At the most reductionist level, we might come to describe human learning, understanding, and problem solving (including mathematics) biologically, particularly at the levels of genetics, evolution, and neuroscience. But cognitive science, the information sciences, linguistics, and developmental and cognitive psychology all provide different and useful ways to describe knowledge structures and their development, including mathematical knowledge of various kinds, at a more holistic level. The idea of consilience suggests that none of these are fundamentally contradictory. Ultimately, we are likely to discover in detail how higher level constructs are encoded or represented in the brains of thinking human beings.

Although we do not yet know most of the specifics of representation at the level of networks of actual neurons or how the human brain as an organ of the body is encoded and evolved genetically, we can still say a lot about mathematical knowledge structures at the psychological level. To do this, we study patterns in verbal and nonverbal mathematical behavior in controlled or partially controlled task environments, from which we seek to draw increasingly reliable inferences about internal cognitive structures and their development.

The societal level, involving as it does variables descriptive of populations of individuals, culturally normative beliefs and expectations, and so forth, is still more holistic. But descriptions at holistic levels do not preclude or contradict more reduc- tive descriptions (see Hofstadter, 1979). Rather, the former may anticipate the latter descriptions, be consilient with them, and eventually be explained in terms of them—as the theory of evolution proved useful before we understood its basis in molecular biology (thus unifying previously disparate areas of study) or the physical field of thermodynamics became well established prior to its reduction to more fundamental principles through statistical mechanics.

Here I want to advocate a unifying theoretical foundation for mathematics education, one that can accommodate the most helpful and applicable constructs from a variety of approaches, including those discussed above, but without the dismissive aspects. Then it becomes feasible to approach currently debated issues in mathematics education as empirical questions, not ideological ones. For this I think that a framework based on the study of representations and representational systems is of great assistance.
Implications for the Construct of Representation

The abstract notion of representation involves a relation between two (or more) configurations, with one representing another in a sense to be specified. In the concrete context of the psychology of mathematical learning and problem solving, we must be able to consider (a) configurations internal to the individual, presumed to be encoded in the brain but mainly to be described at more holistic levels (such as verbal and syntactic configurations, visual imagery, internalized mathematical symbols, rules and algorithms, heuristic plans, schemata, and so forth); (b) configurations external to the individual, generally observable in the immediate environment (such as real-life objects or events, spoken or written words, formulas and equations, geometric figures, graphs, base ten blocks, Cuisenaire rods, or computer-based microworld configurations); and (c) possible representing relations, existing or potential, that involve the individual (but may be external or internal to the individual).

Evidently, the a priori dismissal by the behaviorists of internal configurations as acceptable constructs renders the very notion of representation in this sense inadmissible. Behaviorists have much less difficulty with relations (such as physical linkage) among different configurations that are external, and therefore observable, as long as the relations themselves involve no questionable internal constructs.

Radical constructivists, on the other hand, are deeply reluctant to acknowledge the admissibility of external representational configurations and structures—the inherent unknowability of the external by the individual forbids their discussion. They have much less difficulty with relations among different internal configurations, however (see von Glasersfeld, 1987, 1996).

The parallels here with traditional and reform views in mathematics education are not accidental. To the extent that we dismiss or deemphasize the internal, we tend to focus by default on students’ easily observed productions—their mathematical skills performance, their achievement of behavioral objectives—without addressing the nature of their mathematical understanding or its development. This imbalance has tended to characterize the traditionalist approach. To the extent that we dismiss or deemphasize the external, we focus on students’ cognitive processes and qualitative conceptual understandings, possibly unreliably inferred, to the exclusion of measurable skills attainment or the validity of their mathematics. This imbalance has tended to characterize the reform approach.

Whichever dismissal one adopts, the notion of representation as descriptive of interaction between the internal and the external is effectively banned. We must now set aside the dismissive epistemologies to proceed with concepts that can unify the understandings reached from disparate perspectives.

SOME CONCEPTS IN THE THEORY
OF REPRESENTATION

“Representation is a crucial element for a theory of mathematics teaching and learning, not only because the use of symbolic systems is so important in mathematics, the syntax and semantic of which are rich, varied, and universal, but also for two strong epistemological reasons: (1) Mathematics plays an essential part in conceptualizing the real world; (2) mathematics makes a wide use of homomorphisms in which the reduction of structures to one another is essential.” (Vergnaud, 1987, p. 227)

Representational Systems

In the most general sense, a representation is a configuration that can represent something else in some manner. For example, a word can represent a real-life object, a numeral can represent the cardinality of a set, or the same numeral can represent a position on a number line. The nature of the representing relation between the one configuration and the other depends must eventually be made explicit. Kaput (1998) termed this sort of definition (Kaput, 1985; Palmer, 1978) an abstract correspondence approach in that we have (for now) left open the types of configurations we are discussing and the nature of the representing relation.

The representing configuration might, for instance, act in place of, be interpreted as, connect to, correspond to, denote, depict, embody, encode, evoke, label, link with, mean, produce, refer to, resemble, serve as a metaphor for, signify, stand for, substitute for, suggest, or symbolize the represented one. It might do one (or more) of these things by means of a physical linkage or a biochemical, mechanical, or electrical production process, in the thinking of an individual teacher or student, by virtue of the explicitly agreed conventions or the tacitly agreed practices of a social group or culture, or according to a model developed by an observer.

Rather than distinguish in some fixed and final way the world of representing configurations from that of represented configurations, the relation may frequently be seen as bidirectional. That is, when one configuration represents another, the latter can often be regarded equally usefully as representing the former. In mathematics, for instance, we may take a Cartesian graph as representing an algebraic equation (by depicting its solution set) or the equation as representing the graph (by encoding a relation satisfied by the coordinates of its points).

Written words, numerals, graphs, or algebraic equations are examples of external representations. To be more precise, let us distinguish specific inscriptions of these that are found in books, or produced by individuals doing mathematics—that is, that can be observed and pointed to—from idealized representational configurations that describe socially agreed-on norms. The latter may be thought of as equivalence classes of inscriptions.

What is the nature of the idealized configurations and the representing relations here? The configurations (e.g., algebraic equations) and relations (e.g., the relation between Cartesian graphs and algebraic equations) became established over a period of time, initially through individual inventions and eventually through shared conventions. These conventions became normative among those doing mathematics and are today encoded in the brains of millions of people who have studied mathematics, enabling us to interact coherently with each other. To trace this in detail, it will be important to have a way of moving beyond external representations to describe what individual students, teachers, or mathematicians are doing internally.

The examples mentioned (words, numerals, graphs, or algebraic equations) illustrate the idea that individual representational configurations rarely can be understood in isolation. Whether we are speaking of mathematical or nonmathematical representations, we find they belong naturally to wider systems. Numerals, for instance, belong to a system of base ten Hindu-Arabic notation, and Cartesian graphs to a system of conventions for associating pairs of numbers with points in the plane by means of orthogonal coordinate axes. Thus it is essential to define the notion of a representational system to which individual representations belong—indeed, to begin with the idea of the system.

**Primitive Components.** The building blocks, or primitive components, of a representational system form a class of characters or signs. I use these terms when the intent is not yet to ascribe to them any further interpretation or representing relation.
These may belong to a well-defined set, such as the characters in a system of symbolic logic, the letters in the Roman alphabet, or the bases in a molecule of DNA. We may also work with partially defined or ambiguously defined entities, such as real-life objects and their attributes or spoken words in the English language. In the domain of mathematics, we may consider concrete signs such as numerals and arithmetic symbols or abstract entities such as vectors; in physics, we have constructs such as velocities or forces.

**Configurations.** A representational system further includes ways of combining the elementary signs into permitted configurations. These may be specified by well-defined rules, such as those for creating well-formed formulas (wff’s) in a symbolic logic, or reasonably well-defined lists, such as written words in standard English dictionaries, or they may again be ambiguously defined, such as arrangements of real-life objects or grammatical sentences formed from English words. Single-digit numerals may be used to write multidigit numerals, numerals and operation signs may form mathematical commands or mathematical equations, and so forth. We have still said nothing about the interpretation of such configurations.

**Structures within Representational Systems.** Typically representational systems have higher, more complex structures—such as networks, configurations of configurations, partial or total orderings on the class of configurations, mathematical operations, logical or natural language rules, production systems, and so forth. Rules for moving from one configuration to another, or one set of configurations to another, may create a directed graph structure. Rules of grammar and syntax permit words, designated as parts of speech, to be combined into sentences. Again, we have the possibility of ambiguously defined structures. In formal logic, inferencing rules permit us to obtain theorems from previously established wff’s. Symbol-manipulation rules in algebra or calculus allow us to obtain new formulas from previous ones or to transform and solve equations.

One sense in which we may speak of the meaning of a representational system’s characters and configurations is with reference only to structures within the system. This is illustrated by an example familiar from elementary logic, in which signs for and, or, and not are taken as undefined, acquiring meaning exclusively through the axioms and inferencing rules that combine them in certain ways. This is a syntactic or structural notion of meaning. It complements and contrasts with the semantic notion where the meaning of a representational system’s characters and configurations inheres in the things outside the system that they signify.

**Conventional versus Objective Characteristics of Representational Systems.** External representational systems for mathematics, from logical systems described by axioms and theorems to notational systems for arithmetic, algebra, calculus, and so forth, begin with shared assumptions and conventions (such as the axioms defining an Abelian group or a vector space or the conventions for constructing graphs in Cartesian coordinates). Such systems are structured by their underlying conventions, and when we consider these to be used correctly, we are referring to conformity with conventional norms. For instance at the elementary school level, there is nothing objectively true about the fact that an expression such as $3 + 4 \times 5$ is evaluated by performing the multiplication before the addition and not by performing the addition first. It is a matter of commonly agreed on notation, open to inventive modification.

On the other hand, once a mathematical system with its rules has been established, the patterns in it are no longer arbitrary. There is an important sense in which they are now present to be discovered in the system. Having assumed the conventional properties
of natural numbers, our base ten notational system, the conventional definitions of addition and multiplication, and the conventional definition of a prime number, it is true that 23 is a prime while 35 is not. We invoke here no metaphysical or Platonic notions of absolute truth. Rather we highlight the important and elementary mathematical distinction between that which is conventional and that which is (objectively) no longer so, once the context of mathematical assumptions is established.

Although the mathematical representations we know have originated with human beings, there is no a priori persuasive argument eliminating the possibility of other intelligent life in the universe developing recognizably similar mathematics in representation of similar external, real-world patterns.

Furthermore, representational systems are here defined quite generally, so that they need not be systems where human beings have invented the configurations or established the representing relations. For hundreds of millions of years, the sequences of base pairs in DNA have encoded in a complex way the amino acid sequences that form protein molecules. Not only protein structures, but the phenotypes of organisms, are represented in DNA through subsequent productions. Human scientists have discovered the patterns and are breaking the code, but this should not obscure the important sense in which the biosystem evolved representational capabilities apart from subsequent human knowledge and description of it.

### External and Internal Representation

To this point our examples have mostly been systems of representation (including idealized, socially constructed systems) external to individual learners or problem solvers. Now we want to consider the internal, psychological representational systems of individuals. Such internal systems include their natural language, personal symbolization constructs, visual and spatial imagery, problem-solving heuristics, affect, and so forth. Let us consider how these may be understood in relation to that which is external (Kaput, 1991).

Evidently, I cannot under normal circumstances observe the internal representations of anyone else directly. Even the extent to which introspection permits me to describe my own internal representations is questionable. The latter is best regarded as an empirical issue to be investigated through research. Rather, the idea that individuals have internal systems of representation is an explanatory theory framed at a certain level of description. We are to infer such representation from what individuals do, or are able to do, under varying conditions—that is, from their observable behavior, which may include interactions with observable external representations in their environments.

For example, observation of grammatically consistent spoken English conversation leads us to infer some internally encoded, structured competencies forming a (difficult-to-describe) internal system of language representation. The individual may be able to articulate some aspects of this system through conscious introspection (e.g., she may explain how certain words are used or why they are used in certain ways). Other aspects, although quite stable, are likely to be inaccessible to such introspection (e.g., the native speaker may not be consciously aware of the grammatical rules she uses, nor be able to express them). The term internal representation as I use it is thus not at all synonymous with an individual’s “world of experience” or “experiential reality,” as radical constructivists employ these terms.

Some sources use the expression mental representation in a way that seems more or less in agreement with what is meant here by internal representation. But to avoid misunderstanding, I want to stress that I am not suggesting—even tacitly—any sort of mind–body dualism (cf. Kaput, 1998, p. 267). My expectation is that internal representations are encoded physically. The more reductionist description at the level of
neurons and their interaction in the brain is not yet known in detail, however, nor is it clear that such a level of description will be directly helpful to mathematics educators.

The creation of shared, conventional (external) representational systems is an important thread in the history of mathematics. Most mathematics teaching involves students learning to interpret such systems and to use them to solve problems. Some are mainly notational and symbolic, whereas others display relationships visually or spatially. Although external mathematical configurations have traditionally been (mostly) static, calculator and computer technology can now link them and allow them to change dynamically (Kaput, 1994). But the formal symbolic notations of mathematics, the visual–spatial number lines, complex planes, graphs, and Venn diagrams, the perceived computer-based microworlds and so forth, are also represented and processed internally. It is the internal level that largely determines the usefulness of such external representational systems, according to how the individual understands and interacts with them.

Thus effective teachers continuously make inferences about students’ internal representations, their mathematical conceptions and misconceptions, based on their interaction with or production of external representations. Sometimes one considers the external to represent the internal (e.g., when a student expresses a relationship he has in mind by drawing a graph). At other times, or even simultaneously, one can take the internal to represent the external (e.g., when a student visualizes what is described by a graph or by an algebraic formula). This again exemplifies the bidirectional perspective mentioned above—and, of course, we must be as specific as possible about the direction and nature of the intended representing relation.

An extremely important aspect is that internal configurations of different kinds can represent each other in many different ways (Goldin & Kaput, 1996). An internal visual–spatial image may, for instance, evoke an internal formula configuration, some kinesthetically encoded action sequences, a problem-solving strategy, verbal phrases, feelings of comfortable familiarity or anxiety, and so forth. One way to explore what is involved in a student's understanding of a mathematical concept is to consider the variety of distinct, appropriate (or inappropriate) internal representations she has formed and to try to describe and analyze the representing relations she has developed.

**Interacting Internal Representational Systems**

To characterize the complex cognitions and affect of individuals, one needs a model or framework that permits the description of internal signs, internal configurations, and higher level internal structures of different kinds. Often it is a matter of convenience whether we choose to regard some such system as a single, fairly complex representational system (i.e., having much internal structure) or to see it as comprised of two or more simpler systems with representing relations among them.

**Types of Internal Systems.** Elsewhere I have described in more detail a model based on five types of mature systems of internal representation (Goldin, 1987, 1992, 1998). This framework was developed as a way to characterize problem-solving competency in mathematics and has also proven useful in the study of learning and conceptual development. It connects in obvious ways to the work of others who have focused in depth on just one or two types of representation or who have focused on learning and problem solving in particular mathematical domains.

My viewpoint is that all five need to be taken as psychologically fundamental, extending earlier “dual code” and “triple code” models (Paivio, 1983; Rogers, 1983; Zajonc, 1980). We have (a) verbal–syntactic systems, which include natural language capabilities—lexicographic competencies, verbal association, as well as grammar
and syntax; (b) imagistic systems, including visual–spatial, tactile–kinesthetic, and auditory–rhythmic encoding; (c) formal notational systems, including the internal configurations corresponding to learned, conventional symbol systems of mathematics (numeration, algebraic notation, etc.) and rules for manipulating them; (d) a system of planning, monitoring, and executive control that guides problem solving, including strategic thinking, heuristics, and much of what are often referred to as metacognitive capabilities; and (e) an affective system that includes not only the “global” affect associated with relatively stable beliefs and attitudes, but also the “local” changing states of feeling as these occur during mathematical learning and problem solving.

Relations of meaning and symbolization among internal configurations of different kinds relate these systems to each other in complex ways. That is, the various systems are to be regarded not as separate and isolated but as continually interacting. These internal relations, together with denotative and interpretative relations between internal and external representations, encode the mathematical meanings of the individual’s cognitive and affective activity.

Until relatively recently, the most neglected of these systems by researchers in mathematics education were the imagistic and the affective; for recent work see DeBellis (1996), DeBellis and Goldin (1999), English (1997), Goldin (2000b), Gómez Chacón (2000), Presmeg (1998), and references therein.

**Stages of Development.** Representational systems are not transcribed from outside into human brains like programs being loaded into computers. Over time, they develop in learners, structured by the presence of prior systems. It is here that processes of construction of knowledge become especially important.

The broad model I bring to such development incorporates three main stages, applicable to each system (and often, to subsystems): (a) an inventive/semiotic stage, in which new internal configurations are constructed and first assigned meaning (Piaget, 1969) with reference to previously established representations; (b) a period of structural development, driven by the meanings first assigned, during which the higher structure of the new system is largely built with the earlier system serving as a template; and (c) an autonomous stage, in which the new representational system “detaches” partly or even entirely from its previously essential relation to the prior system(s) and functions flexibly and powerfully with new or more general meanings in new contexts.

**Representation, Pattern, and Communication**

The word pattern, describing the fundamental object(s) of study in mathematics, is already strongly suggestive of some sort of representation. There is a sense in which patterns may be said to “exist,” apart from particular individuals who may detect them or know them (or who, alternatively, may not notice them). We are then speaking of representational structures that are external to the individuals. I still use the word representational here because the pattern has the capability of evoking, and standing subsequently in a certain sort of meaningful relation to, corresponding internal configurations. This contingency is present when a pattern exists, even if it does not always happen:

... we may say, “Mathematics is the classification and study of all possible patterns.” Pattern is here used in a way that not everybody may agree with. It is to be understood in a very wide sense, to cover almost any kind of regularity that can be recognized by the mind.... A bird recognizes the black and yellow bands of a wasp; man recognizes that the growth of a plant follows the sowing of seed. In each case, a mind is aware of pattern. (Sawyer, 1955, p. 12, [emphasis in original])

There is another important sense, though, in which it is the human individuals or communities of individuals (or “minds”) that invent the patterns or construe
them in or impose them on their experiences of the world. Then we are speaking of representational structures internal to individuals in meaningful relation to the external.

How does such meaningful relation come to be powerful? This is the fundamental question we face as mathematics educators. Seminal work in our field has been based on the idea that children’s mathematical ability can be developed through appropriate interactions with well-designed, carefully structured task representations embodying the desired patterns (Bruner, 1960, 1964; Davis, 1966, 1984; Dienes, 1964; Montessori, 1962, 1963, & 1972). In my view, this takes place through the construction of internal representational systems of the types described above, together with multiply encoded cognitive–affective conceptual schemata across the different systems.

Mathematical power consists not only in being able to detect, construct, invent, understand, or manipulate patterns, but in being able to communicate these patterns to others. Thus we can understand mathematics as language, and look at the development of the various types of internal representational systems expressive of mathematics as language learning—that is, occurring through participation in communication and having structural (syntactic) aspects and representational (semantic) aspects.

Each of the five types of internal systems of representation mentioned above permits the individual to produce a vast array of complex and subtle external configurations that other people interpret meaningfully: (a) spoken and written language; (b) iconic gesture, drawing, pictorial representation, musical and rhythmic productions; (c) mathematical formulas and equations; (d) expressions of goals, intent, planning, decision structures; and (e) eye contact, facial expressions, body language, physical contact, tears and laughter, and exclamations that convey emotion. The richness of the resulting communication is what makes the complexity of human social interaction possible.

Thus we have, at least potentially, consilience of the psychological level of description with the sociocultural level, as well as with the neurobiological level.

Ambiguity and Representation

We have noted earlier that with certain exceptions, ambiguity may be a necessary feature in the characterization of a representational system or in its relation to another system. When ambiguity is present in spoken language or in mathematical communication, contextual information is frequently needed to resolve it. Often this requires that one go outside the original system—in practice, we interpret uncertain mathematical expressions, diagrams, problem statements, and so forth when we have information about the objects and context to which they refer. Furthermore, ambiguity in the relation between two representational systems is sometimes resolved with reference to yet a third system.

In mathematics we are used to improving the power of our reasoning by reducing, or eliminating as far as possible, ambiguities in our formal representations. Thus we typically strive for great precision—careful definitions and statements of assumptions, unambiguous notations, and rigorous and detailed proofs. Paradoxically, the very power and flexibility of some of the representational systems we are discussing seem to depend essentially on the presence of ambiguity. Words in natural language that are highly ambiguous out of context convey meanings flexibly and powerfully in a variety of different contexts. Heuristic processes, problem-solving strategies, or critical thinking techniques—highly structured and powerful in the individual—may require considerable contextual input before they make sense in given situation. Even greater ambiguity—and greater power—may be associated with individuals’ internal emotional states.
Affect as Representation

I close this section with the remark that the notion of affect as a representational system is not such a common one. Usually emotion is seen as a concomitant of cognitive processes. It is of course recognized that the individual’s emotional state can enhance cognition (e.g., through mathematical curiosity) or impede it (e.g., through math anxiety). The view I have taken, and pursued in my joint work with DeBellis (see DeBellis, 1996; DeBellis & Goldin, 1997, 1999), is that affective states involve complex structures, including meta-affect (affect about affect or affect about cognition about affect). These carry detailed, context-dependent, rapidly changing information essential to the doing of mathematics (as well as other human activities, of course). Speaking colloquially, feelings have meanings—sometimes fleeting, transient meanings, and sometimes deeper, more enduring ones.

Affect may encode one’s expectations of the nature of the subjective consequences of approaching a mathematical task. It may carry evaluative information regarding the success or failure of a strategic approach to a problem, up to a certain point in time. It may reflect one’s tacit appraisals of the emotional states (actual, or potential) of other people, with whom one has meaningful relationships connected to mathematics (a teacher, a parent, or a friend). It may indicate whether one is meeting the self-expectations flowing from one’s sense of identity in relation to mathematics. And meta-affect stabilizes belief systems (Goldin, in press).

ABSTRACTION, CONTEXTUALIZATION, REPRESENTATION, AND COGNITIVE OBSTACLES

With the above ideas in mind let us consider an alternate way to frame just one of the issues in the current debate, the question of formal or abstract mathematics (valued for its power in the traditional view) versus mathematics in context (valued for its meaningfulness and relevance in the reform view).

Contextualized Understanding

Let us try to understand the characteristics of in-context mathematics, or more precisely of contextualized understanding of mathematics, from a representational perspective. Familiar contexts are encoded internally as representational configurations in common words, images, formal notations, strategies and operations, and (ideally) comfortable affect. The familiar, or common-sense nature of the internal structures—expectations, contingencies, beliefs, as well as competencies—associated with such a context (see Goldin, 1996) means they are likely to be (a) widely shared, (b) based on everyday experiences that are easily referred to, (c) multiply coded in highly redundant ways, (d) developmentally prior to the mathematics being learned in the given context, and (e) culturally encouraged or reinforced. Then these internal structures serve as the templates for the construction of in-context mathematical representations, which may reasonably be said to encode contextualized understandings.

Example. The “Unknown” in Algebra. For students beginning the study of algebra, the notion of a collection of objects is familiar from experience. It is straightforward to develop the idea that one might have such a collection—for instance, a bag of peanuts—and not know how many objects are in it, perhaps because the bag is closed and opaque, and the peanuts haven’t been counted. The construct “an unknown number of peanuts” can thus be visualized, and the action sequence of opening the bag and counting the peanuts imagined. There are many wider contexts in which such a situation might be set. We now have the possibility of introducing the letter x to stand for
this specified, but unknown, number. The students engage in the semiotic act of taking the prior, contextual representation (of the result of the imagined action sequence of counting the peanuts) to be the meaning of the character $x$ in the representational system of formal algebra.

Evidently, with this representing relation established in the concrete context, quite a few algebraic expressions involving arithmetic operations can be written. Their interpretation makes sense with respect to the contextual template. Thus $x + 5$ means the result of counting the peanuts and adding five more, whereas $6x$ refers to the number of peanuts in six identical bags, and so forth. Another letter, $y$, can stand for a different unknown number of not-yet-counted objects, such as raisins in a box.

The verbal descriptions provide another encoding, increasing the redundancy. Familiar, concrete objects might be used with younger children to serve as an external representational system for connection with these constructs.

Because all this is occurring during the inventive–semiotic stage, in which meanings are initially assigned, it is likely that students taught thusly will come to understand the value of an unknown number, encoded in multiple ways, as the real meaning of $x$ in algebra, or the one meaning that is easy to understand, or even the only meaning that is possible, at least for a period of time. That is, $x$ and $y$ always stand for numerical values (their actual values); they must do so; we just don’t know what these values are. The algebraic understanding to this point is entirely in context.

**A Cognitive Obstacle.** Eventually it will be important to abstract from the initial meanings. A small, straightforward abstraction is to see $x$ and $y$ as symbols that could also stand for other specific, unknown values in other concrete contexts (not just a whole number of peanuts in a bag or raisins in a box). We anticipate no important difficulty in this step. But in developing a powerful algebraic representational system, the students at some point need to interpret the letter symbols as variables. That is, $x$ no longer will stand for a specific unknown number, but will be able to flexibly assume any of the values in some numerical domain. The contextualized understanding is likely to make this cognitive representation quite problematic because the “actual” value of $x$ (which, multiply encoded, served as its semantic interpretation) has disappeared entirely. The context now can result in a cognitive obstacle to the more abstract mathematical understanding. It is constraining the desired representation, and a dramatic breakthrough is needed.

This pattern, where the contextualized representations first assist and then constrain the subsequent cognitive development, is quite common in mathematics.

**Decontextualized Representation**

One reform trend associated with radical constructivist methods has been toward teaching most or all mathematics by fostering students’ in-context reinvention of every mathematical concept. The contextualized mathematics is romanticized and the abstract devalued. This is, in my view, a kind of reaction against the widespread tendency toward teaching mathematics as decontextualized representation.

I suggest this term to describe formal mathematical notations and rules of procedure introduced as syntax without semantics, or rules and methods without context—a practice seen often in traditional teaching. The good intention behind such decontextualized representation is to avoid the contextual constraints, to embody that which is abstract in mathematics. But at best, the result is likely to be the construction of an internal, formal system without semantic connections.

The student may, for instance, learn to move the $x$ to the other side of the equation and give it a minus sign, without understanding what such a step means, why it is
valid, or what it accomplishes. The procedure is formal. The period of structural development for the algebraic notational system with accompanying operations, based on a meaningful representational relation with a prior system, has been bypassed. The student may or may not learn to do some algebra in the form of school exercises (i.e., in the original decontextualized format in which the algebra was practiced). But the system might never come to function flexibly and autonomously, as a bona fide abstraction.

Abstraction and Contextualization Processes

Decontextualized representation is not abstraction. In emphasizing the limitations of the former, I argue also against insisting that all mathematics be in context, especially when the contexts are those that will pose natural obstacles to later abstraction.

The process of abstraction is one that involves reaching the autonomous stage in the functioning of a representational system. This can occur after relations with prior systems (involving some context or contexts) have been established through semiotic acts, and after some structural development of the new system. As starting points, we should use those representational contexts that permit maximum ease of structural development and limit our reliance on those that impose the most difficult constraints. Because most initial contexts eventually create some cognitive obstacles, the process requires the progressive detachment of representations from their initial contexts as structure is built.

New semiotic acts then permit the same, familiar representational configurations to acquire new meanings in new semantic domains. This is the process I would like to call contextualization. It is a kind of complement to the abstraction process and in my view equally important to powerful mathematics. Through contextualization, students learn to construct special cases, to see the particular in the general, to move toward the concrete in a new representational situation, and to take these steps spontaneously and flexibly. Through abstraction, they learn to generalize, to see the general in the particular, to move away from inessential details of the concrete representational situation, and to do these things also spontaneously and flexibly.

In short, the representational perspective permits us to relinquish the idea that mathematics in context is somehow the opposite of formal, abstract mathematics. Instead we identify abstraction and contextualization as complementary representational processes. Both are essential to depth of understanding in mathematics, and developing both in students should be our goal as mathematics educators.

REFERENCES


CHAPTER 10

Teacher Knowledge and Understanding of Students’ Mathematical Learning

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It is widely accepted today that teachers should be aware of and knowledgeable about students’ mathematical learning. It is believed that such awareness and knowledge significantly contribute to various aspects of the practice of teaching. In this chapter we critically examine this commonly held belief.

We begin this chapter by interpreting what one might mean by teachers’ knowledge and understanding of students’ mathematical learning. Then we move to examining possible implications of such teachers’ knowledge on instruction. The third part of this chapter examines the validity of the assumption that teacher knowledge and understanding of students’ mathematical learning is essential for good teaching in light of different theoretical perspectives. The fourth part describes pre- and inservice teacher education programs that focus on different aspects of students’ mathematical learning. Finally, we conclude by suggesting issues for further research.

STUDENT UNDERSTANDING IN MATHEMATICS: WHAT KNOWLEDGE AND UNDERSTANDING DO TEACHERS NEED?

In coining the term pedagogical content knowledge, Shulman (1986) contributed greatly to the initiation of the current discussion of what teachers need to know about students’ mathematical learning. In this term, he referred mainly to “an understanding of what makes the learning of specific topics easy or difficult; the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons” (p. 9). In this part
of the chapter we reexamine this issue and further explore what might be implied by the phrase *students' mathematical learning*. We focus on three aspects that have been in the center of researchers’ attention during the last decades: (a) student conceptions, (b) different forms of knowledge, and (c) classroom culture.

**Students’ Conceptions**

In the last three decades many researchers have investigated students’ mathematical ideas and conceptions as well as their development. Results of these studies show that learning mathematics is complex, takes time, and is often not straightforward (e.g., Bishop, Clements, Keitel, Kilpatrick, & Laborde, 1996; Borasi, 1996; Grouws, 1992; Nesher & Kilpatrick, 1990; Schoenfeld, Smith, & Arcavi, 1993; Smith, diSessa, & Roschelle, 1993). The findings indicate that students build their knowledge of mathematical concepts and ideas in ways that often differ from what is assumed by the professional community. In the following sections we describe several lines of that research: theory building, misconceptions, moving from misconceptions to knowledge, and the role of representations.

**Theory Building**

The attempt to develop a comprehensive theory that describes how students learn specific mathematical domains or concepts is rather rare in the field of mathematics education. A prominent example is the van Hiele theory, the most comprehensive theory yet formulated concerning geometry learning. It was developed by Pierre and Dina van Hiele almost half a century ago (Clements & Battista, 1992; Fuys, Geddes, & Tischler, 1988; Hershkowitz, 1990; Hoffer, 1983; van Hiele & van Hiele-Geldhof, 1959). The theory claims that when students learn geometry they progress from one discrete level of geometrical thinking to another. This process is discontinuous and the levels are sequential and hierarchical. The van Hiele theory also suggests phases of instruction that help students progress through the levels.

Several researchers have approached theory building differently from the van Hiele school. They have attempted to construct theories that are not specific to learning in a certain mathematical domain but rather that suggest general principles. One such approach relates to the acquisition of mathematical concepts (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Davis, 1975; Dubinsky, 1991; Sfard, 1991). This approach suggests that there is a chain of transitions from operational to structural conceptions. Some researchers (e.g., Sfard, 1991) further have claimed that operational conceptions are, for most people, the first stage in the acquisition of new mathematical concepts. A main, related claim is that processes performed on certain abstract objects turn into new objects that serve as inputs to higher level processes.

**Misconceptions**

A much more prominent line of research in the field of mathematics education is the study of errors. Whereas theory-building research focuses on general aspects of students’ learning of mathematics, researchers of this type usually focus on specific “local” concepts. Some researchers engage in describing in detail errors that students make in specific topics. Others explore additional dimensions. In this section we briefly describe two such dimensions: sources of students’ misconceptions and the evolution of misconceptions with age and instruction.

**Sources of Students’ Misconceptions.** Many researchers find the study of students’ errors fascinating. They devote their efforts to revealing possible sources of
common students' errors. We illustrate this by using a widely documented error: the tendency to conjoin algebraic expressions (for example, to write the expression $2x + 3$ as $5x$ or $5$). The literature suggests several different reasons for this tendency. One of them has to do with conventions related to not differentiating between conjoining and adding. For example, Stacey and MacGregor (1994) stated that students may draw on previous learning from other fields to their work with algebraic symbols (e.g., in chemistry, adding oxygen to carbon produces $CO_2$). Tall and Thomas (1991) mentioned that because of the similar meanings of $\text{and}$ and $\text{plus}$ in natural language, it is common for students to consider $ab$ to mean the same as $a + b$ because the symbol $ab$ is read as $a \text{ and } b$ and interpreted as $a + b$.

Another explanation that is often given for this error is that students face cognitive difficulties in "accepting lack of closure" and tend to perceive open expressions as "incomplete" (Booth, 1988; Collis, 1975; Davis, 1975). The latter explanation still leaves room for the question, "Why do students feel this?" A typical justification is that students expect the "behavior" of algebraic expressions to be similar to that of arithmetic expressions. Sometimes they expect a specific answer, that is, a final single-terminated answer (e.g., Booth, 1988; Tall & Thomas, 1991); at other times, they interpret symbols such as $+$ only in terms of actions to be performed, as is usually done in arithmetic, and thus conjoin the terms (e.g., Davis, 1975).

Another, somewhat broader explanation for the same behavior relates to the dual nature of mathematical notations: process and object (Davis, 1975; Sfard, 1991; Tall & Thomas, 1991). In algebra, the symbol $5x + 8$ stands both for the process "add five times $x$ and eight" and also for an object. Often, students grasp $5x + 8$ only as a process to be performed and "add" $5x + 8$ in what seems to them a reasonable way and obtain expressions such as $13x$.

We have stated previously that most of the research on students’ errors aims for detailed descriptions of common mistakes in specific mathematical topics. Many instances of common errors, alternative conceptions and misconceptions are described in the research. On the basis of this volume of documented research, several theoretical frameworks attempt to describe general, underlying sources of students’ incorrect responses. Here we briefly describe one theory, the intuitive rules theory (Stavy & Tirosh, 2000). The essential claim of this theory is that irrelevant, external features of the tasks often determine human responses to mathematical and scientific tasks. For instance, students’ responses to comparison tasks embedded in different topics are often of the type "more A–more B" (Stavy & Tirosh, 1996). One example relates to vertical angles. Studies have shown that when children in grades K to 4 are presented with two vertical angles, drawn with the same length of arms, the equality of the angles appear to them as self-evident. However, when the same children are asked to compare two vertical angles, one drawn with longer arms than the other does, they claim that the angle with the longer arms is larger. This judgment exemplifies the effect of the rule "more A–more B" on students’ responses. In this case the difference between the angles in quantity A (the perceived length of the arms) affects students’ judgment about quantity B (the size of the angles). This and other rules bear the characteristics of intuitive thinking: They appear self-evident, are used with great confidence and perseverance, and alternative responses are excluded as unacceptable. The intuitive rules theory explains numerous incorrect responses and has a strong predictive power.

**Evolution of Misconceptions With Age and Instruction.** Another trend in research on error examination is the evolution of misconceptions with age and instruction. For example, Hershkowitz (1987) and Fischbein and Schnarch (1997) investigated the evolution with age and instruction of basic geometry concepts and probability, respectively. In the Hershkowitz study, subjects were students from Grades 5, 6, 7 and
8, as well as preservice and inservice elementary school teachers. The tasks employed in the questionnaires were taken from the primary school geometry syllabus. In her analysis of errors, Hershkowitz identified several patterns of evolution of misconceptions with age and instruction. An expected pattern is that of errors decreasing with age and instruction. For instance, subjects were presented with several shapes and were asked to indicate those that were quadrilaterals. The findings show a great improvement with age in identifying the nonprototypical examples of quadrilaterals (e.g., concave). A deeper analysis reveals that some of these errors have the same pattern of overall incidence from one grade level to the next, as well as for students and for preservice and inservice teachers. For example, when asked to identify right-angled triangles, students, preservice teachers and inservice teachers had difficulty in the identification of those triangles with perpendicular sides not in the vertical-horizontal (prototype) position. This difficulty decreases with age and experience, but the pattern of errors remains rather stable. A somewhat surprising pattern includes errors that increase with age and instruction. For example, subjects were asked to draw the altitude to one side of several given triangles including isosceles, unequal sided, obtuse-angled, and right-angled triangles. Contrary to what might be expected, the number of subjects who made the error of drawing all altitudes inside the triangle increased with age and instruction.

An example from a different domain is that of the intuitive use of heuristics in probability. In a comprehensive series of studies, Kahneman and Tversky (1972, 1973; Tversky & Kahneman, 1982, 1983) found that when estimating the likelihood of events, people tend to use certain judgmental heuristics. When using the representativeness heuristic, for example, people estimate the likelihood of an event based on how similar it is to the process by which the outcomes are generated. For instance, many people believe that in a family of six children, the birth order sequence BGGBGB (B-boy, G-girl) is more likely to occur than either BBBBGB or BBGGGG. In the first case, the sequence BGGBGB may appear more representative of the expected 50–50 ratio of boys and girls in the population than the sequence BBBBGB. In the second case, the sequence BBGGGG does not appear representative of the random process of having children. When using another heuristic, the availability heuristic, people estimate the likelihood of events based on the ease with which instances of that event can be constructed or called to mind. For example, if a student is asked to estimate the probability of a car accident, the frequency of his or her personal contact with this event may influence the estimation. When studying the evolution with age of the use of these heuristics, Fischbein and Schnarch (1997) found that whereas the incorrect intuitive use of the representativeness heuristic decreases with age, the incorrect intuitive use of the availability heuristic gains greater influence.

From Misconceptions to Knowledge

The early research on mathematics learning viewed students’ errors as flaws that interfere with learning and need to be avoided and as misconceptions that need to be replaced with correct knowledge. A newer trend in the field is the focus on what students know and can do, highlighting the useful and productive nature of students’ limited knowledge and the continuity in knowledge between novices and masters (e.g., Smith, diSessa, & Roschelle, 1993). According to the older trend, researchers focused, for example, on how students unsuccessfully compare fractions such as \( \frac{1}{6} \) and \( \frac{1}{8} \), claiming that \( \frac{1}{8} \) is bigger because 8 is bigger than 6. In the newer trend, Mack (1990), for example, showed that the same students solved problems involving comparison of fractions when the problems were meaningful to them and they were allowed to use their informal knowledge. Moreover, Smith et al. (1993) showed fundamental similarities in characteristics of masters’ and novices’ knowledge about fractions.
For example, both groups tended to construct strategies that were tailored to solving specific classes of problems instead of using the more general school-taught strategies.

**The Role of Representations**

The role of different representations in conceptual understanding has also been the focus of attention in the mathematics education community (e.g., Goldin & Janvier, 1998; Janvier, 1987). A prominent observation in the study of fundamental theoretical and practical issues in the domain of representations is that students often respond differently to mathematical problems that are essentially the same but involve different representations. This was found, for example, in relation to the function concept (e.g., Arcavi, Tirosh, & Nachmias, 1989; Even, 1998; Markovits, 1982), as well as in the context of infinite sets. The latter is reported in Tirosh and Tsamir (1996), who found that students’ intuitive decisions as to whether two given infinite sets have the same number of elements depend largely on the specific representations of the infinite sets in the problems. A numerical–horizontal representation in which the two sets were horizontally situated, one next to the other (e.g., \{1, 2, 3, 4, \ldots\} \{1, 4, 9, 16, \ldots\}), yielded high percentages of “different numbers of elements” responses. Most participating students (about 70%), when presented with this type of representation, argued that the given sets were not equivalent, justifying this assertion by part–whole considerations (i.e., “The number of elements in a set is bigger than the number of elements in its subset”). A numerical–explicit representation, in which the two sets were vertically situated and the corresponding elements in the two sets consisted of the same symbols, with a certain variation

\[
\text{(e.g., } \{1, 2, 3, 4, \ldots\} \\
\{1^2, 2^2, 3^2, 4^2, \ldots\}\text{)},
\]

elicited high percentages of “the same number of elements” reactions (about 90%) accompanied with high percentages of one-to-one correspondence justifications (i.e., “each element in one set can be paired with one element in the other set”). Thus, these two modes of representations of basically the same mathematical task elicited different justifications and led to contradictory solutions.

**Different Forms of Knowledge and Kinds of Understanding**

The notions knowledge and understanding are multidimensional. Different forms of knowledge and various kinds of understanding are described in the mathematics education literature (e.g., instrumental, relational, conceptual, procedural, implicit, explicit, elementary, advanced, algorithmic, formal, intuitive, visual, situated, knowing that, knowing how, knowing why, knowing to). The following section presents a brief description of several of these forms, portraying the main themes.

**Instrumental Understanding and Relational Understanding: A Dichotomy or a Continuum?**

In an extremely influential article Skemp (1978) presented his view on the distinction between two kinds of understanding in mathematics: relational and instrumental. Relational understanding is described as knowing both what to do and why, whereas instrumental understanding entails “rules without reasons” (Skemp, 1978, p. 9). Skemp argued that although instrumental mathematics is easier to understand within its own context, its rewards are more immediate and apparent, and one can
often obtain the right answer more quickly and reliably, relational mathematics has the advantages of being more adaptable to new tasks, being easier to remember and capable of serving as a goal in itself. Skemp further asserted that the kind of learning that leads to instrumental mathematics includes the learning of an increasing number of fixed plans by which pupils can find their way from particular starting points to required finishing points. These plans tell them what to do at each choice junction, but there is no awareness of the overall relationship between successive stages and the final goal, and the learner is dependent on an outside guidance for learning each new plan. In contrast, learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can produce an unlimited number of plans for getting from any starting point to any finishing point within the schema. The more complete pupils’ schemas are, the greater their feeling of confidence in their own ability to find new ways of “getting there” without outside help. These schemas, however, are never completed, and the process of constructing them is self-continuing, independent of particular ends to be reached, and a self-rewarding, intrinsically satisfying goal in itself.

Skemp argued that these two kinds of knowledge are so different that there is a strong case for regarding them as different kinds of mathematics. He opposed to instrumental mathematics, hinting that the term *mathematics* ought to be used for relational mathematics only and raised several, severe problems that could occur when pupils whose goal is to understand instrumentally are taught by a teacher who wants them to understand relationally, or vice versa.

Skemp’s work contributes significantly to the long-standing debate on the relative importance of computational skill versus mathematical understanding and to further investigations and discussions on this issue. For example, Nesher (1986) asserted that the dichotomy between learning algorithms and understanding is both superficial and misleading, arguing that research on mathematical performance does not inform us about the relationship between success in algorithmic performance versus success in understanding nor does it give evidence about the contribution of understanding to algorithmic performance. She also contended that the possibility of teaching for understanding in mathematics without attending to the algorithmic and procedural aspects is questionable. In a similar vein, Resnick and Ford (1981) suggested that memorization of certain facts and procedures is important not so much as an end in itself but as a way to extend the capacity of the working memory by developing automaticity of response. They argued that when certain processes can be carried out automatically, without need for direct attention, more space becomes available in the working memory for processes that do require attention.

Other researchers in mathematics education also question the usefulness of instrumental–relational dichotomy and raise various, related issues. Hiebert and his colleagues (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986), for instance, suggested that both conceptual and procedural knowledge are required for mathematical expertise. They defined conceptual knowledge as knowledge that is rich in relationships. The learning of a new concept or a relationship implies the addition of a node or link to the existing cognitive structure, thus making the whole more stable than before. Procedural knowledge, on the other hand, is a sequence of actions that can be learned with or without meaning. Hiebert and Carpenter (1992) suggested that the relationships between conceptual and procedural knowledge may range from no relationship to a relationship so close that it becomes difficult to distinguish between them.

We have shown that different researchers in mathematics education take different points of view on the dichotomy–continuum issue. Whereas Skemp (1978) assumed a dichotomy between instrumental and relational knowledge, and Nesher (1986) and Resnick and Ford (1981) questioned its usefulness, Hiebert and
Carpenter and other researchers have suggested that absolute classifications are impossible.

**Algorithmic, Formal, and Intuitive Dimensions of Mathematics: Interactions and Inconsistencies**

In several of his numerous writings, Fischbein suggested that any mathematical activity requires the use of three basic dimensions of mathematical knowledge: algorithmic, formal, and intuitive (Fischbein, 1983, 1993). These three types of knowledge are essentially different from the types of knowledge described in the previous section. The algorithmic dimension consists of rules, procedures for solving and their theoretical justifications. The formal dimension includes axioms, definitions, theorems, and proofs. The intuitive dimension is a kind of cognition that comprises the ideas and beliefs about mathematical entities and the mental models that are used for representing mathematical concepts and operations. Intuitive knowledge is characterized as the type of knowledge that we tend to accept directly and confidently as being obvious, with a feeling that it needs no proof. This type of knowledge has an imperative power; that is, it tends to eliminate alternative representations, interpretations, or solutions.

Fischbein argued that these three dimensions of knowledge are not discrete; they overlap considerably. Ideally, these dimensions should cooperate in the processes of concept acquisition, understanding, and problem solving. In reality, though, this is not always the case. Both the formal and the algorithmic dimensions of knowledge can become rote for the students. Often there are serious inconsistencies among students' algorithmic, intuitive and formal knowledge. Such inconsistencies could be the source of common difficulties learners encounter in their mathematical activities, such as misconceptions, cognitive obstacles, and inadequate usage of algorithms.

**Knowing About and Knowing To: Knowing Facts versus Knowing to Act**

A rather frustrating phenomenon, often described by both researchers and teachers, is that students who are known to have all the knowledge that is needed to solve a problem are unable to employ this knowledge to solve unfamiliar problems (see, for instance, Schoenfeld, 1988). In an attempt to explain this phenomenon, Mason and Spence (1999) defined a special form of knowing: Knowing to act in the moment. Mason and Spence described and discussed some traditional epistemological distinctions between sorts and degrees of knowledge of mathematics, including knowing that (something is true), knowing how (to carry out some procedure), and knowing why (having some stories to account for something). They argued that education driven by these three types of knowledge, which constitute knowing-about mathematics, sees knowledge as a static object that can be passed on from generation to generation as a collection of facts, techniques, skills, and theories.

Mason and Spence (1999) contended that knowing about is a distant, detached form of knowledge, exhibited rather than used, and that such knowledge does not automatically develop the awareness that enables students to use this knowledge in new situations. They suggested that a fourth form of knowledge, knowing to act in the moment, is the type of knowledge that enables people to act creatively rather than merely react to stimuli with trained or habituated behavior. Mason and Spence claimed that knowing to requires sensitivity to situational features and some degree of awareness of the moment, so that relevant knowledge is accessed when appropriate. They described the interactions among these four types of knowledge, suggesting that knowing to is the critical form of knowing, the type of knowing students need to engage in problem solving where context is novel and resolution nonroutine.
Classroom Culture

An important issue that has received the attention of the mathematics education community in recent years is classroom culture. This new focus signals a shift from examining human mental functioning in isolation (a characteristic of most of the research described in the previous two sections) to considering cultural, institutional, and historical factors. The mathematics education community increasingly embraces the view that cultural and social processes are integral to mathematical activity.

Pimm (1987), for instance, in his examination of the types of interaction commonly found in mathematics classrooms, demonstrated how, in many cases, teacher questioning is aimed at breaking up teacher monologue, making sure students are listening, and ascribing if the particular student questioned has grasped what is being explained. Correspondingly, Pimm revealed how what might seem at first glance as students answering mathematical questions the teacher asks, actually covers a particular type of classroom communication where students aim at guessing what the teacher has in mind.

To illustrate how such classroom norms are supported, we present an episode observed in an algebra lesson (Robinson, 1993). On the board, the teacher wrote two expressions, one simple and the other complex: \(4a + 3\) and \(3a + 6 + \frac{5a}{2}\). Then he asked the students to substitute a fraction in both expressions:

\[
T: \text{Substitute } a = \frac{1}{2}.
\]

\[
S_1: \text{You get the same result.}
\]

Then the teacher asked:

\[
T: \text{Are the algebraic expressions equivalent?}
\]

The students initiated a debate of this issue among themselves:

\[
S_2: \text{No, because we substituted only one number.}
\]

\[
S_1: \text{Yes.}
\]

\[
S_3: \text{It is impossible to know. We need all the numbers.}
\]

\[
S_4: \text{One example is not enough.}
\]

Clearly the students were engaged, on their own initiative, in a genuine and important mathematical discussion, but the teacher ignored the students’ discussion completely and stated:

\[
T: \text{We can determine—it is difficult to substitute numbers in a complicated expression, and therefore we should find a simpler equivalent expression.}
\]

Although the substitution of \(a = \frac{1}{2}\) in the two given expressions might lead naturally to the conclusion that “we should find a simpler equivalent expression” (as was originally planned by the teacher), this was, by no means, the appropriate response for the discussion taking place in that classroom at that moment. Several negative lessons students may easily learn from such experiences are that their mathematical thinking is not valued and only “what the teacher has in mind” is important, that mathematics does not necessarily make sense, and that the teacher is the sole authority for determining the correctness of answers.

Several mathematics educators (e.g., Ball, 1991a; Cobb, Yackel, & Wood, 1989; Hershkowitz & Schwarz, 1999; Lampert, 1990; Schoenfeld, 1994) have attempted in recent years to support the development of a different mathematical culture in the classroom. One of the main characteristics of the revised culture is the alteration of traditional roles and responsibilities of teacher and students in classroom discourse. These researchers and others (e.g., Arcavi, Kessel, Meira, & Smith, 1998) investigate mathematics learning and knowing in these classrooms. They document and
examine, either explicitly or implicitly, the evolution of behaviors that sustain classroom cultures characterized by social norms, such as explanation, justification, argumentation, and intellectual autonomy, as well as sociomathematical norms (a term coined by Yackel & Cobb, 1996), such as what counts as mathematical explanation and justification and what are mathematically different solutions.

WHAT CAN HAPPEN IN THE CLASSROOM?

It is unreasonable to assume that there is a simple connection between teachers’ knowledge and understanding about students’ mathematical learning and the process of instruction. Rather, when applied in practice, such knowledge interacts with a combination of many factors, for example, knowledge about mathematics and about didactics; self-confidence in knowing mathematics and in knowing to teach; personal theories and beliefs about mathematics, teaching, learning, and students; the nature of student assessment (e.g., external–internal, traditional–alternative); the character of the educational system (e.g., centralized–discentralized, goals for teaching mathematics at school); participating parties (e.g., principal, supervisor, parents, colleagues). Still, the contribution of teachers’ knowledge and understanding about student mathematical learning to their instructional practice cannot be ignored. This is illustrated in the following cases.

Knowing and Not Knowing About “Finishing” Open Expressions

Benny, Gilah and Batia are seventh-grade teachers, participating in a research on teaching algebra (Tirosh, Even, & Robinson, 1998). They are teaching algebraic expressions from the same textbook. Benny’s behavior suggests that he is unaware of students’ tendency to conjoin or “finish” open expressions. He does not mention this issue in an interview when asked to describe students’ difficulties related to learning algebraic expressions, nor does he address it in his lesson plans.

When designing the teaching of simplifying algebraic expressions, Benny plans to provide students with a rule of “adding numbers separately and adding letters separately.” During the lesson he states the rule and keeps repeating it. When an incorrect response is given, he often states that this is wrong and repeats the rule. The following fragment describes what happened in his class when he tried to apply his plan.

Benny writes the expression $3m + 2 + 2m$ on the board and asks, “What does this equal?” He immediately follows with the rule: “Add the numbers separately and add the letters separately.” Then he suggests coloring the “numbers”: $3m + 2 + 2m$ (as if 3 and the other 2 are not numbers), and writes $5m + 2$. A student asks, “And what now?” Another student suggests, “$7m$.” The teacher (rather surprised by this answer) says, “No! $5m + 2$ does not equal $7m$,” and he repeats the rule again, “The rule is: add the numbers separately and add the letters separately” (note that this rule may actually lead to $7m$). Then he gives the students another example and colors the (free) numbers: $4a + 5 - 2a + 7$. The teacher emphasizes the rule by dictating it to the students and asking them to repeat it out loud. The rest of the lesson is devoted to work on similar exercises. The students continue to experience difficulties.

In contrast to Benny, Gilah is aware of students’ tendency to “finish” open sentences. When asked during an interview to mention various difficulties related to the learning of algebra, she specifies, among other things, students’ tendency to “simplify” expressions such as $3x + 4$ to $7x$. She further explains, “Students tend to make the expression as simple as possible. They tend to ‘finish’ it [the expression].” In her opinion, this is the main obstacle in teaching how to simplify algebraic expressions. Therefore, she
planned a comprehensive activity, devoted to acquaintance the students with the notions of like and unlike terms, to be taught before the lessons on the simplification of algebraic expressions. She spent time and effort on teaching and directing the students toward the use of this one specific method. In an interview she claims,

I think that differentiating between like and unlike terms should precede the issue of simplifying algebraic expressions. There is a need to work extensively on the topic of like and unlike terms.

Her introductory activity consists of two main parts. In the first one, identifying like terms, students are told that “like terms are terms that have an identical combination of variables” and they receive a variety of examples of like and unlike terms (e.g., $2x^2$ and $4x^2$, $3ab$ and $6ab$, $5a$ and $6a^2$, $2bc$ and $3ac$, $3ab$ and $−2ba$). Then they practice and discuss identifying like and unlike terms. In the second part of this activity, collecting like terms, students are told that “to simplify algebraic expressions, one can collect like terms.” The students then receive several examples that illustrate how to collect like terms, starting with $4a + 2a = 6a$ and gradually reaching more complicated expressions such as $2xy + 4x + 1.5y + 6xy + y = 8xy + 2.5y + 4x$. The examples are accompanied by written descriptions, which highlight the like terms and the result of their collection. After discussing the examples, the students practice simplifying algebraic expressions by collecting like terms. As the class progresses, Gilah and her students keep referring to the notions of like and unlike terms. They use them to determine if and how a given algebraic expression can be simplified.

Like Gilah, Batia’s lesson planning, her teaching, as well as her interviews, indicate that she is aware of students’ tendency to “finish” open expressions. For example, in her written lesson plan she writes, “I expect difficulties in problematic cases [such as] $2x + 3 = 5x$.” Also, in an interview, when asked about difficulties that students commonly encounter when studying algebraic expressions, she mentions, among other things, the tendency to add $2a + 3$ and get $5a$, stating, “They need to get an answer, it does not seem finished to them.” The following interchange illustrates how Batia uses her knowledge about this common mistake in instruction:

**Teacher:** What is $3 + 4x$?
**Student:** $7x$.
**Teacher:** How about $7$?
**Student:** Maybe?!
**Teacher:** Well, let’s see again. $3 + 4x$. What is the operation between $4$ and $x$?
**Student:** Multiplication.
**Teacher:** So, first we have to determine what $4 \cdot x$ could be. Can we know that?
**Student:** No!
**Teacher:** So, can I first add the numbers?
**Student:** No! OK, I got it.

A main difference between Benny and Gilah and Batia was that Benny was unaware of the students’ tendency to finish open expressions. Consequently, Benny was surprised when his students encountered so many difficulties and his teaching decisions were not related to his students’ problems. In his reflection on the lesson, Benny expressed his dissatisfaction and frustration. He explained that he sensed there was a problem, but he did not understand its sources. Gilah and Batia’s students, on the other hand, seemed comfortable with this notion and rarely made mistakes.

Although coming from different starting points regarding understanding of their students’ mathematical learning, both Benny and Gilah chose to provide the students with a rule. Both teachers used some version of the “collecting like terms” approach,
which is commonly used when teaching simplifying algebraic expressions. Benny started to use this method without taking into account the students’ specific mistake. In his class students seemed unwilling to accept expressions including a + sign (such as $3 + 2a$) as final answers. Gilah, on the other hand, as a way to address the specific students’ mistake devoted an extensive period of time to practicing “collecting like terms” before dealing with simplifying algebraic expressions. Indeed, in her class students seemed to have mastered this skill.

At first sight, Gilah’s awareness of her students’ mathematical learning led to successful instruction. It enabled her to plan her teaching accordingly and to navigate the instruction so that students learned what she intended them to. The long-term implications of such a method on students’ general knowledge and conceptions of mathematics is questionable, however. Gilah’s teaching approach consisted of what Davis (1989) referred to as a course in which the student is asked to perform some fragmentary, individual, small rituals. These skills are presented to students as “rituals to be practiced until they can be executed in the proper, orthodox fashion” (p. 117). We join Davis in his claim that when using such an approach, the student sees no purpose or goal in the activity. “Consequently, the student sees no reason why the ritual is performed in one way and not another.” Davis mocked the theory underlying such didactive approaches that assume “if the students spend enough time practicing dull, meaningless, incomprehensible little rituals, sooner or later something WONDERFUL will happen” (p. 118). Gilah seems to emphasize procedural knowledge only, with no explicit consideration of other kinds of knowledge nor of classroom culture.

Batia, who like Gilah was ready to face classroom situations where students “finish” algebraic expressions, did not choose to use one specific approach. Rather she used her rich repertoire of strategies (of which we presented only one), all of which are characterized by short and quick teacher–student interchanges. In such situations, students rarely interact with each other or discuss each other’s ideas. Batia’s understanding of students’ mathematical learning enabled her to make quick relevant responses to students that took their understanding into consideration. Nonetheless, the nature of the discourse in her class, and her exclusive focus during her interviews on the cognitive development of her students, signal that she did not pay explicit attention to classroom culture.

Attention to Student Ways of Learning and Knowing

The two teachers, Magdalene Lampert and Deborah Ball, to whose work we refer in this section are not ordinary teachers. Both are university professors and experienced schoolteachers whose theoretical and practical knowledge (about mathematics, teaching mathematics, students, the educational system, and related factors) is much deeper and broader than that of the average schoolteacher. The classroom culture in their classes is different from those described in the previous section. They (Ball, 1991b; Lampert, 1990) explicitly explain what classroom culture they are aiming for and consciously encourage their students’ intellectual autonomy and their development of specific social and sociomathematical norms. That is, they pay attention not only to students’ individual learning and cognitive development but also to the development of the classroom culture. For example, in one of the lessons cited (Lampert, 1990), the teacher presented her fifth graders with the problem of finding the last digit of 7 to the fifth power. The students offered three conjectures: 1, 9, and 7. The following excerpt illustrates how the teacher navigated the class discussion and how she encouraged the development of norms such as students are to make conjectures, explain their reasoning, validate their assertions, discuss and question their own thinking and the thinking of others, and argue about what is mathematically true.
Even and Tirosh

Teacher: Arthur, why do you think it’s 1?
Arthur: Because 7\(^4\) ends in 1, then it’s times 1 again.
Gar: The answer to 7\(^4\) is 2,401. You multiply that by 7 to get the answer, so it’s 7 \times 1.
Teacher: Why 9, Sarah?
Theresa: I think Sarah thought the number should be 49.
Gar: Maybe they think it goes 9, 1, 9, 1, 9, 1.
Molly: I know it’s 7, ’cause 7 . . .
Abdul: Because 7\(^4\) ends in 1, so if you times it by 7, it’ll end in 7.
Martha: I think it’s 7. No, I think it’s 8.
Sam: I don’t think it’s 8 because, it’s odd number times odd number and that’s always an odd number.
Carl: It’s 7 because it’s like saying 49 \times 49 \times 7.
Arthur: I still think it’s 1 because you do 7 \times 7 to get 49 and then for 7\(^4\) you do 49 \times 49 and for 7\(^5\), I think you’ll do 7\(^4\) times itself and that will end in 1.
Teacher: What’s 49\(^2\)?
Soo Wo: 2,401.
Teacher: Arthur’s theory is that 7\(^5\) should be 2401 \times 2401 and since there’s a 1 here and a 1 here . . .
Soo Wo: It’s 2,401 \times 7.
Gar: I have a proof that it won’t be a 9. It can’t be 9, 1, 9, 1, because 7\(^3\) ends in a 3.
Martha: I think it goes 1, 7, 9, 1, 7, 9, 1, 7, 9.
Teacher: What about 7\(^3\) ending in 3? The last number ends in . . . 9 \times 7 is 63.
Martha: Oh . . .
Carl: Abdul’s thing isn’t wrong, ’cause it works. He said times the last digit by 7 and the last digit is 9, so the last one will be 3. It’s 1, 7, 9, 3, 1, 7, 9, 3.
Arthur: I want to revise my thinking. It would be 7 \times 7 \times 7 \times 7 \times 7. I was thinking it would be 7 \times 7 \times 7 \times 7 \times 7 \times 7.

(Lampert, 1990, pp. 50–51)

Although the teacher does not respond immediately to every student’s question, statement or conjecture as the previous teachers did, she seems extremely attentive to her students’ mathematical thinking. Occasionally she interjects a clarifying question or remark that propels the mathematical discussion forward while allowing enough room for her students to take a principal role in the discussion.

The episode above might create the impression that such extraordinary teachers always understand their students’ ways of thinking. However, there are features that are inherent in the task of hearing and assessing students’ thinking and learning that make this task very difficult (Ball, 1997). The “Shea numbers” episode (Ball, 1991b) is illuminating in highlighting how complicated and challenging it might be for the teacher to understand students’ mathematical thinking. Ball’s third-grade class talked about even and odd numbers. A student named Benny made the observation that even numbers can be “made” from two other even numbers, e.g., 4 + 4 and 6 + 6. Following this, another student, Shea, commented that he had noticed something special about the number six. He claimed that six could be an odd and even number. He further explained that,

I’m just thinking that it can be an odd number, too, ’cause there could be two, four, six, and two, three twos, that’d make six . . . And two threes, that it could be an odd and an even number. Both! Three things to make it and there could be two things to make it (Ball, 1991b).
Ball, who interpreted Shea’s claim as connected to Benny’s observation, thought that Shea’s point was that two odd numbers could also make an even number. She then explained to Shea that Benny’s observation was not that all even numbers are made up of two even numbers. Rather, as Shea just suggested, some of the even numbers, such as six, can be made up of two odd numbers. However, this was not what Shea suggested. As later became apparent, he claimed that if splitting up fairly into two groups (i.e., an even number) makes an even number, then splitting up fairly into three groups (i.e., an odd number) makes an odd number. According to Shea’s definition, six is indeed both an even and odd number. Viewing sensitivity and attention to students’ thinking as critical attributes of a teacher’s role, and caring about the development of a classroom culture where explanation, justification, argumentation, and intellectual autonomy are norms, Ball eventually, with the help of other students, came to understand Shea’s mathematical thinking.

LEARNING PERSPECTIVES

At the beginning of this chapter, we suggested that it is widely accepted that knowledge and understanding of students’ mathematical learning is important for teaching. This section examines how this assumption fits with three main learning perspectives. Following Greeno, Collins, and Resnick (1996), we focus on behaviorism, constructivism, and situationism perspectives. There are various different versions of each. For our purposes, we present several main features of each that will allow us to clarify our claim that the assumptions about what teachers need to know are interrelated with the learning perspectives adopted.

Behaviorism

Behaviorism focuses on observed behaviors as the only means to study learning. It views knowledge as an organized accumulation of associations and skill components. Learning is the process in which associations and skills are acquired. Learning environments designed according to behaviorist principles are organized with the goal of teachers, the source of knowledge, transmitting efficiently facts and procedural knowledge to students. Usually, the teacher presents correct procedures and provides opportunities for practice. A basic assumption is that any practice of a wrong association tends to strengthen it. Therefore, it is essential to prevent students from making mistakes or from being exposed to errors made by their peers. Consequently, students rarely interact or collaborate with each other. Classrooms are viewed as a collection of individual students. Often, programmed instruction and computer-based drill and practice programs are designed to provide well-organized information and procedural training to each individual student, while taking into consideration his or her correct responses to the small steps of a prescribed course of study.

Constructivism

Piaget, a founder of the constructivist perspective, demonstrated that children often understand mathematical concepts in a way quite different from adults. According to constructivism, children’s knowledge is not only quantitatively different from that of the adult/expert but also qualitatively different. Constructivism focuses on characterizing the cognitive growth of children, especially their growth in conceptual understanding. A basic assumption is that knowledge is not communicated but constructed and reconstructed by unique individuals; that is, knowledge is gained by an active process of construction rather than by passive assimilation of information or rote memorization. Learning is understood as a process of conceptual growth often
involving reorganization of concepts in the learner’s mind and growth in general cognitive abilities, such as problem-solving strategies, and metacognitive processes. Constructivist learning environments are designed to provide students with opportunities to construct conceptual understanding and to foster problem-solving and reasoning abilities. When teaching mathematics, the teacher should form an adequate model of the students’ ways of viewing an idea and then construct a tentative path on which students may move to construct a mathematical idea more consonant with accepted mathematical knowledge.

**Situationism**

This perspective focuses on the situated character of learning and knowing. Rather than asking what kinds of cognitive processes and conceptual structures are involved, the situative perspective asks what kinds of social engagements provide the proper context for learning to take place. Learning is perceived as a process that takes place in a participation framework, not in an isolated individual mind. Learners do not gain a discrete body of abstract knowledge, which they then apply in other contexts. Rather, knowing is viewed as the practices of a community and the abilities of individuals to participate in those practices; learning is the strengthening of those practices and participatory abilities. As a situated activity, learning’s central characteristic is “legitimate peripheral participation,” a process coined by Lave and Wenger (1991). This is a process by which the learner becomes a full participant in the sociocultural practices of a community. Learners are regarded as apprentices and teachers as masters.

In the situationism view, an important part of learning concepts entails learning to participate in the discourse of the community in which those concepts are used. Mathematical learning environments are designed to foster students’ learning to participate in practices of inquiry and reasoning and to support the development of students’ personal identities as capable and confident learners and knowers. Classroom discourse is organized so that students learn to explain their ideas and solutions to problems, rather than focusing entirely on whether answers are correct. Small groups of students interact with each other: They formulate and evaluate questions, problems, hypotheses, conjectures and explanations, and propose and evaluate evidence, examples, and arguments presented by other students. Particular attention is given to those norms of discourse involving respectful attention to others’ opinions and efforts to reach mutual understandings based on mathematical reasoning.

To summarize, both behaviorism and constructivism focus on acquisition of knowledge. The first conceives acquisition of knowledge as transmission, the second as construction. Situationism conceptualizes learning as initiation to a practice and not as “acquisition of knowledge.” In a way, both behaviorism and situationism focus on behaviors, but there is a substantial difference between these two approaches. Behaviorism deals mainly with small units of simple behaviors, whereas situationism deals with large chunks of complex practices. Hence, learning environments designed according to each learning perspective are different. The situative learning environment emphasizes social engagement, whereas the other two address individuals.

**What Teachers Need to Know About Student Learning**

At the beginning of this chapter we presented three main aspects of student mathematical learning: student conceptions, different forms of knowledge, and classroom culture. It is generally agreed on as important for teachers to be knowledgeable about these features. In this section we examine each in light of the three learning perspectives described above.
Knowing About Student Conceptions

Behaviorists state explicitly that it is impossible for anyone (including teachers) to know what goes on in the students’ mind. They direct teachers toward determining the correctness of the students’ responses, not the students’ conceptions (this includes misconceptions). In contrast, the essence of constructivism is what goes on in the students’ mind. Constructivists claim that a main goal for the teacher is to attend to and understand students’ thinking to design appropriate ways to foster knowledge construction. The situative perspective attends to students’ ability to participate in shared mathematical activities.

Knowing About Forms of Knowledge

Behaviorists focus mainly on skills and procedural knowledge. Consequently, this is what teachers are apt to emphasize in instruction. The constructivist perspective emphasizes the development of different forms of knowledge such as conceptual knowledge (including knowing that and knowing why), problem-solving strategies, and metacognitive abilities. Consequently, teachers should be knowledgeable about different forms of knowledge. Knowing to is a central feature of participation. However, because the situative perspective does not concentrate on knowledge, it is questionable whether knowing about different forms of knowledge, as we described them earlier, is relevant for teachers. What might be important is attention to participation in complex activities, which involves the use of different forms of knowledge.

Knowing About Classroom Culture

The versions we have described of both behaviorism and constructivism1 focus on the individual student, not on building a community of learners. In contrast, the latter is the essence of the situative perspective. Teachers represent the community of practice, exemplify valued practices, encourage the development of desired norms, and guide students as they become increasingly competent practitioners.

Navigating Between Perspectives

It is clear that each learning perspective approaches teacher knowing about student learning differently. We join Sfard (1998) in arguing that choosing and being completely loyal to one learning perspective is counter-productive in educational practice. Adherence to one theoretical perspective might seem an advantage because it eliminates confusion and contradictions, but the task of teaching is much too complex to be reduced to clear-cut global principles or applied in all circumstances. We believe that understanding student conceptions, both those documented in the research literature and those known from experience, would assist teachers to adjust instruction to where their students are in their mathematical understanding. Also, it is important for teachers to be aware that knowing mathematics cannot be reduced to one simple form of knowledge. Furthermore, teachers should be aware that classroom culture is inseparable from learning mathematics because learning always occurs in a specific sociocultural environment. Teacher understanding of the interrelations between classroom norms and mathematics learning is essential for designing an appropriate learning environment.

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1Social constructivism does take account of the social aspect of learning, yet it centers on the individual learner in a social context and not on the class as a community.
TEACHER EDUCATION: WHAT AND HOW

Current research and professional rhetoric (e.g., Barnett, 1991; Cobb & McClain, 1999; Even & Markovits, 1993; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; National Council of Teachers of Mathematics, 1991; Rhine, 1998; Simon & Schiffter, 1991) recommend that attention be paid to students’ mathematics learning and thinking in teacher education and professional development programs. This recommendation is based on the view that awareness to and understanding of students’ mathematics learning and thinking are central to good teaching and that such awareness and understanding does not happen automatically. Consequently, the development of such awareness and understanding need to be part of (preservice and inservice) teacher education curriculum. We do not attempt to provide here a survey of programs that adapt such practice. Rather, we limit ourselves to discussion of what it might mean for teacher education to focus on student mathematical thinking and learning. We organize our discussion around the three aspects of student mathematical learning that serve as our foci points throughout this chapter: student conceptions, different forms of knowledge, and classroom culture.

Educating About Student Conceptions

Many teacher education programs center on developing teachers’ knowledge about students’ mathematical conceptions. Some concentrate on teaching specific theories and models of students’ mathematical thinking. Others aim at developing awareness that students often think differently about mathematical concepts than what might be expected. A pioneering project entitled Cognitively Guided Instruction (CGI) has focused on enabling inservice elementary school teachers to understand their students’ thinking by using a specific research-based model of children’s mathematical thinking (Fennema et al., 1996). The researchers presented teachers with a model of children’s thinking about basic addition, subtraction, multiplication, and division word problems. The model distinguishes between several problem types and identifies the relative difficulty of each category. During workshops, teachers learned to recognize differences among word problems, to identify the solution strategies that children might use to solve different problems and to organize these strategies into hierarchical levels of thinking. The findings indicate fundamental changes in the beliefs and instruction of the participating teachers. The teachers’ role evolved from demonstrating procedures to helping children build on their mathematical thinking by engaging them in a variety of problems and encouraging them to talk about their mathematical thinking. Such changes in instruction were later directly related to changes in students’ achievements.

While the CGI Project aims at professional development of elementary school teachers, the Manor Project focuses on the development of a professional group of secondary school mathematics teacher–leaders and inservice teacher educators (Even, 1999a). Part of the Manor Program centers on deepening and expanding the participants’ understanding about students’ conceptions and ways of learning different topics in mathematics. The aim is to assist participants to look at mathematics learning “from the student’s point of view” to examine what might be the meaning of the widespread constructivist claim that students’ ideas are not necessarily identical to the structure of the discipline nor to what was intended by instruction and that students construct and develop their own knowledge and ideas about the mathematics they learn.

In contrast with the approach of the CGI, the Manor Program participants are not provided with explicit research-based models of children’s thinking in specific mathematical topics. Research on student thinking at the level of junior and senior high
school mathematics does not seem to support the existence of such models. Rather, similar to the Integrating Mathematics Assessment (Rhine, 1998) and the Mathematics Classroom Situations (Even & Markovits, 1993; Markovits & Even, 1999) approaches, the aim is for the participants to become acquainted with research-based key features of student thinking in different mathematical topics (i.e., cognitive development and aspects of mathematical thinking in algebra, analysis, geometry, and probability). The purpose is to challenge and expand the participants’ understanding of students’ ways of making sense of the subject matter and the instruction.

The Manor Program focuses on deepening the academic background of the participants and in line with the model proposed by Leinhardt, Young and Merriman (1995), it emphasizes the synthesis of theoretical and practical sources of knowledge. To help the participants become familiar with relevant research literature, a large part of the program includes reading, presentations, and discussions of research articles on students’ mathematical conceptions and ways of thinking and on classroom cultures that support and promote the development of mathematical reasoning.

Participants then are directed to examine the theoretical knowledge acquired from reading and discussing research in the light of their practical knowledge. The participants also are guided to build on and interpret their experience-based knowledge using research-based knowledge. To do so, the participants are asked to choose one of the studies presented in the course and replicate it (or a variation of it) with their own students. Intellectual restructuring depends on deep processing of experiences (Desforges, 1995), which is more likely to occur if the activity requires personal involvement and presenting the ideas and reasoning to others (Chinn & Brewer, 1993). Therefore, the participants are required to write a report that describes the students’ ways of thinking and difficulties and to compare the results with those of the original study.

It appears (Even, 1999b) that for the participants, acquaintance with research in mathematics education via discussion of research articles supports the development of what were initially intuitive, naive, and implicit ideas about student mathematics learning, into more formal, deliberated, and explicit knowledge. Replicating a study further expands theoretical knowledge and helps to develop better understanding of the issues raised and discussed in the articles they read. Redoing a ministudy with real students provides opportunities for examining theoretical matters by particularizing them in a specific context. For example, reading and discussing research contributed to learning in general about how students construct their own knowledge. The ministudy made general theoretical ideas more specific concrete, and relevant, illustrating what the constructivist view might mean in a practical context. By conducting a ministudy with real students, the participants learned that what they thought they knew about their students was not necessarily a good representation of the students’ knowledge and abilities (similar results are reported by Lerman, 1990, and by D’Ambrosio & Campos, 1992). Depending on their background and the specific project they chose to work on, some participants learned that, contrary to expectations, students can successfully work with sophisticated mathematical ideas that seemed too difficult. Others found that even well-planned teaching might not produce the kind of learning they expected.

Educating About Different Forms of Knowledge

A thorough review of preservice teacher education programs and inservice professional development projects suggests that these programs and projects usually do not declare learning about various forms of mathematics knowledge as their principle aim. Many of those programs and projects, however, state that designing opportunities for teachers to develop deeper understandings of the mathematics they are to teach
and enhancing teachers’ understanding of their students’ mathematical thinking are two of their main aims. Thus, although learning about various forms of mathematics knowledge is not listed as an explicit aim of such programs, highlighting instrumental and relational knowledge and procedural and conceptual understanding are implicit goals of many of them. Here we shall briefly describe a 1-year preservice elementary school teacher program, Students’ Thinking About Rationals (STAR), which concentrates on participants’ subject matter knowledge and pedagogical content knowledge of rational numbers (Tiross, 2000). One aim of this program is to familiarize prospective teachers with Fischbein’s (1993) framework of the three basic dimensions of mathematics knowledge (described in the first part of this chapter). It was believed that this framework could support teachers in their attempts to foresee, interpret, explain, and make sense of students’ mathematics learning. More specifically, this framework was introduced, discussed, and used as a means that could assist teachers in their attempts to predict possible students’ mistakes in various rational numbers tasks and to hypothesize about possible sources of given mistakes.

Fischbein’s framework was used on many occasions in the course. We present here one example relating to division of fractions. Participants were presented with four division expressions and were requested to (a) calculate each of these expressions, (b) list common mistakes seventh-grade students might make after completing their studies on fractions, and (c) describe possible sources for each of these mistakes. One of the expressions was $\frac{1}{4} \div \frac{1}{2}$. At the beginning of the course, all participants calculated this expression correctly. Most of them argued that the (only) common mistake students would make is $\frac{1}{4} \div \frac{1}{2} = \frac{2}{4}$, which will originate from a bug in the algorithm (e.g., $\frac{1}{4} \div \frac{1}{2} = \frac{4}{2} \div \frac{1}{4} = 2$). During the course the instructor uses Fischbein’s framework to exemplify that the same error may have other sources. She demonstrates that such a response could derive from the commonly held intuitive belief that in division, the dividend should always be greater than the divisor (and therefore $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2} \div \frac{1}{4} = 2$), from inadequate formal knowledge (e.g., division is commutative and therefore $\frac{1}{4} \div \frac{1}{2} = \frac{1}{2} \div \frac{1}{4} = 2$) or from other sources. By the end of the course most participants were acquainted with Fischbein’s framework and used it to guide their attempts to describe common incorrect responses.

Educating About Classroom Culture

Rarely do preservice or inservice teacher education programs state explicitly that they focus on educating teachers about classroom culture and its role in learning mathematics. Because the focus on sociocultural aspects is relatively new among mathematics educators, it is only natural that most of the emphasis is currently centered on examining sociocultural aspects of student learning and not yet on educating teachers about it. Below we describe some pioneering work in this direction.

In their work with inservice elementary school teachers, Cobb and McClain (1999) emphasized that one of their goals is to help teachers to locate “students’ mathematical activity in social context by attending to the nature of the social events in which they participate in the classroom” (p. 29). These researchers acknowledged the need for teachers to learn about the social aspects of mathematics learning and used episodes from classrooms to serve as a basis for conversations with teachers about the role of the teacher in supporting the development of sociomathematical norms.

Lampert and Ball (1998) work for several years with preservice elementary school teachers towards pedagogical inquiry. Among the various aspects of teaching to which they attended, they designed tasks to help prospective teachers consider classroom culture, stating explicitly that classroom culture is “one of the core dimensions of practice and hence an important idea for prospective teachers to learn” (p. 111). Lampert and Ball created multimedia records of practice; a comprehensive record
of information of various kinds (video and text) about what occurred in the third- and fifth-grade mathematics classes they were teaching during the 1989–1990 school year. Preservice teachers explored the records of practice in the multimedia environment, aiming to identify items that exemplify key elements of the culture of the classroom and formulating conjectures and explanations about the teacher’s role in establishing and maintaining these elements of classroom culture. In doing so, Lampert and Ball treated what constitute classroom culture and how it can be developed in a classroom as content to be learned by prospective teachers. They design opportunities for the prospective teachers to engage in learning this content and to organize their ideas conceptually.

LOOKING TO THE FUTURE

In this chapter we discussed teachers’ knowledge and understanding of students’ mathematical learning. Three main relevant issues are

- What should teachers know and understand?
- How should they learn?
- When should they learn?

In the preceding sections we focused on the first two questions (What? and How?) in light of the information provided by the research literature. Much less is known about the third question, “When,” (e.g., during preservice education? during inservice professional development?). We approached the “What?” and “How?” questions by referring to three aspects: (a) student conceptions, (b) different forms of knowledge, and (c) classroom culture.

There are other issues that need to be examined such as, “What do teachers need to know about these aspects?” and “What are promising ways for teacher learning about them?” For example, regarding student conceptions, a spontaneous solution may be to choose the most salient ones. However, students’ conceptions may differ according to the curricula they study, the classroom practices they experience, and other factors. The extent to which mathematical ways of thinking and difficulties are embedded in a particular approach to learning and teaching still needs to be studied. For instance, it is possible that the tendency to conjoin open expressions will be found only in classes that use the traditional approach to teaching algebra. It might not be found in classes that use curricula that attempt to provide students with a broader context, one in which not completing the expression makes sense, offers some advantage, and does not simply remain another formal exercise.

A similar issue emerges in relation to educating teachers about forms of knowledge. Currently, there is no single theoretical framework that is widely accepted by the mathematics education community. Should such consensus be reached? Should we wait until this line of research is more advanced before we make decisions regarding its inclusion in teacher education? If one feels that we should not wait for more information, decisions should be made regarding which and how many frameworks will be used. In the meantime, we need to obtain more information about the impact of focusing on different forms of knowledge in teacher education.

With respect to research on classroom culture, we feel that the literature does not provide enough critical analyses of problematic aspects, of advantages and disadvantages of adapting the current advocated classroom culture. Missing are analyses that take into account the complexity of actual mathematics instruction that needs to consider various (and sometimes conflicting) factors, facets, and circumstances. Even if we adopt the vision of a desired classroom culture as advocated today in reform
documents (e.g., Australian Education Council, 1990; National Council of Teachers of Mathematics, 1991, 2000), we are still faced with questions concerning “How?”: Is it necessary for teachers to experience a desired classroom culture as learners? Is it sufficient? Do they need to observe such classrooms? Is it enough? Do they need to actually experience teaching in such classrooms as student teachers?

Finally, although we raised many issues in this chapter regarding teacher knowledge and understanding of students’ mathematical learning that still need to be explored, we would like to stress that our research community has made huge progress with respect to this issue in the last decade. This research has advanced our understanding of the complex nature of teacher knowledge in general, of teacher knowledge and understanding about student mathematical learning in particular, and of the interrelations of this kind of knowledge with instructional practice. We look forward to seeing what exciting research this millennium will bring.

REFERENCES


CHAPTER 11

Developing Mastery of Natural Language: Approaches to Theoretical Aspects of Mathematics

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INTRODUCTION

Mathematical theories\(^1\) and the theoretical aspects of mathematics\(^2\) represent a challenge for mathematics educators all over the world. Neither abandoning them in favor of curricula designed to work with “the average student” nor insisting on the traditional methods for teaching of them are good solutions. Indeed, the former represents negligence of on the part of the school in passing scientific knowledge to new generations; the latter is hardly productive and impossible in today’s school systems. This

\(^1\)The expression *mathematical theories* includes the usual theories based on systems of axioms and developed according to the rules of deduction (from Euclid’s geometry to theory of groups); this expression is largely common among mathematicians when they speak about their discipline. In this chapter, we refer to theories taught (or approached) in secondary schools in most countries: Euclid’s geometry, mathematical analysis, elementary theory of probability, and so forth.

\(^2\)The expression *theoretical aspects of mathematics* refers (at a low level of schooling) to reflective activities about concepts (comparison of definitions, making properties explicit, etc.). At higher levels of schooling, it refers to the use of specific systems of signs endowed with their syntactic rules (for instance, the algebraic language), the comparison between different proofs of the same theorem, and so forth.
presents an interesting and important area of investigation to mathematics education research, but on what theoretical aspects of mathematics should the effort be concentrated? Why must they be implemented in curricula? How can we ensure reasonable success with them in classroom activities?

This chapter is based on three categories of studies regarding theoretical aspects of mathematics in school, which we and some of our Italian colleagues have undertaken in recent years: (a) studies concerning the nature of mathematical theories and theoretical aspects of mathematics in a school setting (Bartolini Bussi, 1998; Boero, Chiappini, Pedemonte, & Robotti, 1998; Mariotti, Baronlini Bussi, Boero, Ferri, & Garuti, 1997); (b) studies based on experimental work exploring the possibility of approaching theories in school from fourth to eighth grade (see Bartolini Bussi, 1996; Bartolini Bussi, Boni, Ferri, & Garuti, 1999; Boero, Garuti, & Mariotti, 1996; Boero, Pedemonte, & Robotti, 1997; Douek, 1999b; Douek & Scali, 2000); and (c) studies related to analyses of competencies (and difficulties) of students entering university mathematics courses without the intent of becoming mathematicians (see Ferrari, 1996, 1999).

Despite these studies’ focus on different aspects of mathematics education, mastery of natural language in its logical, reflective, exploratory, and command functions emerges from them as one of the crucial conditions in approaching more-or-less elementary theoretical aspects of mathematics. Indeed, only if students reach a sufficient level of familiarity with the use of natural language in the proposed mathematical activities can they perform in a satisfactory way and fully profit from these activities. The reported teaching experiments also show how teachers must have a strong commitment to increasing students’ development of linguistic competencies by way of producing, comparing, and discussing conjectures, proofs, and solutions for mathematical problems.

Theoretical positions and educational implementations concerning different functions of natural language in the teaching and learning of mathematics are widely reported in mathematics education literature. Communication in the mathematics classroom in particular has received much attention from mathematics educators in the last two decades (cf. Steinbring, Bartolini Bussi, & Sierpinska, 1996). These studies influenced our own research. The contributions of this chapter are intended to join the research on natural language in mathematics education through in-depth analysis of the specific functions that natural language plays in relationship to the theoretical side of mathematical enculturation in school and related implications for education.

Taking all these factors, as well as our studies and their outcomes, our chapter is organized according to a “what, why, and how” schema.

What and Why

We consider theoretical aspects of mathematics that are relevant both to mathematics (as a cultural inheritance) and to education in general, taking into account the needs resulting from the complexity of today’s society. In particular, we consider the following factors.

• Mastery of specialized systems of signs (with their rules and specific features) in mathematical activities, as a prototype of skills that are commonly required in computer environments: From this point of view, algebraic language is not an isolated case, interesting only for mathematics, but one of a wide set of artificial languages that enters different domains of human activity.

• Construction of mathematical objects (concepts, procedures, etc.) and their development into systems in a conscious, gradual, intentional process: Here, we focus
on transmission of mathematics as a system of “scientific” concepts, according to Vygotsky’s seminal work (see Vygotskij, 1992, chapter VI).

- Construction of theories: Since the ancient Greeks, Western “rationality” has depended on reason (e.g., deductive reasoning), through which mathematical knowledge is organized in theories.

Bearing this perspective in mind, we focus on natural language, considered in some of its crucial functions in theoretical work in mathematics. We consider its functions

- as a mediator between mental processes, specific symbolic expressions, and logical organizations in mathematical activities; in particular, we consider the interplay between natural language and algebraic language and the natural language side of the mastery of connectives and quantifiers in mathematics (see Natural and Symbolic Languages in Mathematics).
- as a flexible tool, the mastery of which can help students manage specific languages (“command function”) and which is the natural environment to develop metalinguistic awareness (see Natural and Symbolic Languages in Mathematics).
- as a mediator in the dialectic between experience, the emergence of mathematical objects and properties (i.e., concepts), and their development into embryonic theoretical systems (see Natural Language, Mathematical Objects, and Early Development where we consider natural language primarily within individual or interpersonal argumentative activities).
- as a tool in activities concerning validation of statements (finding counterexamples, producing and managing suitable arguments for validity, etc.; see Linguistic Skills, Argumentation, and Mathematical Proof).

All these functions are relevant for developing theories and theoretical aspects of mathematics because (according to the above Vygotskian perspective about “scientific” concepts) theoretical work in mathematics includes, in particular, managing different systems of signs according to specific transformation rules (first function) and coherence constraints needing metalinguistic awareness (second function); connecting components of a concept as a system (Vergnaud, 1990) and linking concepts into a system (third function); and deriving the validity of a statement from shared premises (fourth function).

How

We consider briefly both the issue of teachers’ preparation, designed to make them aware and competent in enhancing students’ natural language skills, as well as some methodological aspects of classroom activities aimed at promoting these skills. In particular, we consider the problem of teaching mathematics in multilanguage classes.

NATURAL AND SYMBOLIC LANGUAGES IN MATHEMATICS

The Language of Mathematics

In this section, we attempt to clarify the relationships between ordinary language and the specific languages and notation systems of mathematics, with an emphasis on advanced mathematics education. We argue that mathematics learning involves management of different linguistic varieties (registers) at the same time and that some degree of metalinguistic awareness is required to control the notation systems of
mathematics, rather than specific proficiency in single languages (cf. first and second function of natural language as described in Section 1).

By language of mathematics, we mean a wide range of registers that are commonly used in doing mathematics (cf. Pimm, 1987). According to Halliday (1985), a register is “a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings.” In other words, a register is “variety according to use, in the sense that each speaker has a range of varieties and chooses between them at different times.” A thorough investigation on the evolution of Halliday’s definition has been carried out by Leckie-Tarry (1995). To achieve the specific goal of comparing the word component and the symbolic component of mathematical language, the idea of register seems suitable in comparison with other constructs. Advanced mathematical registers share a number of properties with literate registers, whereas to communicate in any classroom, one cannot avoid adopting everyday conversational registers. The differences between all these registers are profound, and mastery of them may require a deeper linguistic competence than the one usually displayed by students. Most mathematical registers are based on ordinary language, from which they widely borrow forms and structures, and may include a symbolic component and a visual one.

In principle, much of mathematics could be expressed in a completely formalized language (for example, a first-order language) with no word component, that is, with no component borrowed from ordinary language. Languages of this kind have been built for highly specialized purposes and are rarely used by people (including advanced researchers) who are doing or communicating mathematics. We argue that ordinary language plays a major function in all the registers that are significantly involved in doing, teaching, and learning mathematics.

**Ordinary Language in Mathematical Registers**

Some difficulties generally arise from the differences in meanings and functions between the word component (i.e., the words and structures taken from ordinary language) of mathematical registers and the same words and structures as are used in everyday life. The difference is not noteworthy in children’s mathematics, in which forms and meanings have been almost completely assimilated into ordinary language, but it grows more and more manifest in the transition to advanced mathematics. The needs of highly specialized languages have characteristics that clearly distinguish them from ordinary ones. For example, in mathematical registers, some words take on meaning that differs from the ordinary meaning, or new meaning is added to the standard one. This is the case of words such as power, root, or function. The use or the interpretation of some connectives may change as well, which implies that the meaning of some complex sentences (e.g., conditionals) may significantly differ from the standard one. Ordinary language and mathematical language also may differ with regard to purposes, relevance, or implications of a statement. For example, in most ordinary registers a statement such as “That shape is a rectangle” implies that it is not a square, for if it were, the word square would be more appropriate for the purpose of communication. This additional information is called an implication of the statement and is not conveyed by its content alone, but also by the fact that it has been uttered (or written) under given conditions. Also, a statement such as “2 is less or equal than 1000” is not acceptable in ordinary registers (because it is more complex than “2 is less than 1000” and conveys less information), whereas it may be quite appropriate in some reasoning processes to exploit some properties of the “less or equal” predicate. In general, a relevant source of trouble is the interpretation of verbal statements within mathematical registers according to conversational schemes (i.e., as it were within standard language). For more examples of this, see Ferrari (1999).
These remarks suggest not only that some degree of competence in ordinary language is required in any mathematical register, but also that working with different mathematical registers may require something more on the side of metalinguistic awareness, to manage the transition between the different conventions. The idea of metalinguistic awareness has been applied to the exploration of the interplay between language proficiency and algebra learning by MacGregor and Price (1999). In their paper, they focus on word awareness and syntax awareness as components having algebraic counterparts. In our opinion, it is necessary to consider a further component of metalinguistic awareness, namely the awareness that different registers and varieties of language have different purposes.

Ordinary Language and Algebraic and Logical Symbolisms

The relationships between the word component and the symbolic component in mathematical registers are not only complex but have been developing through the years as well. For many centuries, suitable registers of ordinary language have been the main way of expressing fundamental algebraic relationships. The invention of algebraic symbolism has provided us with a powerful, appropriate tool for treating algebraic problems and for applying algebraic methods to other fields of mathematics and to other scientific domains (physics, economics, etc.). A widespread idea among mathematics teachers is that algebraic symbolism, once learned, is enough to treat a wide range of pure and applied algebra problems. Moreover, it is also a common belief that students’ symbolic reasoning skills develop first, with the ability to solve word problems developing later. For evidence regarding this theory, see Nathan and Koedinger (2000), who outlined the SPM (symbol precedence model), which induces a significant number of teachers to fail in predicting the behaviors of two groups of high school students dealing with a set of algebra problems. We argue that all the opinions that underestimate the role of natural language in learning do not fit the actual processes of algebraic problem solving and will give both theoretical reasons and experimental evidence.

Mathematically speaking, algebraic symbolism can be regarded as (part of) a formal system designed to fulfill specific purposes, among which we mention the opportunity of performing computations correctly and effectively.

Even if in the past algebraic symbolism was introduced as a contraction of ordinary language, and in some cases (such as “3 + 5 = 8” and “three plus five equals eight”) it may be treated like that, there is plenty of evidence suggesting that the relationships between ordinary language and algebraic symbolism are more complex.

First of all, algebraic symbolism has a very small set of primitive predicates (in some cases, the equality predicate only); this requires the representation of almost all predicates in terms of a small set of primitives; for example, in the setting of elementary number theory, the predicate “x is odd” does not have a symbolic counterpart and cannot be directly translated; its symbolic representation requires a deep reorganization resulting in an expression such as “y exists, such that x = 2y + 1,” that, in addition to the equality predicate, involves a quantifier and a new variable that does not correspond to anything mentioned in the original expression. In other words, there are plenty of algebraic expressions that are not semantically congruent (in the sense of Duval, 1995 to 1991) to the verbal expressions they translate. The lack of semantic congruence may induce a number of misbehaviors such as, for example, the well-known reversal error (for a survey and references see Pawley & Cooper, 1997).

Another major source of trouble lies in the fact that ordinary language employs a lot of indexical expressions (such as “this number,” “Maria’s age,” “the triangle on the top,” “the number of Bob’s marbles”) that are automatically updated according to the context but are not available in algebraic symbolism. So, in a story in which at the
beginning Bob has 7 marbles and then he wins 5 more, the expression “the number of Bob’s marbles” automatically updates its reference from 7 to 12. The same would not happen for a mathematical variable: if, at the beginning of the story, one defines \( B = \) the number of Bob’s marbles, then at the end, Bob has \( B + 5 \) (and not \( B \)) marbles. These features of algebraic symbolism have been recognized as sources of analgebraic thinking (Bloody-Vinner, 1996).

These peculiar characteristics of algebraic symbolism constitute its main strength (because they allow algebraic transformations, i.e., the possibility of transforming an algebraic expression in such a way to both preserve its meaning and to produce a new expression that is easier to interpret or useful to suggest new meanings). But it also shows the intrinsic limitation of algebraic language in comparison with natural language: algebraic language can fulfill neither a reflective function nor a command function. In other words, the structure of symbolic expressions and the fact that symbolic translations of verbal expressions are not semantically congruent to them, even though they preserve reference or truth value, imply that they cannot be used to organize one’s processes or to reflect on them because the nonreferential and non–truth-functional component of their meaning (such as, for example, connotation or the use of metaphors), which often are crucial in reasoning, are lost. Also, the use of indexicals that update their meanings according to the context is a fundamental tool for reflection and control, as in the following sentence:

The number we have is undoubtedly divisible by three and by two because it is the product of three consecutive numbers and therefore (looking at the sequence of natural numbers) one of them is even, one of them is a multiple of three. We could generalize this property to the product of any assigned number of consecutive natural numbers.

For further reflection on the relationships between natural language and algebraic language from the perspective of approaching algebra in school, see Arzarello (1996).

Another case in which many teachers would prefer to make a prevalent (possibly, exclusive) use of a specialized language is the case of quantifiers, especially in the approach to mathematical analysis. Definitions and proofs are frequently written with an extensive use of the symbols for quantifiers, up to the level of a kind of pseudo-logic formalization, as in the following example (definition of continuity at \( x_0 \); see Fig. 11.1).

\[
\forall \varepsilon > 0, \exists \delta: |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon
\]

FIG. 11.1. Current writing of the definition of continuity.

The student example in Fig. 11.1 is strongly encouraged as well. In this case, the necessity of using natural language depends on the links through natural language that we need to establish between the logic structure of a statement and its interpretation in the given content field. We can consider the following examples. When approaching mathematical analysis, many students consider a “decreasing” function to be the opposite of an “increasing” function, and they define the negation of the existence of a limit when \( x \) approaches \( c \) (“for every \( t > 0 \) a positive number \( d \) can be found, such that . . .”) in this way: “for no \( t > 0 \) a positive number \( d \) can be found, such that . . .”. The fact that “increasing” covers only one part of the functions that are not decreasing or that “for no \( t > 0 \)” covers only one part of the opposition to “for every \( t > 0 \),” could be a matter of symbolic transformations of formal expressions, performed according to syntactic rules. Unfortunately, in this way, novices lose all contact with meaning. Furthermore, it is not easy to move from pseudo-logic formalizations such as the one shown in Fig. 11.1 to syntactically correct expressions needed for symbolic transformations. An alternative possibility is to explain the mistake with the help of
common life examples; natural language is the necessary mediator for this kind of explanation not only because the presentation of common life examples requires it, but because translation from one situation to the other and related reflective activity need natural language as a crucial tool.

**Some Experimental Data**

Let us come now to experimental data concerning some of the issues discussed in the previous sections. We report some evidence on the role of natural language in the solution of mathematical tasks we have collected from groups of freshmen students. The choice to present data at the university level seems most significant to our goals because at that level students are widely required to understand, use, and coordinate highly specialized mathematical registers, including symbolic systems such as algebraic and logical symbolism. Our evidence shows that there are ordinary language skills (related to the first function in the list of the first section of this chapter) that are well correlated with students’ performances in algebraic problem solving tasks. We have also found evidence that superficial use (i.e., with little metalinguistic awareness; second function) of natural language in mathematical work (according to everyday life linguistic conventions, such as conversational schemes and so on) can conflict with specific semantics and conventions in mathematics, resulting in student failure.

**Example 1.** The first experiment was carried out in 1994–95 and involved 45 freshman computer science students at the Università del Piemonte Orientale. The students had been offered an optional entrance test. One of the problems was a simple middle school (or lower high school) arithmetic problem (a version of the well-known “king-and-messenger problem”):

A king leaves his castle with his servants and travels at a speed of 10 km a day. At the end of the first day, he sends a messenger back to the castle to be informed of the queen’s health. The messenger, who travels at a speed of 20 km a day, goes to the castle and departs immediately with the news. When the messenger overtakes the king, who keeps traveling at 10 km a day?

Students were asked to explain their answer, but there was no explicit mention of a written text. Nevertheless almost all of them wrote down an argument in words (sometimes with the addition of diagrams or other graphics). The papers were roughly classified according to the kind of language adopted in the arguments. Three levels were identified:

- **Level 0 (L0):** no verbal comment, rambling words, poorly organized sentences (10 students)
- **Level 1 (L1):** well-organized, semantically adequate, simple sentences; few compound and no conditional sentences (24 students)
- **Level 2 (L2):** a good number of well organized, semantically adequate compound sentences, including conditional ones (11 students)

Thirteen students left university during October (after 2–3 weeks of university courses); among them, eight were L0, and five were L1. Twenty-five students passed the algebra exam before May 1995; among them, 11 were L2, and 14 were L1. No L0 student passed the algebra exam before May. In other words,

- all students who passed the algebra exam before May were L1 or L2;
- all L2 students passed the algebra exam before May; and
- gifted students were almost equally distributed between L1 and L2.
Students in L2 seemed to have a sort of insurance against failure. Nevertheless, it was not a necessary condition for proficiency in algebra. By comparison, the L0 seems to be a sufficient condition for failure. For more details see Ferrari (1996).

The following year, a similar experiment involving two tests was carried out with another group of freshman computer science students at the same university. For the first test, a problem analogous to the one assigned the previous year was given, the instructions were exactly the same, and answers were classified according to the same criteria. After 1 month, another problem of the same type was given, but on that occasion we asked students to write an explanation of their answers. Thirty-seven students took part in both the tests. Table 11.1 shows the outcomes.

It appears that the formulation of the task influenced students’ linguistic behaviors. It is noteworthy that although the results of the first test were well correlated with the results of the final examinations (although not as well as in the 1994 test), the results of the second test were not. The significant increment of the number of students classified as L2 suggests that a good share of them possess some academic linguistic skills but normally do not use them if not asked to do so. It seems that what is crucial is not so much the ability to produce high-quality texts if prompted, but the ability to use language as a tool in various situations. Notice that the first case students learned (or acquired) some knowledge or skills about language as a subject, but did not use it as a semiotic tool (i.e., as a tool to use to represent ideas) to think about the problems and to communicate with others (or with oneself). In the second case, students seemed to show some form of metalinguistic awareness because they used language to represent their thought processes and were able to make choices about how to communicate information partly expressed in another register. These results, if confirmed, have strong implications for the teaching of both language and mathematics.

**Example 2**

Let us now consider a conflict between ordinary language and mathematical language. Situations of this kind may occur often, especially at the advanced level. In the following problem, we report, as an example, the case of two terms designating the same referent (an unusual situation in everyday-life registers):

Problem: Is it true that the set $A = \{ -1, 0, 1 \}$ is a subgroup of $(\mathbb{Z}, +)$?

There are students who claim that $A$ is closed under addition and show that if an element of $A$ is added to another (different) element of $A$, the result belongs to $A$. They do not take into account the case $x = y = 1$ nor $x = y = -1$, the only evaluations that lead to discover that $A$ is not closed under sum (i.e., that sum is not a function from $A \times A$ to $A$). In other words, they misinterpret the definition of subgroup. This happens despite students’ knowledge of different representations (such as addition tables) that point out that an element can be added to itself. Moreover, if explicitly prompted, they seem aware that each of the variables $x$, $y$ may assume any value in $A$. Most likely, they are hindered by the need to use two different variables to denote the
same number, which does not comply with the conventions of ordinary language according to which different expressions (in particular, atomic ones) usually denote different things. Notice that phenomena of this kind involve the pragmatic dimension of languages because they concern language use more than the simple interpretation of symbols (students seem aware that each of the symbols $x$, $y$ may denote any element of $A$ but nonetheless use them to denote pairs of different elements only).

**Some Comments**

The experimental data we have presented can be interpreted in a unified way if we accept an analysis of the “language of mathematics,” taking into account the role of natural languages and avoid overvaluing the role of specific symbolism.

First of all, we remarked that students do not use languages flexibly; they often do not apply their linguistic skills to the resolution of problems (Example 1) or apply conversational schemes improperly (Example 2). Thus, a first goal for mathematics education that comes out from our evidence is the need to teach not only languages (from ordinary to specific symbolic ones), but also the flexible use of them. In this perspective, ordinary language (which is far more flexible than specific mathematical symbolisms) should play a major role.

Second, we remarked that students often lose their contact with meaning. As shown in Example 2, the meaning of the definition of subgroup (as far as it could be grasped through alternative representations or students’ knowledge or their previous mathematical experience) is neglected and a stereotyped interpretation is adopted. In this regard, ordinary language, as a reflective and command tool in the interplay between semantic and syntactic aspects of algebraic and logic activities, could help students in keeping themselves in touch with meanings. Of course, ordinary language alone cannot guarantee meaningful learning, but it can act as mediator between everyday experience and the specific needs of mathematical thinking, in particular, the need to interpret and apply patterns of reasoning that are different from the customary ones. Natural language is designed to represent a wide range of everyday-life meanings and patterns of reasoning and embodies most of them, but it is also a flexible tool that can be used to express different meanings (for example, truth-functional semantics) and different logics (for example, mathematical logic). Thus, a flexible mastery of ordinary language (which cannot be achieved by means of everyday-life experiences alone but should include scientific communication) should be a necessary step to mathematical proficiency.

The available data suggest one main educational implication: the need for developing mastery of natural language in mathematical activities as the key for accessing control of algebraic problem-solving processes. These data stress also the necessity of considering opportunities and limitations of that particular, specialized “mathematical verbal language” that includes mathematical symbols, peculiar and often stereotyped mathematical expressions (“for every $t > 0$ a positive number $d$ can be found, such that . . . , etc.”) as a mediator between the flexibility of ordinary language and the specific needs of mathematical activities.

**NATURAL LANGUAGE, MATHEMATICAL OBJECTS, EARLY DEVELOPMENT INTO SYSTEMS**

Recent literature has widely considered the issue of the constitution of mathematical knowledge through language according to different theoretical orientations; in particular, for reasons of proximity with our own research, we may quote Sfard’s constitution of mathematical objects through linguistic processes (Sfard, 1997) and
the “grounding metaphors” theoretical construct by Lakoff and Nunez (1997). In this section, we consider a specific issue: the role of natural language in the constitution of mathematics concepts through argumentation (i.e., we consider some aspects of the third function of natural language, as described in the first section).

**Concepts**

Vergnaud (1990) defined a concept as the system consisting of three components: the reference situations, the operational invariants (in particular, theorems in actions), and the symbolic representations. This definition can be useful in school practice because it allows teachers to follow the process of conceptualization in the classroom by monitoring students’ development of the three components. From the research point of view, Vergnaud’s definition raises some interesting questions, in particular, questions about the progressive constitution of the three components: How does an experienced situation become a “reference situation” for a given concept? What are the relationships between the acquisition of the symbolic representations of a given concept and the construction of its operational invariants?

In the Vygotsky elaboration on “scientific” concepts (see Vygotsky, 1992), consciousness, intentional use, and developing concepts into systems are considered crucial features of “scientific” concepts. Vygotsky’s elaboration suggests other questions: How can consciousness about symbolic representations and operational invariants of a given concept be attained (as a condition for their appropriate, intentional use in problem solving)? How can concepts be developed into systems?

As we will see in the next subsections, the study of argumentation (i.e., the use of natural language in argumentative activities) in relationship with conceptualization can play an important role in tackling the above-mentioned questions.

**Argumentation**

Research about argumentation has been strongly developed over the last four decades. Different theoretical frameworks have been proposed, based on different research perspectives: from the analysis of the pragmatics of argumentation (argumentation to convince; Toulmin, 1958), to the analysis of the syntactic aspects of argumentative discourse (polyphony of linguistic connectives in Ducrot, 1980). A few studies have considered the specific, argumentative features of mathematical activities; we may quote the cognitive analysis of argumentation versus proof by Duval (1991) and more recent studies about the interactive constitution of argumentation (Krummheuer, 1995) and interactive argumentation in explanation and justification (Yackel, 1998). In our opinion, if we want to consider the role of argumentation in conceptualization, we need to reconsider what argumentation can be in mathematical activities, focusing not only on its syntactic aspects or its functions in social interaction but also on its “logical” structure and use of arguments belonging to “reference knowledge” (Douek, 1999a).

We use the word argumentation both for the process that produces a logically connected (but not necessarily deductive) discourse about a subject (defined in Webster’s dictionary as “1. the act of forming reasons, making inductions, drawing conclusions, and applying them to the case under discussion”), as well as the text produced by that process (Webster’s dictionary: “3. writing or speaking that argues”). The discourse context will suggest the appropriate meaning. Argument will be “a reason or reasons offered for or against a proposition, opinion or measure” (Webster’s); it may include linguistic arguments, numerical data, drawings, and so forth. So an argumentation consists of some logically connected “arguments.” If considered from this point of view, argumentation plays crucial roles in mathematical activities: It intervenes in conjecturing and proving as a substantial component of the production processes (see Douek, 1999c, and the next section); it has a crucial role in the construction of
basic concepts during the development of geometric modeling activities (see Douek, 1998; 1999b; Douek & Scali, 2000).

**Argumentation and Conceptualization**

With reference to Vergnaud’s and Vygotsky’s elaborations about concepts, we consider conceptualization as the complex process that consists of the construction of the components of concepts considered as systems, of the construction of the links between different concepts, and of the development of consciousness about them. The main purpose of this section is to analyze possible functions of argumentation in conceptualization.

**Experiences, Reference Situations, and Argumentation**

An experience can be considered as a reference situation for a given concept when it is referred to as an argument to explain, justify, or contrast in an argumentation concerning that concept. The criterion applies both to basic experiences related to elementary concept construction and to high-level, formal, and abstract experiences. As a consequence of our criterion, to become a reference situation for a given concept, an experience must be connected to symbolic representations of that concept in a conscious way (to become an argument intentionally used in an argumentation). In this way, a necessary functional link must be established between the constitution of reference experiences for a given concept and its symbolic representations. Argumentation may be the way by which this link is established (see Douek [1998, 1999b] and Douek & Scali [2000] for examples and Bernstein, 1996, for a general perspective about recontextualization of knowledge).

**Argumentation and Operational Invariants**

Argumentation allows students to make explicit operational invariants and ensure their conscious use. This function of argumentation strongly depends on teacher’s mediation and is fulfilled when students are asked to describe efficient procedures and the conditions of their appropriate use in problem solving. The comparison between alternative procedures to solve a given problem can be an important manner of developing consciousness. The inner nature of concept as a “system” is enhanced through these argumentative activities: Different operational invariants can be compared and connected with each other and with appropriate symbolic representations, thus revealing important aspects of the system.

**Argumentation, Discrimination, and Linking of Concepts**

Argumentation can ensure both the necessary discrimination of concepts and systemic links between them. These two functions are dialectically connected: Argumentation allows us to separate operational invariants and symbolic representations between similar concepts (for an example, see Douek [1998, 1999b]: Argumentation about the expression *height of the Sun* allows one to distinguish between *height* to be measured with a ruler and *angular height of the Sun* to be measured with a protractor). In the same way, possible links between similar concepts can be established.

**Some Examples**

The examples come from a primary school class of 20 students, a participant in the Genoa Project for Primary School. The aim of this project is to teach mathematics, as well as other important subjects (native language, natural sciences, history, etc.), through systematic activities concerning “fields of experience” from everyday life
(Boero, Dapueto, Ferrari, Ferrero, Garuti, Lemut, Parenti, & Scali, 1995). For instance, in first grade, the “money” and “class history” fields of experience ground the development of numerical knowledge and initiate argumentation skills, as well as the use of specific symbolic representations.

We consider data coming from direct observation, the students’ texts, and videos of classroom discussions.

**Grade II: Measuring the Height of Plants in a Pot With a Ruler**

We analyze a classroom sequence consisting of five activities concerning the same problem: Students had to measure wheat plants (grown in the classroom) in their pot using their rulers. Students had already measured wheat plants taken out of the ground in a field; now they had to follow the increase in time of the heights of plants of the classroom pot. The difficulty was in the fact that rulers usually do not have the zero mark at the edge, and students were not allowed to push the ruler into the ground (to avoid harming the roots). The children had to find a general solution (not one that worked for a specific plant). In particular, they could use either the idea of translating the numbers written on the rulers (by using the invariability of measure through translation), an act we call the **translation solution**, or the idea of reading the number at the top of the plant and then adding to that number the measure of the length between the edge of the ruler and the zero mark (by using the additivity of measures), an act we call the **additive solution**.

The first activity was a one-on-one discussion with the teacher to find out how to measure the plants in the pot. The ruler had a 1-cm space between the zero mark and its edge. The purpose of this discussion was for the students to arrive at and describe solutions. The main difficulties met by students were as follows.

First, students found it difficult to focus on the problem. The teacher, T, used argumentation (in discussion with the student, S) to focus on the problem, as in the following excerpt: (with the help of the teacher, this student had already discovered that the number on the ruler that corresponds to the edge of the plant is not the measure of the plant’s height of the plant):

S: We could pull the plant out of the ground, as we did with the plants in the field.

T: This would not allow us to measure the growth of our plants.

S: We could put the ruler into the ground to bring zero to the ground level.

T: But if you put the ruler into the ground, you might harm the roots.

S: I could break the ruler, removing the piece under zero.

T: It is not easy to break the ruler exactly on the zero mark, and then the ruler would be damaged.

Once focused on the problem, the question remained of how to go beyond the knowledge that the measure line on the ruler was not the height of the plant. Usual classroom practices on the “line of numbers” (shifting numbers or displacing them by addition) had to be transferred to a different situation. A change of the ruler’s status was needed: Instead of a tool for measurement it became an object to be measured or transformed (e.g., cut or bent) by the imagination. With the help of the teacher, most students overcame difficulties using different methods. In particular, some of them imagined putting the ruler into the ground to bring the zero mark to the level of the ground, but because this action could harm the roots, they imagined shifting the number scale along the ruler.

S: I could put the ruler into the ground . . .

T: If you put the ruler into the ground, you could harm the roots.
S: I must keep the ruler above ground . . . but then I can imagine bringing zero below, on the ground, and then bring one to zero, and two to one . . . It is like the numbers slide downward.

Other students imagined cutting the ruler. They were not allowed to do so, so they imagined taking a piece of the ruler (or the plant) and bringing it to the top of the plant.

At the end of the interaction, 9 students out of 20 arrived at a complete solution (i.e., they were able to dictate an appropriate procedure): Four were translation solutions, four were additive solutions, and one was a mixed solution (with an explicit indication of the two possibilities, addition and translation). Example include the following

• Rita’s translation solution: To measure the plant we could imagine that the numbers slide along the ruler, that is, 0 goes to the edge, 1 goes where 0 was, 2 goes where 1 was, and so on. When I read the measure of the plant, I must remember that the numbers have moved: If the ruler gives 20 cm, I must think about the number coming after 20—21.

• Alessia’s additive solution: We put the ruler where the plant is and read the number on the ruler, which corresponds to the height of the plant, and then add a small piece, that is, the piece between the edge of the ruler and 0. But before that we must measure that piece.

Four students moved toward a translation solution without being able to make it explicit at the end of the interaction. The other seven students reached only the understanding that the measure read on the ruler was the measure of one part of the plant and that there was a “missing part,” without being able to establish how to go on.

The second activity consisted of an individual written production. The teacher presented a photocopy of Rita’s and Alessia’s solutions, asking the students to say whose solution was like theirs and why. This task was intended to provide all students with an idea about the solutions produced in the classroom. With one exception, all 13 students who had produced or approached a solution were able to recognize their solution or the kind of reasoning they had started. Six out of the other seven students declared that their reasonings were different from those produced by Rita and Alessia.

The third activity was a classroom discussion. The teacher worked at the blackboard, and the students worked in their exercise books, where they had drawn a pot with a plant in it. They used a paper ruler similar to the teacher’s, effectively putting into practice the two proposed solutions, first the translation solution and then the additive solution. The students discussed problematic points that emerged. In particular, they noted that while the translation procedure was easy to perform only in the case of a length (between the zero mark and the edge of the ruler) of 1 cm (or eventually 2 cm), the additive solution consisted of a method that was always easy to use. Another issue they discussed concerned interpretation of the equivalence of the results the two solutions provided (“why do we get the same results?”).

Here is an excerpt from the discussion, concerning the starting point of the comparison of the two solutions:

Angelo: Rita’s method is similar to Alessia’s method.
Many students: No, no . . .
Ilaria: Rita makes the numbers slide, but Alessia . . . she does not make it slide . . . The two methods are not similar.
Jessica: Rita says to make the numbers slide, but Alessia moves a piece of the plant.
T: Wait a moment, please. Jessica probably has recognized an important point. She says, “Rita makes the numbers slide”… the measure of the plant for Rita is always the same, they are the numbers that slide. While, as Jessica says, Alessia has imagined taking a piece of the plant and bringing it to the edge of the plant, where we can measure it.

Angelo: I didn’t say that it is the same thing; I said that the two methods are a little bit similar.

Giulia: It is like Alessia overturned the plant, she would bring a piece over… a small piece was moved, and the plant seems to be hanging on… so we could measure it.

The fourth activity was an individual written production in which students had to explain why Rita’s method worked and why Alessia’s method worked.

With three exceptions (who remained rather far from a clear presentation, although they showed an “operational” mastery of the procedures), all the students were able to produce the explanations the task demanded. Half of the students added comments about the two methods; most of them explained in clear terms the limitation of the “translation” solution. Here is an example:

Marco: Rita’s reasoning works, because it makes the ruler like a tape measure, because zero goes at the edge of the ruler. She imagined the same thing and made the numbers slide. Where I see that the edge of the plant is at 8 cm, she says “to slide” and sees that the ruler slid by one centimeter, and so she sees that 8 became 9. But this method works easily only if the ruler has a space of 1 cm between zero and the edge. Alessia’s reasoning works because she adds the piece of plant she carries to the height of the plant and adds 1 to 10 and she sees that it makes 11. One difference is that Rita leaves the piece of the plant where it is, while Alessia carries it up to 10.

The fifth activity consisted of the classroom construction (guided by the teacher) of a synthesis to be copied into students’ exercise books. In the following classroom activities, students recalled “measure of the plants in the pot with the ruler” as a reference when they had to measure the length of objects that were not accessible in a direct reading of their length on the ruler.

**Some Comments About the Available Data**

The available data show experimental evidence for the potential of natural language involved in argumentative activities, as stated previously. In particular, during the interactive resolution of the problem (the first activity), the argumentative activity with the teacher allowed students to grasp the nature of the problem and transformed the experience into a possible “reference situation” for the involved operational invariants of the measure concept. The teacher’s arguments compelled students to move from imagining physical actions (putting the ruler into the ground or cutting the ruler) to operations involving operational invariants of the measure concept (translation invariance or additivity of measure of length).

During the classroom discussion (the third activity), argumentation allowed students to make explicit the two different operational invariants (translation invariance and additivity) of the measure of lengths and the systemic links between them and with other concepts (addition and subtraction) in the conceptual field of the additive structures (Vergnaud, 1990).

During the last individual activity (“explain why…”: a typical task requiring argumentation), students attained a first level of understanding about the potential and limitations of the operational invariants involved.
In this section, we refer to the second and fourth functions of natural language as described in the first section. In particular, we attempt to support the idea that rich argumentative processes (including management of analogies, metaphors, and questioning) constitute the core of the activities of conjecturing and proving, whereas sophisticated metalinguistic awareness and linguistic competences are needed to obtain socially acceptable products (proofs). To achieve our goal, we need to discuss the relationships (and the distinction) between proof as a product (submitted to social rules of conformity to cultural models) and proving as a process and also to take a critical position about the widespread conviction (shared by many mathematics teachers and some mathematics educators) that learning to prove mainly concerns the development of formal logic skills. In particular, the image of proving that we want to highlight is that of a complex, culturally rooted but also creative practice that requires a highly developed mastery of natural language in its reflective, control, and command functions. An example from university mathematics students provides some evidence for our position.

Formal Aspects of Proof and Metalinguistic Awareness

With regard to the didactical transposition (Chevallard, 1985) of proof in school mathematics, Hanna (1989) developed a comprehensive perspective to frame further theoretical investigations and educational developments. Her paper analyzes the complex interplay between the manner of presentation of mathematical results and the mathematical ideas that are to be communicated. She argued that “To a person only partially trained in mathematics . . . it might easily appear that the manner of presentation . . . is the core of mathematical practice.” This belief may induce people “to assume that learning mathematics must involve training in the ability to create this form” and then overestimating formalism, whereas “when a mathematician evaluates an idea, it is significance that is sought, the purpose of the idea and its implications, not the formal adequacy of the logic in which it is couched.”

The overestimation of formal aspects of mathematical proof, which has its objective reasons, also has its price because it may make students become symbol pushers. Arriving at the educational implications of her analysis, Hanna argued that formalism should be regarded as a tool to be used in all its rigor when it is necessary (e.g., “when there is a danger that genuine confusion might develop”) but to be interpreted with some tolerance in many other situations. In particular, this means that the use of formalism requires some metalinguistic awareness, which is more than the simple knowledge of certain rules governing formal languages (e.g., the rules of algebraic formalism) to standard, everyday-life linguistic competence.

Mathematical Proof and Argumentation as Linguistic Products

In this section, we consider “mathematical proof” to be what—in the past and today—is recognized as such by those working in the mathematical field. This approach covers Euclid’s proof, as well as the proofs published in high school mathematics textbooks and current mathematicians’ proofs, as communicated in specialized workshops or published in mathematics journals (for the differences between these two forms of communication, see Thurston, 1994). We could try to go further and recognize some common features across history—in particular, the functions of making clear or validating a statement by putting it into an appropriate frame; the reference to an
established knowledge (see the definition of theorem as “statement, proof and reference theory” in Mariotti et al., 1997); and some common requirements, like the enchainment of propositions, which, however, may differ according to the different historical periods and different contexts (for example, a junior high school context is different from a university graduate course context).

To compare mathematical proof and argumentation as linguistic products, we adopt the following criteria of comparison, inspired in part by Duval’s analysis (see Duval, 1991): the role of context (and, in particular, the existence of a “reference corpus”) in the development of reasoning and the form of reasoning.

One of the most relevant points of discrimination between argumentation and mathematical proof as products is the language they adopt and the context they work within. The context of argumentation includes all the levels of context, as identified by Leckie-Tarry (1995) in the frame of a functional perspective, that is, the context of situation (the concrete, physical, and social context the exchange taking place within), the context of text (the information provided by the texts involved, e.g., the text of the problem under scrutiny and other connected texts), and the context of culture. The context (and consequently the reference corpus) of mathematical proof is quite different: The context of situation is usually banned, and even the context of culture is strongly bounded. Not all the knowledge available can be used, but only parts of it according to their operational status. For example, when writing the proof of some propositions from Euclid’s Elements, one could use any proposition that has already been proven (i.e., one occurring before in the list of propositions) but not other ones; more generally, a mathematical proof often refers to “straightforward” arguments, but not all arguments that may appear straightforward to a nonexpert (e.g., measure, for plane geometry figures) are allowed.

Also from the viewpoint of “language genre,” argumentation and proof (as products) are different. Argumentation may use a wide range of linguistic registers, including conversational (and situation-dependent) ones, whereas mathematical proof, for various reasons, is compelled to use more explicit and institutionalized ones only. In particular, argumentation may freely use a wide range of metaphors, whereas in mathematical proofs only few metaphors are allowed (e.g., large–small, before–after) according to conformity criteria.

According to Leckie-Tarry’s classification of contexts and registers, this means that the understanding and production of mathematical proofs belong to the literate side of linguistic performances and require prior deep linguistic competence.

In the preceding comments, we stressed relevant points of the difference between argumentation and mathematical proof as products. Now we point out some aspects of at least partial superposition. If we focus on the part of context Duval named “reference corpus,” which includes not only reference statements but also visual and, more generally, experimental evidence, physical constraints, and so forth assumed to be unquestionable (i.e., “reference arguments,” or, briefly, “references,” in general), we understand that no argumentation (individual or between more participants) would be possible in everyday life if there were no reference corpus to support the steps of reasoning. The reference corpus for everyday argumentation is socially and historically determined. Mathematical proof also needs a “reference corpus,” which is socially and historically determined as well. We may add that the reference corpus is generally larger than the set of explicit references, both in mathematics and in other fields.

In mathematics the knowledge used as reference is not always recognized explicitly (and thus appears in no statement); some references can be used and might be discovered, constructed, reconstructed, and stated afterward. The example of Euler’s theorem discussed by Lakatos (1976) provides evidence about this phenomenon in the history of mathematics. The same phenomenon also occurs for argumentation concerning areas of study other than mathematics.
In general, we could hold no exchange of ideas, whatever area we are interested in, without exploiting implicit shared knowledge. Implicit knowledge, of which we are generally unconscious, is a source of important “limit problems” (especially in nonmathematical fields, but also in mathematics). In the “fuzzy” border of implicit knowledge, we can meet the challenge of formulating more and more precise statements and evaluating their epistemic value. Again Lakatos (1976) provided us with interesting historical examples about this issue in mathematics. Another crucial point of superposition between argumentation and mathematical proof is the relevance of “content” (semantic arguments) in the validation of statements. This fact contrasts a widespread idea in the community of mathematics educators, according to which advanced mathematical proofs are near to the model of formal derivation, and checking their validity is a formal logic exercise.

In the reality of today’s university mathematics, if we consider some basic theorems in mathematical analysis (e.g., Rolle’s theorem, Bolzano-Weierstrass’ theorem, etc.), we see that their usual proofs in current textbooks are formally incomplete (if considered as formal derivations), and completion would bring students far from understanding; for this reason semantic (and visual) arguments frequently are exploited or evoked to fill in the gaps existing at the formal level. We add that the validity of most of published mathematical proofs is based on the semantics of symbolic (linguistic, algebraic, etc.) expressions that constitute them and that checking for validity is based on strategies that usually are very different from checking the validity of a formal derivation. As Thurston (1994) noted,

![Image](https://via.placeholder.com/150)

Our system is quite good at producing reliable theorems that can be solidly backed up. It’s just that the reliability does not primarily come from mathematicians formally checking formal arguments; it comes from mathematicians thinking carefully and critically about mathematical ideas.

The Processes of Argumentation and Construction of Proof

We have proposed distinguishing between the process of construction of proof (proving) and the product (proof). What comprises the “proving” process? Experimental evidence has been provided about the hypothesis that in many cases “proving” a conjecture entails establishing a functional link with the argumentative activity needed to understand (or produce) the statement and recognizing its plausibility (see Mariotti et al., 1997). “Proving” requires an intensive argumentative activity, based on “transformations” of the situation represented by the statement. Experimental evidence about the importance of “transformational reasoning” in proving has been provided by various, recent studies (see Arzarello, Micheletti, Olivero, & Robutti, 1998; Boero et al., 1996, 1999; Harel & Sowder, 1998; Simon, 1996). Simon defined transformational reasoning as follows:

the physical or mental enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated.

It is interesting to compare Simon’s definition with that of Thurston (1994):

People have amazing facilities for sensing something without knowing where it comes from (intuition), for sensing that some phenomenon or situation or object is like something else (association); and for building and testing connections and comparisons, holding two things in mind at the same time (metaphor).
Metaphors can be considered as particular linguistic outcomes of transformational reasoning. For a metaphor, we may consider two poles (a known object and an object to be known) and a link between them. In some cases the “creativity” of transformational reasoning consists of the choice of the known object and the link, which allows us to know some aspects of the unknown object as suggested by the knowledge of the known object (“abduction”; cf. Arzarello et al., 1998). We note that mathematics “officially” concerns only mathematical objects. Usually, metaphors where the known pole is not mathematical are not acknowledged. But in many cases the process of proving needs these metaphors, with physical or even bodily referents (cf. Thurston, 1994). In general, Lakoff and Nunez (1997) suggested that these metaphors have a crucial role in the historical and personal development of mathematical knowledge (“grounding metaphors”).

An Example From University Mathematics Students

Forty-three fourth-year university mathematics students had to generalize a proposition (the sum of two consecutive odd numbers is divisible by four) then prove the generalized proposition. Students attended a mathematics education course and were accustomed to reporting their reasoning verbally. Here are two examples of students’ texts. They represent characteristic and opposite behaviors in the texts of many students.

Example 1

Excerpts from the text of Student 1 contain seven large, spatially organized pieces, such as the two reported below, and many arrows, connecting lines, and encirclings.

I have some difficulties understanding in what direction I must generalize. It might be: “by adding two odd or even consecutive numbers I get a number divisible by 4” [she performs some numerical trials]. This does not work. I will try to generalize in another way (see Fig. 11.2).

I was looking for something that could help me … but I have nothing.

Let’s see how we can generalize the problem in another way:

The even numbers:

\[
\begin{align*}
2k & \quad 2k + 2 \quad 2k + 4 \quad 2k + \varepsilon \quad 2k + 8 \ldots \\
\text{Divisible by 2} & \\
(6k + 6) &= 3 \cdot 2(k + 1) \\
\text{Divisible by 2 and by 3} & \\
(8k + 12) &= 4(2k + 3) \\
\text{Divisible by 4} & \\
(10k + 8) &= 2(5k + 4) \\
\text{Divisible by 2} &
\end{align*}
\]

FIG. 11.2. An excerpt from a student’s protocol (even numbers).
[Other trials, with a rich spatial organization; including two consecutive even numbers, two consecutive odd numbers; here she gets divisibility by 4; then three, four, five, six, and seven consecutive odd numbers. By performing calculations, she gets the following formulas:

\[
3(2K + 3); \\
8(K + 2)10K + 25 = 5(2K + 5); \\
12K + 36 = 12(K + 3); \\
14K + 49 = 7(2K + 7).
\]

Is the result of the addition of \( n \) consecutive odd numbers \((n\ odd)\) divisible by \( n \)? \((2K + 1) + (2K + 3) + \cdots (2K +)\)? What must I put here? (E) (see Fig. 11.3).

Let's look at the example:

\[
(2k + 1) + (2k + 3) + (2k + 5) \\
(2k + 1) + (2k + 3) + (2k + 5) + (2k + 7) \\
+ (2k + 9)
\]

\(3\) numbers
\(4\) numbers
\(5\) numbers

\[
(3 \cdot 2) - 1 \\
(4 \cdot 2) - 1 \\
(5 \cdot 2) - 1
\]

FIG. 11.3. An excerpt from a student’s protocol: divisibility by \( n \).

The student performs an unsuccessful trial by induction, then considers \( n \) numbers in general.]

\( n \) numbers: \((2K + 1) + (2K + 3) + \cdots (2K + (2n - 1)) = 2nK + 1 + 3 + 5 + \cdots + (2n - 1) = 2nK + (I\ am\ thinking\ of\ the\ anecdote\ of\ ‘young\ Gauss’): \( (F) \) it makes \( 2n \cdot \frac{n}{2} = n^2 = 2nK + n^2 + n(2K + n) \) OK!!

[Trials performed by applying the preceding formula \( 2nK + n^2 \) in the cases \( n = 2, n = 4, n = 6, n = 8 \): The student arrives at: 4\( K + 4 \) divisible by 4;

\[
8K + 16 \text{ divisible by } 8; \\
12K + 36 = 12(K + 3); \\
16K + 64 = 16(K + 4) \text{ divisible by } 16.
\]

Then if I add \( n \) consecutive odd numbers \((n \ even)\), I get divisibility by \( 2n \). Let us try a proof:

\[
(P) \quad (2K + 1) + (2K + 3) + \cdots (2K + 2n - 1) = 2nK + (1 + 3 + \cdots 2n - 1) = 2nK + (2n \cdot n)/2 = 2nK + n^2. \cdots = 2n(K + n/2).
\]

\( n \ even \) implies that \( n/2 \) is an integer number: so I get divisibility by \( 2n \).

Example 2

In the excerpts from the text of Student 2, spatial organization is almost linear, like that in the following transcript.

Student 2 starts her work by checking (on numerical examples: 3 + 5; 5 + 7; 101 + 103) the validity of the given property and then proves it. Then she writes.

What does it mean “to generalize”? It means considering a property in which there are some closed variables (two odd numbers or divisibility by 4) and getting a property
in which variables are open. I change the number of odd consecutive numbers to add. For instance, I consider \( n \) cross odd numbers \( 2n + 1, 2n + 3, 2n + 5, 2n + 7 \) and make the addition: \( 2n + 1 + 2n + 3 + 2n + 5 + 2n + 7 = 8n + 16 = 8(n + 2) = 4(2n + 4) \). Then I find a number that is divisible by 8, so it is divisible by 4. I perform the addition of six consecutive odd numbers [similar calculations] = \( 12n + 36 = 6(2n + 6) \). Then I find a number divisible by 12, so it is divisible by 6. I try with 8; [similar calculations] = \( 8.2n + 64 = 8(2n + 8) \) Then I find a number that is divisible by 8, so it is divisible by 4. Following my reasoning, for an even number \( K \) of odd consecutive numbers, I get: \( 2n + 1 + 2n + 3 + \ldots + 2n + 5 + \ldots = K(2n + 1) = 2K(n + K/2) \); but \( K \) is an even number, so it is divisible by 2 and \( (n + K/2) \) is an integer number. Then \( 2K \) is divisible by 4 (because \( K \) is odd). So I have found that the given property is still valid if I add up an even number of odd consecutive numbers.

**Analysis of Student Behaviors.** We attempt to show the complexity of the argumentative activity needed to fulfill the task; such complexity involves different functions of natural language in interaction with other symbolic systems.

The task called for elementary content reference knowledge: elementary arithmetic, algebraic language, and its rules of calculation. Concerning algebra, we may remark that the process of formalization (i.e., the passage from content to formula) was not easy for many students, especially when they wanted to write the sum of an even number of odd consecutive numbers, I get: \( 2n + 1 + 2n + 3 + \ldots + 2n + 5 + \ldots = K(2n + 1) = 2K(n + K/2) \); but \( K \) is an even number, so it is divisible by 2 and \( (n + K/2) \) is an integer number. Then \( 2K \) is divisible by 4 (because \( K \) is odd). So I have found that the given property is still valid if I add up an even number of odd consecutive numbers.

**Metamathematical Knowledge.** Summing up the analyses performed, we may say of metamathematical knowledge that shared explicitable knowledge was much narrower than the actual knowledge used globally by the students. We found that more than half of students referred explicitly to methods for solving problems of this kind, but, as an example, “organization of data” was never mentioned even in partial explicitations of methods although it was a key strategy for 12 students and useful for nine of them. The implicit problem-solving methods we could detect globally were change of representation, interpreting calculations in words and vice versa, and visually organizing data and calculations. We also detected many changes of mathematical frames: arithmetic, algebra, series, and so forth.

Natural language played a crucial role in the management of metamathematical knowledge; in particular, it accomplished command and reflective functions about changes of frame, changes of representation, and other factors.

**Algebraic-Syntactic or Semantically Based Steps of Reasoning.** We have listed numerous breaks during calculations, which were needed to reinterpret the mathematical content of calculus in words. Wording was a crucial tool for reinterpretation. This can be seen as a sign of the primacy of semantic content over algebraic calculation during the process of conjecture and proof construction. As an example, we can consider the need of Student 1 to express algebraic propositions in words when seeking to recognize possible conjectures. This attitude displays the search for a semantically consistent grasp of the algebraic signs. We can interpret it by saying...
that constructive work in mathematics cannot evolve within formal expression alone. Natural language is revealed here as a crucial tool for thinking.

**Proof as Product and Proof as Process.** As remarked above, analyzing the text of Student 1 and other students who performed well, we can observe frequent changes of strategy, organization of data, and calculations, as well as a frequent effort to interpret the problem with words. Some of these useful forms disappeared in the final drafts of the proof, in which the logical link between the propositions became a priority (see P in Example 2). Also justification of the research method disappeared from the products, whereas examples of the interwoven presence of metamathematical arguments in mathematical reasoning were frequent in the construction stage.

Student 2 is considered skillful; her presentation is close to that of a final presentation. Nonetheless, this approximation to a formally correct mathematical text (cf. Hanna, 1989) seems to bear negative consequences on the productivity of this student’s work: Her research is linear, and no change of strategy is found at any level. There are long repetitive arithmetic calculations, quite surprising for the only student in the group who usually managed algebraic tools very well; more remarkably, the student arrived algebraically at a strong conjecture and interpreted it in words as being much weaker. Finally, she did not produce a complete proof. The same happened with other students producing texts that were similar to a final presentation.

**Conclusions**

We have seen that important reference knowledge remained implicit in the students’ proving processes and that some of the references concerned the content, whereas others related to the metaknowledge about the activity to be performed. Natural language played a crucial role in the management of the complex game involving explicit as well as implicit reference knowledge. We also have seen how non-standardized, appropriate representation of explicit reference knowledge played an important role in the students’ performances, under control of natural language. We have seen that when elaborating a productive process, many students found syntactic arguments insufficient, and so semantically rooted arguments (expressed in words) became critical. Finally, we have collected some experimental evidence about the negative consequences of subordinating the proving process to the requirements of proof as a final product, as a “standardized text.”

**Some Educational Implications**

Let us come back to the argumentative process of proof construction as distinct from the final result. In our opinion, an important part of the difficulties of proof in school mathematics comes from the confusion of proof as a process and proof as a product, which results in an authoritarian approach to both activities. Frequently, mathematics teaching is based on the presentation (by the teacher and then by the student when asked to repeat definitions and theorems) of mathematical knowledge as a more or less formalized theory based on rigorous proofs. In this case, authority is exercised through the form of the presentation (see Hanna, 1989); in this way school imposes the form of the presentation over the thought, leads to the identification between them, and demands a thinking process modeled by the form of the presentation (eliminating every “dynamism”). This analysis may explain the strength of the model of proof, which gives value to the idea of the linearity of mathematical thought as a necessity and a peculiar aspect of mathematics. The use of natural language is molded to such constraints of linearity, losing its potential for creativity (e.g., producing and using metaphors, managing explorations, etc.).
If a student (or a teacher) assumes linearity as the model of mathematicians’ thinking without taking the complexity of conjecturing and proving processes into account (consider the example of Student 1), it is natural to see “proof” and “argumentation” as extremely different. On the contrary, giving importance to nondeductive aspects of argumentation required in constructive mathematical activities (including proving) can develop different potentials. Regarding the possibility of educating ways of thinking other than deduction, Simon (1996) considered “transformational reasoning” and hypothesized:

transformational reasoning is a natural inclination of the human learner who seeks to understand and to validate mathematical ideas. The inclination... must be nurtured and developed.... school mathematics has failed to encourage or develop transformational reasoning, causing the inclination to reason transformationally to be expressed less universally.

We are convinced that Simon’s assumption is a valid working hypothesis, needing further investigation not only regarding the role of transformational reasoning in classroom discourse aimed at validation of mathematical ideas but also its functioning and its connections with other “creative” behaviors (in mathematics and in other fields).

In this way, argumentation seems an activity suitable to promote both the improvement of linguistic skills (e.g., forcing the transition from conversational, oral registers to more abstract, written ones) and the development of mathematical reasoning. In particular, through argumentation in social contexts and the activity of writing down reports of the related discussions, experiential knowledge progressively becomes textual, and some of the arguments that were implicit in the context of the social situation become explicit in the context of individual texts and then, in the context of the shared culture of the class. These remarks join the same conclusions of the previous section which concerned the early approach to theoretical aspects of mathematics.

The passage from argumentation to proof about the validity of a mathematical statement should be constructed openly because of limitation in the reference corpus (see Mathematical Proof and Argumentation as Linguistic Products). This passage could be supported by using different texts, such as historical scientific and mathematical texts, and different modern mathematical proofs (see Boero et al., 1997, for a possible methodology).

**HOW CAN STUDENTS DEVELOP NATURAL LANGUAGE COMPETENCIES IN MATHEMATICAL ACTIVITIES?**

Research perspectives and problems about the role of natural language in mathematical activities considered in the preceding sections raise important questions related to teachers’ preparation and educational implications for classroom work. We now consider some aspects of this *problematique* and some didactical implications.

**Teachers’ Preparation**

We have already remarked that the development of linguistic competencies in mathematical activities strongly depends on a teacher’s mediation. Therefore, teachers’ difficulties must be taken into consideration. Some obstacles come from the widespread belief that natural language is not an efficient tool in developing and communicating mathematical knowledge because of its redundancy and lack of precision. Many reasons are advocated to support this idea: the supposed prominence of mathematics
as a formal system, the need for a purely syntactic treatment of mathematical relations, the difficulty many students have in managing and understanding natural language with a sufficient level of precision, and (last but not least) the theory of Piaget (who considered communication as the main function of natural language). Many mathematics teachers (encouraged by current textbooks) are tempted to reduce the relevance of natural language in classroom work: Tasks are formulated primarily via images or stereotyped linguistic expressions; completely nonverbal answers (diagrams, formulas, etc.) are allowed, and verbal explanations are represented at the blackboard with significant support from nonverbal tools (diagrams, schemas, algebraic expressions, etc.). Add to this the increasing number of foreign students in classrooms in many countries, with its obvious consequences: Mathematics (and mathematical formalisms) is universal, so we should try to teach it by reducing its verbal aspects. Add also the fact that poorly paid teachers (a common situation in many countries of the world) may prefer to reduce the “wasted time” required to correct students’ homework and classroom problem solutions with a strong commitment to use technical, synthetic formalisms. Finally, the quantity of mathematical content that can be presented to students by using these formalisms is much greater than in the case of a verbal presentation (cf. Boero, Dapueto, & Parenti, 1996).

All the reasons considered above make a different perspective (concerning the development of linguistic skills related to the use of natural language as a crucial issue in mathematics education) difficult to accept by both prospective and inservice teachers (cf. Morgan, 1998, for an in-depth analysis of constraints influencing teachers’ choices).

According to our experience, it is insufficient to produce good theoretical arguments against the dismissal of natural language in mathematics class: The discussion of well-chosen examples of students’ behaviors seems to be necessary. Discussion should put into evidence the crucial function of natural language in mathematical activities, according to preceding considerations. Discussion also should focus on the quality of students’ performance in relation to their current mastery of natural language in mathematical activities.

**Promoting Verbal Activities in Mathematics Classes**

Even if prospective and inservice teachers accept the relevance of verbal language in mathematical activities, the educational problem related to the choice and management of suitable classroom activities remains. We can consider the role of the teacher as an indirect mediator (when he or she selects and uses students’ linguistic productions), as a direct semiotic mediator (when he or she provides students with appropriate linguistic expressions to fit their thinking processes), and as a cultural mediator (when he or she provides students with important “voices” as linguistic models of theoretical behaviors in mathematics).

Let us consider the following example: Students are asked to find the side of the square that has twice the area of a given square. They are required to produce a verbal report about their trials to solve the problem. Sometimes this verbal report follows the steps of the activity; sometimes it is written during the activity and reflects the ongoing reasoning. Here is an example from a seventh-grade student:

**Daniele:** “I think that I can double the side of the square, then everything will be double, and the area will be double”

[Daniele produces a partial drawing.]

“No, I didn’t get a double area by doubling the side of the square. The area becomes four times larger. I must find a smaller increase. I could take one time and one half the length of the side.

[Daniele draws with careful measurements and calculations.]
“No, it doesn’t work. It is bigger than I need. I should decrease once more. I could take 1.4 times larger. I can make the calculation without making the drawing. It is sufficient to multiply 1.4 by the length of the side, and then multiply by itself.”

[calculations: $1.4 \times 2 \times 1.4 \times 2 = \cdots = 7.84$].

“Double area means 8. I am near, but I haven’t obtained 8.”

The teacher selects some texts produced by students and for each of them invites students to identify similarities and differences with their own solutions. In this way, the content and expression of the chosen text are put under scrutiny. Following is an excerpt taken from the discussion about the preceding text:

**Sabrina:** Daniele has come up with a good result. The side with length 1.4 times 2 (namely, 2.8) gives a surface that is very near to the area 8, or the double of the initial area.

**Pietro:** But Daniele has not reached the solution. The solution is to find the square of double area.

**Elena:** And Daniele has put his numbers by chance, why one time and one half, and then 1.4, and not other numbers?

**T:** Daniele, try to explain why you chose those numbers.

**Daniele:** I said to Elena that if you see that doubling the sides of the square gives a four-times bigger area, then you must decrease the side to decrease the area. I have chosen 1.5 because it is a number in the middle between 1 and 2 (which does not work).

The same activity is repeated for three texts (usually starting from the least successful of the chosen texts). The teacher tries to encourage students to use increasingly precise expressions. Then the teacher presents a long excerpt taken from Plato’s “Menon”: The well-known episode of the dialogue between Socrates and Menon’s slave about the problem of doubling the square (cf. Garuti, Boero, & Chiappini, 1999).

Similarities and differences are found in the students’ productions; Plato’s text is discussed as a model of using dialog to identify and overcome a mathematical mistake. The three phases of the dialogue are put into evidence (production of the mistake and identification of it by counter-examples; attempts to overcome the mistake; finally, solution of the problem guided by the teacher). A discussion follows about a common mistake of students and the nature of that mistake. Finally, students are asked to produce an “echo” to Plato’s dialogue by writing a dialogic treatment of the chosen mistake. Here is an example of a high-level individual production by Daniele (the chosen mistake concerns the idea that by dividing an integer by another number, one always gets a number smaller than the dividend).

**Socrates (SO):** Tell me, my boy, what is the result of $15 \div 3$?

**Slave (SL):** Five.

**SO:** Is it smaller than 15?

**SL:** What a question! That is clear!

**SO:** And yet, how much is it $20 \div 5$?

**SL:** Obviously 4, Socrates.

**SO:** Then is it smaller than 20?

**SL:** Exactly.

**SO:** Then, what can you say about the divisions?

**SL:** I think that they are always smaller than the dividend.

**SO:** Are you sure?

**SL:** Yes, because “to divide” means “to break in equal parts.”
SO: Now perform this division: 15 ÷ 1.
SL: Uhm, . . . it makes 15.
SO: But 15 is equal to the dividend.
SL: It is true.
SO: Why is it equal?
SL: Because dividing by one is how to give an amount to one person, it remains equal.
SO: So does your theory still work?
SL: Not completely. Now I see that in some cases it does not work.
SO: Are you still sure you are right?
SL: Yes . . . perhaps . . . no . . . perhaps there is one case in which the result is larger . . . or perhaps not . . . My Zeus, I understand nothing! (Five minutes elapse).
SO: What is the result of 2 ÷ 0.5?
SL: These are difficult questions. I am no longer able to answer.
SO: Take this square [drawing] and divide it into small squares!
SL: This way? [The drawing is divided into 16 pieces by drawing three horizontal and three vertical lines, all equally spaced.]
SO: Yes, good. Now the unit is the small square [drawing]. How much is 0.5 compared with 1?
SL: One half.
SO: Now make one half of the small square.
SL: Done.
SO: Do the same for all the small squares.
SL: Just a moment. Done.
SO: How many halves?
SL: 1, 2, 3 . . . 32, Socrates.
SO: How many unit squares at the beginning?
SL: Sixteen, Socrates.
SO: Then you got a result greater than the starting number.
SL: Uhm . . . Of course.
SO: And how is one half written as a fraction?
SL: Uhm . . . perhaps 1/2.
SO: Good! Are you able now to divide a number by a fraction?
SL: Yes, surely!
SO: Then divide 2 by 1/2. How many times is 1/2 contained in 2?
SL: According to the preceding rule, I must invert the fraction and then multiply. OK, it makes 4.
SO: How can you represent this?
SL: I’ll try . . . Two squares . . . [drawing]. One half twice for each [drawing]. It works: 4.
SO: Good!
SL: I understand: The division is not only “breaking into equal parts”, but also seeing how many times a number is contained in another!
SO: Make an example by yourself!
SL: 1:1/4. [He performs and illustrates it]

By comparing the initial texts with the final productions, we observed how different roles played by the teacher had left traces in the students’ productions (for further details, see Garutti et al., 1999); including an increased precision in linguistic expression (with the use of appropriate terms) and the assimilation of a dialogic treatment of the mistake (according to the cultural model provided by Plato’s dialogue).
The Problem of Teaching Mathematics in Multilanguage Classes

Teaching mathematics in a multilanguage classroom is a difficult task, and most teachers tend to bypass this problem by using the universal, technical languages of mathematics (arithmetic and algebraic languages, but also diagrams, arrows, etc.) to create common tools of communication between the students and the teacher. But if natural language is necessary because of its reflective and command functions, in personal mathematical activities as well as in social interaction, the fact of privileging technical languages of mathematics can damage the development of mathematical knowledge. In a multilanguage class the problem of verbal communication and production cannot be avoided in other subjects (such as sciences or history), and thus a separatist position on the part of the mathematics teacher (“unlike the other teachers, I can avoid the necessity of communication in natural language”) can result in general damage to the cultural preparation of students.

The perspective of teaching mathematics to multilanguage classes can rely on many studies performed in the last 20 years concerning the relationships between linguistic competencies and mathematical performances (for a partial survey, see Cocking & Mestre, 1988). An interesting emerging trend is to design teaching situations in which language diversity can help mathematical understanding: Indeed, it happens rather frequently that different languages use expressions which are particularly appropriate for speaking about specific concepts and properties or that some linguistic expressions allow one to grasp specific aspects of the mathematical knowledge that those expressions carry. In Boero and Radnai Szendrei (1998), an example is provided, concerning the manner of speaking about numbers in Hungarian and in other languages. In particular, consider the different situations in Italy and in Hungary concerning the learning of natural numbers: In the Italian language, the names of natural numbers (one, two, three, four . . .) are also used to indicate the days of the month (only the first day is commonly named ‘first of . . . ’); in the Hungarian language, all the days of the month are named with the ordinal adjective (first, second, third). In Italy and in Hungary, the relationships between cardinal and ordinal aspects of natural numbers are therefore different in the first mathematical experiences of pupils. Classroom discussions about these differences may result in an improved flexibility in managing the different “meanings” of numbers (and these discussions are possible, for instance, with those Hungarian classes in which Italian and Hungarian students learn mathematics in Italian).

Another example is the comparison between French and English (or Italian) regarding the names of numbers such as 85 or 75. In French, 85 is quatre-vingt-cinq (four twenties plus five), and 75 is soixante-quinze (sixty plus 15). When a native speaker of English or Italian (or many other languages) first encounters this system of numbers, it may be difficult to grasp because of the tendency to compare it to one’s own language, in which tenths follow tenths from 10 to 100. Comparison with one of these languages can help French-speaking beginners grasp the regularity of the tenths sequence and learn basic facts in the transition from units, to tenths, to hundredths (e.g., the fact that the same, basic sequence is repeated with values that are 10, 100, etc. times bigger.) French numbers then can be useful in the classroom for all the students to experience basic facts concerning the additive and multiplicative relationships between numbers: quatre-vingt means that 80 can be reached by repeating 20 four times. This can have positive effects on students’ mental calculations. As happens frequently in mathematics, difficulty can result in the opportunity to develop knowledge. This example suggests the opportunity to perform cross-cultural investigations to detect specific, linguistic aspects of basic mathematics in different languages that can be exploited as an opportunity for developing mathematical skills.
CONCLUSION

The analyses performed in this chapter provide theoretical reasons and experimental evidence for the complex functions fulfilled by natural language in mathematics, especially with regard to the interplay between natural language and algebraic language and the role of argumentation related to some theoretical aspects inherent in the systemic character of mathematical knowledge (systemic links between concepts and deductive organization of mathematical theories).

Educational implications can be summarized by saying that these complex functions cannot be fulfilled without appropriate instruction. Spontaneous classroom discussion and negotiations are insufficient to reach the level of sophistication and mastery of natural language needed to use it in an efficient way. Teachers must fulfill the complex role of mediation, including both the exploitation of students’ individual productions and the use of cultural models.

REFERENCES


The United States would not displace Germany as the world’s technological leader until after World War II. In the first half of the century, those who wanted a leading-edge scientific education went to Germany. During World War II, Germany was the only adversary to deploy ballistic missile technologies; it had prototype jet engines; and much of the urgency behind America’s Manhattan Project was the fear that Germany would be the first to invent atomic weapons. In the end it was not the physical destruction of losing a war but its racial policies [our emphasis] that cost Germany its scientific and engineering leadership. The physical damage could be repaired. The human damage could not. America had gained the Einsteins, the Fermis, and their intellectual descendents. It seized global scientific and technological leadership.

—(Thurow, 1999, p. 20)
partisan, and central to debates of what democratic societies want for their children and schools.

How access to school mathematics is framed politically and socially often is very subtle and can require careful analysis from multiple disciplinary perspectives. A case in point can be found in the United States between 1954 and the early 1970s. In his classic analysis of race and education in the United States The Mis-Education of the Negro, Woodson (1990) argued that one pressing opportunity-to-learn problem facing African American children in the mathematics classrooms of the segregated southern school systems was teachers’ knowledge of mathematics and their limited ability to provide appropriate mathematics pedagogy. Twenty-one years after Woodson’s volume, Brown v. Board of Education (1954) ended de jure segregation in United States education and promised to change the democratic access to education for African American children. The hopes and dreams of many parents for true access to education were built on the belief that better qualified teachers and resources would be a product of Brown. This important Supreme Court decision was followed by a large-scale effort to improve mathematics education in the United States. Initiated as part of a larger set of Cold War policy initiatives, and specifically in response to the launch of Sputnik by the Soviet Union, the “new math” movement sought to improve school mathematics in the United States. However, the mathematics reform did little to address the concerns of parents of color (Tate, 1997). Those responsible for the reform argued that their efforts should be limited to “college-capable” students (Devault & Weaver, 1970; Kliebard, 1987; National Council of Teachers of Mathematics [NCTM], 1959). The code words college capable provided a subtle indicator to the education community that only a select few communities and students were to be provided true access. Thus, a major reform effort was tacitly built on an elitist position that ignored how a students’ race or ethnic background was a key predictor of opportunity and access in mathematics education.

Some might suggest this type of access denial in school mathematics is not possible in a highly competitive global market place. However, in Margo’s (1990) economic history, Race and Schooling in the South, he warned against such thinking:

It is frequently said that the growth of competitive, market economies and social progress go hand in hand. Freely mobile workers can always leave if they are treated unfairly; discrimination is unprofitable for private firms (and sometimes governments). Competition ultimately makes it costly for societies to maintain rigid social norms in the face of long-run economic growth and structural change. The economic history of black Americans, however, offers little evidence in support of these claims. Before the Civil War the southern economy grew at the same rate as the rest of the country; there is no evidence that slavery was incompatible with industrialization (Fogel, 1989; Goldin, 1976). From the end of the Civil War until World War II the southern economy lagged somewhat behind the rest of the nation, but the South still experienced modern economic development, as labor shifted out of agriculture. Yet employment in the South was more segregated in 1950 than in 1900. Segregationist ideology, like slavery, was not incompatible with economic growth or structural change. Both took a concerted political effort to fight, and in the end neither was overcome without bloodshed. (p. 132)

It is naïve to assume that global economic competitiveness will result in true democratic access to mathematics across demographic groups in the United States. Unfortunately, the lack of access to a quality education, and more specifically a quality mathematics education, has the possibility of limiting human potential and individual economic opportunity. It is clear in the United States and in many other countries that mathematics acts as an academic passport for entry into virtually every avenue of the labor market and higher education opportunity. As the global market moves forward, the pressure to advance the mathematical skills of workers across the world will heighten. How will the increased demand for workers with technological skills influence educational access policy development in mathematics education across the world?
One purpose of this handbook is to provide an international perspective on research and development in mathematics education with a focus on democratic access. This is a challenging charge in that every scholar is a product of social conditions that have greatly influenced his or her world view. We are no different. Like many mathematics education scholars trained in the United States, we have worked to develop a skill set in mathematics and psychology. Disciplined inquiry in mathematics education is a relatively new field of study. The original paradigmatic boundaries of educational research were borrowed from scholarship in psychology. Landsherre (1988) noted that educational research was first known as experimental pedagogy. Theoretical principles of experimental pedagogy were isomorphic to that of experimental psychology, a term credited to Wundt in around 1880 (Wundt, 1894). According to Landsherre, experimental pedagogy was introduced around 1900 with experiments initiated by Lay and Meumann in Germany; Binet and Simon in France; Rice, Thorndike, and Judd in the United States; Claparede in Switzerland; Mercante in Argentina; Schuyten in Belgium; Winch in England; and Sikorsky and Netschajeff in Russia. From 1900 to the present, the study of educational problems developed quickly, and three major research movements emerged: (a) the child study movement, in which the research was strongly associated with applied child psychology; (b) the progressive movement, in which philosophy took precedence over principles of science, life experience over scientific method; and (c) the scientific research movement, with a logical positivist approach to educational problem solving (Landsherre, 1988).

Historically, research in mathematics education has been more closely aligned with the scientific research movement (Romberg & Carpenter, 1986). As Kilpatrick (1992) pointed out, the traditional paradigmatic boundaries of mathematics education are drawn from two fields of study—mathematics and psychology. As a result, scholarship in mathematics education has made unique contributions to our understanding of student cognition, teacher learning, curriculum design, and assessment. Thus, most discussions related to “democratic access” are limited to a focus on classrooms, and more specifically, individual student cognition. There are exceptions, but for the most part, studies in mathematics education are framed with a theory from the psychological paradigm. The concern we have with the psychological paradigm, and paradigms in general, is that its adherents often fail to consider alternative interpretations. Instead, the concepts and theories of the “accepted” paradigm guide the interpretation of the social problem and problem solving. Secada (1991) questioned the philosophical underpinnings of cognitive research on mathematics learning, teaching, and teaching/learning. He argued:

A danger in conducting this kind of research lies in the stress placed upon the individual and the submerging of that individual’s race, social class, gender, and other characteristics that locate that individual as a member of our society and of groups within that society. By excluding characteristics of diversity, we can create technically sophisticated models of the learning and teaching of mathematics. Tacit claims of universal applicability, however, must be tempered by the degree to which this research transforms problems of affect, course taking, underachievement, and careers into problems within the individual. Since cognitivist models of learning and teaching are seen as universally applicable to individuals, deviance from those models is interpreted as being due to individual differences. The alternative, that such differences are a function of the individual’s membership in a social group and that said membership is constructed through a complex web of social forces, cannot be addressed at present. We are in danger of creating models that further legitimate the characterization of minority students, who are becoming an increasingly larger portion of our population, as deficient. (Secada, 1991, p. 45)

Secada’s critique of cognitive research in mathematics education included another important observation. Many studies in the mathematics teaching domain, attend to teachers’ knowledge and beliefs in terms of content knowledge, but fail to study
what teachers’ think, believe, and do as a function of their diverse student populations. Moreover, mathematics education research at the intersection of teaching and learning is similarly lacking. Secada’s conclusions are closely associated with equity research that is often classified as multicultural education (see Banks & Banks, 1995). The philosophical precepts of multicultural education are most strongly linked to the progressive education movement of educational research (Grant & Tate, 1995). Multicultural education as an educational philosophy and ideology was born out of the Civil Rights Movement in the United States during the 1960s and early 1970s. It was initially conceptualized as an educational effort to counter racism in schools (Baker, 1973; Banks, 1975; Grant, 1975). Subsequently, it expanded to become the umbrella phrase for a school reform movement that addresses the nature and extent of access to educational opportunity, democratic decision making, and social action.

A great deal of the multicultural movement within mathematics education is defined by its opposition to the belief that mathematics should be confined to an elite group thought to possess the requisite talent denied the majority of the population (Cuevas, 1984; Ethington, 1990; Research Advisory Committee, 1989). Specifically, multicultural approaches in mathematics education have focused on (a) working with culturally diverse students to improve affective factors (e.g., self-esteem and attitude), (b) adding more diversity to the mathematics teaching workforce, and (c) introducing multicultural elements into mathematics textbooks (McLeod, 1992; National Research Council, 1989; Valverde, 1984). This body of literature delineates an approach to mathematics education principled on all students learning the traditional curriculum in traditional classrooms and being successful in society as it is currently configured. Recall that multicultural education is a product of the progressive movement of educational scholarship. Ernest (1991) argued that followers of the progressive perspective treat the problems of ethnic minorities as they perceive them, and consequently the solutions proposed are only partial, with a number of weaknesses, including the following. (1) The culture-bound nature of knowledge, mathematics in particular, is not acknowledged, and so the solution fails to address the cultural domination of the curriculum. (2) There is insufficient recognition of institutional racism in society, and so these root problems are not addressed. (3) The problems of overt and institutional racism are avoided in the classroom, with the aim of protecting the sensibilities of the learners. The outcome is a denial of these problems and their importance, despite their impact on children. . . . (4) Multicultural education is seen to be the solution to the problems of black children, and is not seen as a necessary response to the nature of knowledge and to the forms of racism that exist in society, and hence of importance for all learners, teachers, and members of society. (5) Through this limited perception and response to problems, this perspective is palliative, and tends to reproduce cultural domination and the structural inequalities in society. (p. 271)

A challenge for the field of mathematics education is to articulate the language and meaning of democratic access in the formation of interpretations that make up the theoretical perspective(s) of our scholarship. We submit that the field of mathematics education is in need a of “democratic access” hermeneutics, or theory of interpretation. The purpose of this chapter is to provide the beginnings of such a theory for mathematics education.

**OUTLINE OF DISCUSSION**

We have organized our remarks into four major sections. The first section is a selected review of the tracking (i.e., mathematics course-taking opportunities) literature and a closely related literature—school restructuring. These literatures provide insights
into how schools and school systems organize themselves to distribute knowledge and opportunity to learn in mathematics education. The central thrust of this section is on the strengths and limitations of the methodologies found in these literatures and the major findings from these literatures concerning the nature and extent of student access to mathematical knowledge. We will specifically focus on how opportunities to learn vary on racial and economic dimensions. After reviewing two organizational features strongly associated with access—tracking and school restructuring—the second major section is a review of several important studies that examine the school as mediator of opportunity to learn. The school context is often overlooked in mathematics education research and methodological design. The next major section is a review of two distinct sets of literatures that center directly on classroom practice—one generic (i.e., nonmathematics classrooms) and mathematics classrooms. The purpose of this section is to illustrate how the mathematics education literature, and more particularly future research on access, can be informed by a classroom-based literature on equity. The final section is an argument for the need to develop a theory of democratic access in mathematics education.

The literature reviewed for this chapter is largely from research conducted in the United States. This is an obvious limitation for a volume seeking an international perspective. We contend, however, the ways that access to mathematics are limited or extended vary depending on the social, economic, and cultural ideologies that make up the fabric of a country. Nonetheless, we hope our analysis of democratic access and school mathematics in the context of the United States will prove instructive to others compelled to examine practice in a particular country. Democratic access in mathematics education is a global problem, yet very few of the international comparative studies focus on internal—within country—access problems across demographic groups. A recent case in point is the Third International Mathematics and Science Study (TIMSS). We have valuable comparative information about achievement, classroom instructional practice, and curriculum, yet little or no discussion is devoted to how various demographic groups within each country are systemically regulated in and out of mathematical opportunity. It is safe to say that this is not the intent of these kinds of international studies. It is nonetheless clear that we need better information across countries about traditionally underserved populations. It is the intent of this chapter to begin such a discussion in the context of the United States with the hope it will move colleagues in mathematics education to focus similar attention on other countries. Thus, our goal is catalytic.

ORGANIZING THE COURSE OF STUDY: SEPARATE BUT EQUAL?

Two correlates of student mathematics achievement in many national mathematics assessments have been (a) increased time on task in high-level mathematics and (b) the number of advanced courses taken in mathematics. Research indicates that African Americans, Hispanics, and students of low socioeconomic status are less likely to be enrolled in higher level mathematics courses than are middle-class White students (Oakes, 1990; Secada, 1992). Furthermore, African American and Hispanic students, as demographic groups, are consistently outperformed by White students on national assessments of mathematics achievement (Tate, 1997). The positive relationship between mathematics achievement and course taking exists across multiple data sets (e.g., National Assessment of Educational Progress, Scholastic Achievement Test (SAT), and American College Testing Program (ACT); Tate, 1997). For example, Hoffer, Rasinski, and Moore (1995) reported on the relationship between the number of mathematics courses that high school students of different racial and socioeconomic backgrounds
completed and their achievement gain from the end of Grade 8 to Grade 12. The findings indicated that when African Americans and White students who completed the same courses were compared, the differences in average achievement gains were smaller, and none were statistically significant. Moreover, none of the socioeconomic status (SES) comparisons showed significant differences among students taking the same number of courses. These findings suggest that much of racial and SES differences in mathematics achievement in Grades 9 through 12 are a product of the quality and number of mathematics courses that African American, White, Hispanic, high- and low-SES students complete during high secondary school. Thus, the organization of school knowledge in the form of tracks is central to democratic access and academic progress (see also Smith, 1996).

Course-taking opportunities in U.S. schools are typically organized into two kinds of tracking systems, curricular and ability. Comprehensive high schools offer a wide range of mathematics courses associated with a different set of postsecondary options—college preparatory, vocational, and general education. No student could take all of the courses, and it is assumed that counselors or teachers will oversee the selection process, matching students to course options that reflect their “ability and needs.” To this end, students in most high schools are categorized by curricular tracks, each track involving a course sequence and, ultimately, a different set of opportunities to reason with mathematics. The college preparatory track has the highest status and affords the student a greater opportunity to reason with mathematics. Furthermore, there is evidence indicating that the quality of teaching varies across tracks. Ingersoll and Gruber (1996) reported that the amount of out-of-field teaching is not distributed equally across different kinds of classes and groups in schools. Both student achievement levels and type or track of class were related to access to qualified teachers. In each case, the pattern was the same—low-track and low-achievement classes frequently have more out-of-field teachers than do high-track and high-achievement classes. Also, teachers in lower track classrooms have shorter interactions with students and expect less of them than do teachers in academic tracks. Consequently, lower track teachers develop less supportive relationships with pupils.

Many high schools and middle schools also assign students into ability tracks. The assignments provide various levels of instruction to students across the different ability tracks. This version of tracking is more difficult to identify because the practice differs across the United States. For instance, schools may offer two courses in geometry. Both may have the same title, but the content covered in each course could vary dramatically. Another strategy is to offer students of different abilities entry into the college-preparatory mathematics courses at different times in their academic careers (e.g., 1st year of high school versus the 3rd year). Furthermore, the organizational structure of the school may include many tracks or just a few; schools may have tightly or loosely organized curricular or ability grouping; and schools may or may not connect tracks to a block of subjects or mathematics only. What is clear is that students are organized in ways that may prohibit democratic access to mathematics.

Tracking is a form of segregation. Segregation in education has a long legal history in the United States. In one of the earliest cases, Roberts v. City of Boston (1850), the plaintiff sought to desegregate Boston’s public schools to achieve access to the all-White schools of the city. The Roberts suit was eventually rejected by the Supreme Court of Massachusetts, but Black leaders lobbied the legislature for a law against segregated schools and succeeded in acquiring a law prohibiting segregation. Over the next hundred years, few school systems made the effort to desegregate; instead most continued to operate segregated schools until and even after the Brown v. Board of Education (1954) decision. In Brown the United States Supreme Court stated that in the field of public education, the doctrine of separate but equal was unconstitutional.

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In sum, maintaining segregated schools districts violated the Equal Protection Clause of the Fourteenth Amendment (Bell, 1992). As a matter of law, the Supreme Court replaced the accepted doctrine of “separate but equal” with the “equal opportunity for all,” with respect to public education.

Paul Green (1999) argued that the 1960s and 1970s witnessed a gradual erosion of support for equal access and equal educational opportunities. One artifact of this period was a series of legal challenges to tracking and ability grouping on the grounds that these practices resulted in intraschool segregation. The first legal challenge to tracking was initiated by plaintiffs in Hobson v. Hansen (1967) who alleged that tracking in the Washington, D.C. school system perpetuated racial segregation of students because African Americans were disproportionately represented in vocational and lower academic tracks. Like many other school systems in the United States, Washington, D.C. used a combination of standardized tests and teacher recommendations to sort students into ability groups. According to the school superintendent, students were sorted on the basis of ability and educational need, not on the basis of race (Hobson v. Hansen, 1967). The court did not concur, and it ordered that the system of tracking be abandoned in the school district. Building on the legal precedent of Brown v. Board of Education (1954), the Court indicated that the track system was unconstitutional and deprived African American and poor children their right to equal opportunity with White and more affluent children. In the written opinion of the court, Judge J. Skelly Wright stated: “Even in concept, the track system is undemocratic and discriminatory. Its creator [Superintendent Hansen, our addition] admits it is designed to prepare some children for white-collar, and other children for blue-collar jobs” (Hobson v. Hansen, 1967, p. 407). The ruling was built on two major findings. First, African American students were consistently assigned to the lower track at a greater rate than Whites, thus segregating the student body. Second, the lower track was deemed an inferior educational experience in comparison with the academic track.

Moses v. Washington Parish School Board (1972) was the next major legal challenge to tracking. Located in Louisiana, Washington Parish schools remained segregated until 1965. In reaction to court-ordered desegregation rulings, the school board implemented a plan to group students by ability. Like Hobson, the plaintiffs argued that ability grouping resulted in the segregation of students within the district. Specifically, the plaintiffs claimed that the use of IQ tests to determine track placement put African American students at a disadvantage to Whites, who had received a superior education. The Fifth Circuit Court concurred, stating that the use of standardized achievement tests for classification purposes deprived African American students of their constitutional rights. As in Hobson, the court indicated that homogeneous grouping was educationally detrimental to students confined to lower tracks, and African Americans constituted a disproportionate number of students in the lower tracks.

Another case before the Fifth circuit court, McNeal v. Tate County School District (1975) signaled a narrowing of the grounds to litigate questions about tracking. In McNeal, the court ruled that testing could not serve as the instrument to sort into track placement in a desegregated system until the district had resolved the products of de jure segregation. The court argued this provision was needed to ensure that the track assignment methodology, in this case IQ testing, was not based on the present results of past discrimination. This decision left open the possibility that curricular and ability segregation in public education might be constitutional. Schools could legally assign students to tracks that resulted in segregated racial groups as long as the segregation is a de facto outcome rather than an explicit goal of district policy.

Tracking litigation in the 1980s and early 1990s was strongly influenced by a neoconservative judicial perspective. In her Harvard Law Review article, Kimberle Crenshaw (1988) described the perspective:
Neoconservative doctrine singles out race-specific civil rights policies as one of the most significant threats to the democratic political system. Emphasizing the need for strictly color-blind policies, this view calls for the repeal of affirmative action and other race-specific remedial policies, urges the end of class-based remedies, and calls for the Administration to limit remedies to what it calls “actual victims” of discrimination. (p. 1337)

During this time period (1980–1992), the federal courts often deferred to school districts that used organizational practices and pedagogical policies such as tracking and ability grouping (see e.g., Quarles v. Oxford Municipal Separate School, 1989; Montgomery v. Starkville Municipal, 1987). In his historical analysis of tracking, Green (1999) noted that the judicial retreat from equal access and equal educational opportunity ended with the election of President William Clinton in 1992. He observed that “The Justice Department’s new assistant attorney general for civil rights, Patrick Deval, decided that tracking was the segregation tool of the 1990s . . . As a result, the 1990s saw cases challenging the harmful effects of policies and practices of tracking and ability grouping” (p. 245). In particular, four cases, People Who Care v. Rockford Board of Education (1994), Vasquez v. San Jose Unified School District (1994), Simmons v. Hooks (1994), and Coalition to Save Our Children v. State Board of Education (1995), produced rulings in favor of detracking school districts. The Rockford case was particularly important for the purposes of this chapter.

Rockford differed from past tracking litigation—for example, Hobson, Moses, and McNeal—in the area of supporting evidence. Before this case, litigators employed somewhat simplistic attacks that focused on discriminatory intent of the sorting mechanisms used by school districts to assign African American students to lower tracks. Although this approach led to victories in all three cases, the products of the verdicts were mixed. Once a “biased” sorting mechanism was eliminated (e.g., IQ testing) the practice of tracking could be resumed.

In Rockford, tracking expert, Jeannie Oakes (1990) accumulated a set of evidence on inequities that influence all lower track and minority students. She accumulated data from the district, such as curriculum guides, district reports, enrollment figures (disaggregated by grade, race, track, and school), standardized test scores, teacher recommendations for course enrollment, discovery responses, and deposition testimony. This wealth of quantitative and qualitative evidence convinced the court that placement practices skewed enrollments in favor of White students over and above what could be reasonably attributed to measured achievement. Thus, research methodology played a key role in the dismantling of an undemocratic system of opportunity to learn.

A National Portrait of Access

One of the methods employed to investigate questions related to democratic access involves the use of large-scale survey data. Typically, these data sets are nationally representative surveys (such as High School and Beyond or National Education Longitudinal Study [NELS]) that sample thousands of students at several hundred schools. The goal of nationally representative survey data is to be able to gain insight into the behavior and practices of schools, educators, and students nationwide. The organizational and structural barriers inhibiting democratic access to school mathematics are evidenced in a wide variety of data sets collected in the United States. Many important access issues are analyzed, for example, dropout rates, school completion rates, course taking, and so forth. It is beyond the scope of this chapter to review all of the access constructs embedded in the many nationally representative surveys. Instead, we focus on two organizational constructs: tracking and school restructuring. Historically in the United States, traditional approaches to mathematics education have been
closely aligned with a philosophy of elitism and social stratification that has resulted in tracking systems and other school practices that provide many students of color or of low-socioeconomic status with few opportunities to learn higher level mathematics (Oakes, 1990). Thus, an important way to investigate the question of democratic access in the United States involves the close examination of tracking practices and other related organizational features of schooling. This is also true in other countries that have differentiated curriculum opportunities in school mathematics. This scholarship moves beyond the traditional paradigmatic boundaries of mathematics education—mathematics and psychology—to include the sociology of education. Our purpose is to understand this literature from a methodological perspective. One central question is, “What are the methodological strengths and limitations of access scholarship in these two lines of inquiry?”

Most studies of track placement using large-scale survey data examine the relationship among student demographic characteristics, track placement, and achievement. Although students’ prior achievement is the strongest predictor of track placement, other background characteristics have also been found to affect where students end up in the tracking hierarchy (Oakes, Gamoran, & Page, 1992). For example, the findings regarding social class and track placement have been fairly consistent. “Researchers find that higher social class is associated with placement in more advanced courses or the college preparatory track” (Lucas, 1999, p. 41). This relationship holds even after previous academic achievement has been controlled for.

Although the findings on social class have been largely unambiguous, the results on the relationship between race and track placement have been less consistent. “Whether racial and ethnic status is an advantage, a disadvantage, or irrelevant to secondary school curricular location remains unclear” (Lucas, 1999, p. 41). Several large-scale survey studies have found that African American students are more likely to be placed in the lower mathematics tracks, even after prior achievement has been accounted for (Braddock & Dawkins, 1993; Catsambis, 1994; Dauber, Alexander, & Entwisle, 1996). Others, however, have found no racial differences in track placement once prior achievement has been considered (Alexander & Cook, 1982). Still others have found a positive effect of being African American on track placement. For example, Jones, Vanfossen, and Ensminger (1995) found that Black students were 2 times more likely than non-Black students to be on the academic track in high school.

There are those who suggest that these inconsistent results on the relationship between race and track placement are indicative of the problems with this type of research. One of the primary critiques of this research, as it relates to the question of equity, has to do with the aggregation of data across school districts (Useem, 1992b). According to Oakes (1994),

neither the negative impact of minority status nor discriminatory placement practices are obvious in analyses of large-scale survey data, particularly when the data are aggregated across school systems. Such analyses tend to obscure between-system differences in track assignments resulting from the composition of the student population in schools. (p. 87)

Oakes (1994) argued that differences in the racial makeup of districts explains the results of studies that have shown a minority advantage in track placement. She noted that students who would likely not be in the top track in predominantly White districts on the basis of achievement are in the top track in predominantly minority districts. Thus, when data are aggregated, the results seem to show an advantage in track placement for minority students after controlling for ability. However, Oakes (1994) contended that “this aggregation can mask considerable discrimination against minority students in high-track placement in both [predominantly White and predominantly minority] systems” (p. 88). Thus, the use of large-sample survey data
may, in some cases, distort rather than clarify the picture of inequity as it relates to tracking.

Researchers considering tracking are not the only ones who have used large-sample survey data to look at issues of democratic access in mathematics. Other relevant research using large-scale data sets has been conducted by Valerie Lee and her colleagues. In particular, they have conducted several studies investigating the practices or organizational features of schools that are associated with more equitable distribution of student achievement across different socioeconomic levels. In this work, the researchers have found several factors that appear to be related to greater equity. These include smaller school size, a "core curriculum," collective responsibility on the part of teachers, authentic instruction, and practices associated with school restructuring (Lee & Bryk, 1989; Lee, Croninger, & Smith, 1997; Lee & Smith, 1995, 1996, 1997; Lee, Smith, & Croninger, 1997). Practices associated with school restructuring include a collective set of goals, commitments, and practices enacted throughout the school; small learning groups for teachers and students; teacher opportunity to learn and collaborate with colleagues; shared governance linked to teacher teams; and a variety of learning opportunities for all members of the school community (Newmann & Wehlage, 1995).

As with the tracking research, however, this work is not without limitations caused by the use of survey data. One such drawback is the fact that definitive causal connections cannot be made on the basis of these data. For example, Lee and Smith (1996) suggested, on the basis of their results, that collective responsibility on the part of teachers leads to greater student learning. They conceded, however, that the structure of the data prevents them from completely ruling out an alternate causal explanation for their results, that teachers are more willing to accept responsibility for students who are already academically successful. Similarly, in the studies done by Lee and Smith (1993, 1995) on restructuring practices it is assumed that the reforms led to gains in student achievement. Without data on the timing of implementation, however, the authors could not rule out the possibility of a different order of events.

These restructuring studies also offer other limitations of large-scale data surveys. The data gathered in these studies came primarily from principal’s reports of restructuring practices (Lee & Smith, 1993, 1995). This allowed the researchers to count the number of practices reported by the principals but offered no information on the level of implementation of the practice or the number of students affected. The researchers themselves indicated a need to conduct a more intense investigation of a smaller number of schools and classes to more fully understand the effects of restructuring (Lee & Smith, 1993).

Although both the tracking and school restructuring literatures contribute to our understanding of issues related to equity, these large-scale studies have limitations in what they can reveal about the processes and conditions that lead to equity or inequity in schools. In fact, several researchers have warned against the overreliance on large-scale studies because of the possible factors obscured by such work (Garet & DeLany, 1988; Oakes, 1994; Oakes et al., 1992; Useem, 1992a). Specifically, these scholars charge that much of the variation among schools and districts is lost through this approach. So, what have we learned about democratic access in mathematics when the methodological approach calls for a smaller number of schools or districts?

Research on Democratic Access: Is Less Better?

There are several studies on tracking that take another approach to the investigation of democratic access. Rather than attempting to examine national access issues by sampling a large number of schools across the United States, these studies have as
their goal the creation of more detailed portraits of a smaller number of schools. Because of the much smaller number of sites, a greater variety of data sources can be employed. The studies included in this section drew data from school documents and records, interviews with parents and school personnel (teachers, administrators, or guidance counselors), questionnaires, and classroom observations. In one study by Oakes (1995), she also had access to court documents on the school districts involved.

One thing that is revealed in these multisite studies is the variability between schools and districts in tracking policies and the effects of this variability on students. For example, Useem (1992a) found:

substantial variations in ability group assignment policies in middle school mathematics among school districts that lead to inequities and arbitrary elements in students’ placement. . . . Students who would be deemed fully qualified for accelerated math in one system could easily find themselves rejected for that same track in an adjacent school district. (p. 346)

Whereas Useem (1992a) looked at differences among districts, Oakes and Guiton (1995) considered the differences among three schools in the same area. They found that the racial placement patterns varied substantially by school. For example, Latino students with the same achievement scores had very different chances of being placed in college prep math depending on the school in which they were enrolled (Oakes & Guiton, 1995). The authors suggested that decisions on course offerings and student assignment were the result of a complex dynamic “that has important commonalities across schools but that does not operate identically at all schools or for all students within schools” (Oakes & Guiton, 1995, p. 30).

Another characteristic of work done at this level is the potential to augment our understanding of trends that can be seen in the large-scale research. In terms of the tracking research, this would mean a better understanding of the origins of differential placement patterns in schools. For example, the results of interviews conducted by Oakes and Guiton (1995) offer some insight into track placements at the three schools involved in their study. The researchers found in their conversations with teachers at these schools that student race had come to signal ability and potential. For example, “educators at all three schools characterized Latinos as having poor basic skills, little interest in school, and being ‘culturally disinclined’ to aspire to postsecondary education” (Oakes & Guiton, 1995, p. 19). This perception of ability, in turn, influenced student placement, leading to the disproportionate placement of White and Asian students in the higher tracks and of Latino and African American students in the lower tracks. “At each school, racial groups had become identified in most educators’ minds with particular tracks” (Oakes & Guiton, 1995, p. 19).

Similarly, a study conducted by Useem (1992b) speaks to the origin of the relationship between track placement and socioeconomic status. The results of her study, in which interviews were conducted with parents from two school districts, suggest that “the strong correlation between parents’ level of education and their children’s placement in mathematics can be explained, in part, by the degree of the parents’ involvement in their children’s education” (p. 275). She found that parents with baccalaureate and graduate degrees appeared to pass on their educational advantages to their children by

being much more aware of the implications of academic choices made in schools, by being more integrated into schools affairs and parent-information networks, by having a greater propensity to intervene in educational decisions that are made for their children in school, and by the greater likelihood that they will exert influence on their children over the choice of courses. (Useem, 1992b, p. 275)
Thus, these studies demonstrate how the inclusion of a wider range of data sources, such as interviews, can lead to greater understanding of the reasons behind the larger patterns in student track assignment.

Another closely related line of research on tracking has gone beyond the study of placement practices to consider the experiences of students in the various tracks. One of the most frequently cited examples of work of this type is Oakes’ (1985) study of 25 middle and high schools. She documented several differences in the quality of education that students of different tracks were provided. For example, she found differences in the types of knowledge presented to students. In particular, students in the high-track classes had access to “high-status” mathematics knowledge (ideas and concepts), whereas students in the low-track classes repeated the same basic computational skills year after year. Furthermore, the goals of the teachers in the two tracks were different. Teachers in high-track classes hoped to develop competent and autonomous thinkers, whereas the emphasis in low-track classes was on conformity to rules and expectations. There also was a greater emphasis on control and less emphasis on learning in the low-track classes (Oakes, 1985). These and other differences between tracks, combined with the fact that poor and minority students are disproportionately to be found in the low tracks, point to the ways that schools systematically disadvantage certain segments of the student population. “Within subject areas, if access to certain kinds of knowledge or organizational arrangements is restricted for some students and enhanced for others, schools cannot be said to be providing equal education or even equal opportunity” (Oakes, 1985, p. 165).

In a more recent study, Oakes (1995) considered the tracking policies of two school districts and their effects on students. The results, in terms of the differences between tracks, were similar to her earlier study.

Students in lower-track classes had fewer learning opportunities. Teachers expected less of them and gave them less exposure to curriculum and instruction in essential knowledge and skills. Lower-track classes also provided . . . students with less access to a whole range of resources and opportunities: to highly qualified teachers; to classroom environments conducive to learning; to opportunities to earn extra “grade points” that can bolster their grade point averages; and to courses that would qualify them for college entrance and a wide variety of careers as adults. (p. 687)

Not surprisingly, she found that the academic achievement of students in the lower tracks suffered. The achievement gap between students in different tracks widened over the years of schooling. Students who were placed in lower level courses consistently showed smaller gains in achievement over time than their peers (with the same preplacement test scores) who were placed in high-level courses. Furthermore, Oakes (1995) demonstrated that the track placements in both districts were racially skewed. African American and Latino students were much less likely than White and Asian students with the same test scores to be placed in high-track classes. Oakes (1995) concluded from these findings that “grouping practices have created a cycle of restricted opportunities and diminished outcomes, and exacerbated the differences between African American and Latino and White students” (p. 689).

However, the results of a multisite study by Gutierrez (1995) suggest that tracking may not be the only productive lens through which to understand issues of democratic access in schools. In her study of eight secondary mathematics departments, Gutierrez (1995) focused on the features of departments that were most and least successful in getting large numbers of students to advanced mathematics. Her work is pertinent to this discussion because her focus was on schools with significant minority populations. She found that the presence or absence of a formal tracking structure did not seem to distinguish between departments that were more or less successful in advancing students in mathematics. As a result she suggested that “tracking is not
the pivotal policy on which student advancement depends” (p. 163). She asserted that tracking is not unimportant, but that “we need to look beyond tracking for solutions to the problems of access and equity in mathematics that many poor and minority students face” (p. 164).

Looking beyond tracking, yet still focused on an organizational structure, Gutierrez (1995) found that the following departmental characteristics were related to high levels of student advancement: a rigorous common curriculum, active commitment to students, commitment to collective enterprise, and innovative instructional practices. It is important to note that these elements are similar to those identified in the work of Lee and her colleagues (Lee et al., 1997a; Lee & Smith, 1993, 1995, 1996). Through the use of interviews and other data sources, however, Gutierrez’s findings move a step beyond those from the large-scale surveys. For example, Gutierrez (1995) was able to outline in detail what each of these factors entails. She is able to suggest what practices have to be in place for there to be a “common and rigorous curriculum,” or what it actually means to have an “active commitment to students.”

Furthermore, Gutierrez (1995) also addressed the relationship between the factors, suggesting that they are not stand-alone elements. Rather, the four characteristics are intertwined. Her results suggest that the presence of one or two of these factors will be insufficient to create departments that are successful at advancing large numbers of students. This finding is of particular significance within the context of mathematics education reform, for she noted that innovative instructional practices in the absence of the other three components will not lead to more students progressing to advanced levels of mathematics. She stated:

This finding challenges the calls from math educators to incorporate instructional practices like those presented in the NCTM Standards without addressing other aspects of the math teaching environment. . . . Without a clear understanding of the context in which these practices will be placed, innovative practices may serve to continue the patterns of inequality found in mathematics at many high schools. (Gutierrez, 1995, p. 188)

Outlined in the previous section on large-scale survey studies were two lines of inquiry: tracking and effective school restructuring practices. The research included in this section on multisite studies has also followed those same two lines of inquiry. It has been possible to see how the greater variety of data sources employed in the multisite studies fine tune and augment the results found with large-scale survey methodology.

Yet these multisite studies are not without their own limitations. As noted earlier, Gutierrez (1995) observed the problematic nature of focusing strictly on tracking to understand access to mathematics in schools. Furthermore, she also offered the potential limitations of her own work. Describing her study as offering a panoramic view of math departments and student outcomes, she cited the need to look more closely at classrooms. “If we believe that unequal opportunities to learn math occur partly through instruction, we must also explore the classroom as the unit of analysis” (Gutierrez, 1995, p. 200).

**DEMOCRATIC ACCESS: UNDERSTANDING THE ROLE OF SCHOOL CULTURE**

Before taking the advice of Gutierrez and switching the focus to the classroom, we review several important studies that centered on the school as the unit of analysis. These studies were selected because they moved beyond a focus on organizational features such as tracking, restructuring, and departmental thrusts. Rather, the entire school setting was the focal point of the research. Cole and Griffin (1987) argued that “We need to know much more about the conditions that lead to effective school
cultures and optimized student careers in school systems. Much work in the United States focuses on school effectiveness factors such as ‘management generalizations,’ but the tie to value factors that may be the pump-primers for effective implementation is not sufficiently understood” (p. 86).

McQuillan’s (1998) investigation of the culture of an urban high school provides insight into the value factors that influence democratic access in classrooms. In particular, he described the role of the “myth of educational opportunity” in this school. McQuillan suggested that our culture promotes the belief that students are offered educational opportunities and that it is an individual student’s responsibility to take advantage of those opportunities. What McQuillan found at Russell High School, however, suggested that students are, in fact, denied educational opportunities. He noted that there are many factors, some unintentional, that led to students’ lack of opportunity. Rather than a “malicious conspiracy,” this lack of opportunity resulted from the reactions of teachers, administrators, and guidance counselors to constraints (as they saw them) within the system. Teachers lowered their expectations in response to what they perceived as student resistance. Guidance counselors could not offer “guidance” because they spent all of their time creating schedules. Administrators had to focus their energies on creating an orderly school. McQuillan (1998) stated:

In the context of Russell High, all of these reactions made perfect sense. But as a consequence, the institution did not allow itself to enact educational opportunity. The structure of the school system and the organization of Russell High were ineffective, and therefore inequitable. (p. 81)

This study looks beneath the surface of an individual school to understand a status quo that allowed students to go uneducated. “Students, teachers, and administrators had struck their compromises. It was all routine and unquestioned, but it wasn’t educational” (p. 98). McQuillan argued, in part, that these compromises and the maintenance of the status quo were possible because the culture of the school was intricately tied to the myth of educational opportunity.

Fine’s (1991) study of an urban high school reveals similar routine and unquestioned mechanisms that allow the maintenance of an inequitable status quo. In particular, she focused on the ways through which the institution pushes students out of school, denying them whatever opportunities might come with a high school diploma.

The ease with which most of these students were accorded educational outcomes likely to guarantee them poverty and unemployment, enacted by well-intentioned educators, offers sobering evidence of the smooth functioning of public education as a system of injustice. The fact that this process is public, pronounced, and used to threaten the student body speaks to the entrenchment of, and commitment to, unequal outcomes inside public schools. (Fine, 1991, p. 100)

A third study we included in this section because of its focus on value factors in the school setting is Lipman’s (1998) investigation of two junior high schools. She looked closely at the culture and ideology of the schools in relation to their implementation of restructuring. She found that restructuring in these two schools failed to bring about fundamental changes in the education of African American students. Lipman (1998) posited that this failure resulted from several interrelated factors:

I found that educators’ cooperative efforts and opportunities to change policy were profoundly influenced by their ideological dispositions, the culture of the school and school district, and the structuring of race, class, and power in multiple contexts. On balance, I have concluded that, independent of their intent, educators at Gates and Franklin were engaged in a complex process of reproducing African American student failure and disempowerment through educational reform. (p. 291)
One of the factors that prevented substantive reform was the prevailing belief system of the teachers concerning their African American students. “The discourse surrounding African American and ‘at-risk’ students exerted a particularly potent influence on how educators defined the problem of African American low achievement and disengagement from school and on the actions that made sense to them” (Lipman, 1998, p. 291). Lipman indicated that most of the teachers at both schools offered explanations of low student achievement that were consistent with the deficit model, ascribing failure to “deficiencies in students’ social and economic condition, their families and culture” (p. 73). Lipman further described how this prevalent view of students, in combination with other factors, produced an enactment of reform that did little to change the educational lives of African American students.

McQuillan (1998), Fine (1991), and Lipman (1998) examined individual schools and revealed the specific structural and cultural features of the school and community that support inequity. Thus, these studies offer insight into the complex nature of inequity in schools. They offered a warning against simplistic explanations. For example, all three studies illustrate that the origin of inequity in schools is not as simple as intentional and conscious discrimination on the part of educators. Each of the studies documents the work of well-meaning educators. Yet the teachers’ beliefs played a role in the enactment of inequality in all three cases. Thus, the nature of inequity is multidimensional. What these in-depth studies offer, then, is greater insight into the combination of forces that prevent students from receiving democratic access in schools.

Just as the multisite studies offered understandings not possible with the large-scale survey studies, the single-site studies offer insights different from those of the previous two types. One significant limitation of these studies from a mathematics education perspective is the fact that they offer little insight into the specific nature of mathematics teaching in these schools. Although the researchers documented classroom practices and classroom episodes are described in the studies, the focus of these episodes is rarely on mathematics, and the episodes are typically used to exemplify a larger theme within the culture of the school. Thus, although these studies reveal much about the culture of the school in general, they say little about mathematics in particular.

It would be tempting, then to dismiss these single-site studies as irrelevant to mathematics education, but to ignore these studies altogether would be a mistake for they offer examples of the context in which mathematics education takes place. They point to the relationship between the larger ethos of the school and what happens inside individual classrooms, a relationship that is important to consider as we shift our focus from the school to classroom practice.

**DEMOCRATIC ACCESS: A CLASSROOM PERSPECTIVE**

The research to which we have paid particular attention at the classroom level has a focus on gender, SES status, race, and the intersectionality of these characteristics. We have included gender in this section because it intersects in important ways with the other demographic characteristics. Thus, our focus is less on gender and more on how gender interacts with the other demographic variables in relationship to democratic access to knowledge in the classroom. We have organized our discussion of classroom research into two major sections. The first section includes major studies of equity and access with a focus on gender, race, SES status, or some combination of these constructs. Note that these studies were not centrally focused on mathematics instruction. They are reviewed because each provides insight into some of the major debates about pedagogy in the equity literature. The second section is a review of equity-related studies within the mathematics education literature. These studies were
selected because they examined classrooms and teachers that attempted to provide access to mathematics as defined in recent mathematics reform documents (NCTM 1991, 1993, 2000).

Classrooms and the Equity Question

One distinction that recurs throughout this section has to do with the nature of how inequity is described. In particular, there are studies that document differential treatment of groups of students in the same classroom and those that reflect differential student responses to the same treatment. An example of this distinction can be seen in the contrast between two studies on gender. Sadker and Sadker’s (1986) study of more than 100 fourth, sixth, and eighth grade classes offers evidence of differential treatment of boys and girls in the same classes. The researchers found that male students received more attention from teachers and were allowed more time to speak than were female students. Moreover, teachers were significantly more likely to give precise feedback (in the form of praise, criticism, or remediation) to male students. The authors concluded that “students in the same classroom, with the same teacher, studying the same material were experiencing very different educational environments” (Sadker & Sadker, 1986, p. 513).

A study by Guzzetti and Williams (1996) of the participation patterns of boys and girls in physics classes also revealed differences in students’ educational experiences based on gender. These differences, however, were in student response rather than teacher treatment. Their results suggest that refutational discussions, considered an important part of the learning experience in these classes, favored male over female students. Girls rarely volunteered to speak and said that they felt intimidated by these discussions in which they were to argue their ideas with other class members. The researchers also found that girls’ participation was not facilitated by the use of cooperative learning unless they were in groups of all girls (Guzzetti & Williams, 1996). These findings suggest how the same treatment can evoke differential classroom experiences. Furthermore, they would seem potentially important because of the apparent similarities between the nature of the discussions described in these classes and those that are advocated in mathematics reform (NCTM, 2000).

There is also evidence that studies such as Sadker and Sadker’s (1986) that focus only on gender are missing a significant part of the picture in terms of students’ differential treatment by teachers. Specifically, the results of other studies point to the fact that students of the same gender can have very different classroom experiences, a finding that suggests that these experiences are linked to the intersection of gender and race. For example, Irvine (1990) described a study of 63 classrooms that focused on teacher verbal feedback, student initiating behaviors, and public response opportunities for students in lower elementary (K–2) and upper elementary (3–5) classes. She found it important to take into account student race, gender, and grade level to understand what was happening in the classes. She discovered that White female students at both grade levels received significantly less teacher feedback (positive, negative, or neutral) than students in other race or gender categories. However, the case for Black female students was slightly different. Irvine (1990) argued that Black girls became less salient in the classroom as they moved up in grade levels. There were no significant differences in teacher feedback between Black girls and White and Black boys in lower elementary school, but differences appeared in upper elementary, at which point Black female students received significantly less teacher feedback than they had in lower elementary and less than Black and White male students. “By the time Black females enter the upper elementary grades, they seem to have joined their White female counterparts in their invisibility, thereby resulting in fewer interactions with teachers” (Irvine, 1990, p. 71).
Irvine suggested that these changes may reflect growing feelings of indifference on the part of teachers toward Black female students in upper elementary grades.

A study by Grant (1984) indicated that, from an academic standpoint, this lack of salience may start even earlier for Black female students. In a study of six first-grade classrooms, Grant found that none of the six teachers singled out a Black female student as having an outstanding academic ability. Furthermore, the teachers seemed to focus more on nonacademic than academic criteria when assessing Black female students (Grant, 1984). This occurred despite the fact that the teachers were specifically asked by the researcher about students’ academic performance.

Grant (1985) also considered the experiences of Black male students in the same classrooms. She found three recurrent themes emerged more often in evaluations of Black boys than other groups of students: poor academic skills, potential behavior problems, and feelings of teacher uncertainty or lack of understanding with regard to Black students. Grant’s (1985) observations revealed verbal feedback patterns on the part of the teacher that were generally consistent with the teacher’s perceptions of Black boys. For example, Black boys were more likely than other students to be given qualified praise for academic work, suggesting that the work was good but did not measure up to the highest standards or to the work of other students in the class. In terms of feedback for behavior, teachers gave less positive and more negative feedback to Black male students than to other groups of students and responded more quickly to minor misbehaviors on the part of black male students. Grant also observed more instances involving Black boys than other groups of students in which the teacher seemed to underestimate the attention level of the student. The results of these studies seem to lend support to Irvine’s (1990) assertion that

Any consideration of the school experiences of black children must take into account the gender of the children. It seems nonproductive to posit that black males are more at risk than black females, or vice versa. What is pertinent is that both groups, for different reasons, experience discrimination and isolation with a similar outcome—poor academic achievement. (p. 79)

Earlier we indicated that many of studies included in this section could be classified as describing differential treatment of students by teachers or differential responses to the same treatment. There is another type of study included here that fits neither of these two categories. The studies in this third category do not compare the experiences of one group of students to another. Instead, these studies focus only on one demographic group of students and the ways that the members of this group are advantaged or disadvantaged by the experiences that they encounter in classrooms.

One direction that such studies have taken is to focus attention on the mismatches between the home culture of the student and the culture of the classroom. A study by Mohatt and Erickson (1981) of two Native American classrooms suggested that participation can be made easier for students when classroom organization and interaction patterns are more “culturally congruent” with those of the students’ home culture. Similar results reflecting enhanced student participation can be seen in studies of the KEEP program (Kamehameha Elementary Education Program; Au & Jordan, 1981; Vogt, Jordan, & Tharp, 1993). The focus of the KEEP program was on improving the academic success of native Hawaiian children. Research identified sources of conflict for Hawaiian children between the cultures of home and school. Changes were then made in instructional practices, classroom organization, and motivation management to make these elements of classroom practice more compatible to Hawaiian culture (Vogt et al., 1993). The results of these changes suggest that both participation and greater student achievement can be facilitated by more “culturally compatible” or “culturally appropriate” pedagogy (Au & Jordan, 1981; Vogt et al., 1993).
Ladson-Billings (1995) pointed out, however, that these studies equate student success with achievement within the existing system. “The goal of education becomes how to ‘fit’ students constructed as ‘other’ . . . into a hierarchical structure that is defined as a meritocracy. . . . It is unclear how these conceptions do more than reproduce current inequities” (Ladson-Billings, 1995, p. 467). In contrast, she outlined a different pedagogical model that “not only addresses student achievement but also helps students to accept and affirm their cultural identity while developing critical perspectives that challenge inequities that schools (and other institutions) perpetuate” (Ladson-Billings, 1995, p. 469). She referred to this model of teaching as “culturally relevant pedagogy” (Ladson-Billings, 1994, 1995). Whereas the culturally appropriate pedagogy of the KEEP program (Au & Jordan, 1981; Vogt et al., 1993) reflected a fairly narrow set of changes in reading instruction, culturally relevant pedagogy is broader in scope. Ladson-Billings (1995) posited that the practice of culturally relevant teachers is not reflective of one teaching technique but can include a range of teaching behaviors. The teachers who worked with Ladson-Billings were chosen because of their effectiveness in educating African American students. Thus, the characteristics of culturally relevant pedagogy can be viewed as those that lead to the creation of more equitable learning environments for African American students. The characteristics of culturally relevant pedagogy center around the teacher’s conceptions of self and others; the manner in which social relations are structured in and out of the classroom; and the teacher’s conceptions of knowledge (Ladson-Billings, 1994, 1995). These characteristics include, among others, the belief that all students can succeed, connectedness with all students, encouragement of a “community of learners,” and willingness to help students develop necessary skills (Ladson-Billings, 1994).

**Equity and Mathematics Reform**

The final group of studies included in our analysis of democratic access in mathematics classrooms was selected for two reasons. First, the studies were part of large multiyear projects focused on classroom-based research. Second, each project was an effort to reform school mathematics in a manner that was consistent with the teaching practices and curricular goals found in the NCTM reform documents. In 1980, NCTM, a professional organization of mathematics teachers, supervisors, and college professors, released *An Agenda for Action*, which described a 10-year reform process. A central goal of *An Agenda for Action* was to move the focus of school mathematics from a strictly basic skills curriculum to a more balanced approach that included a problem-solving conception of mathematics content and pedagogy. Subsequently, but not as a direct result of *An Agenda for Action*, NCTM sponsored the development of the *Curriculum and Evaluation Standards for School Mathematics* (1989), the *Professional Standards For Teaching Mathematics* (1991), and the *Assessment Standards for School Mathematics* (1995). These documents were a product of extensive literature reviews and a series of technical reports that described the key ideas and issues in the field of mathematics. This set of reform documents and the movement to reform school mathematics are important from an equity perspective. Past reform efforts have failed to significantly improve democratic access to mathematics for African American, Hispanic, and low-SES students (Tate, 1996). Thus, a close examination of more recent reform efforts is critical.

**Cognitively Guided Instruction (CGI).** CGI was developed by researchers at the University of Wisconsin. The CGI studies are built on Carpenter and Moser’s (1983) analysis of young children’s learning of addition and subtraction. Subsequent studies were conducted to determine if teacher knowledge of children’s thinking would affect teachers’ instructional decision making and student learning (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). These
studies indicated that knowledge obtained from research on learners’ thinking can be used by teachers in a way that has an impact on students’ learning. The CGI studies support the idea that knowledge of students’ thinking, when integrated, robust, and a part of the established curriculum, can affect the teaching and learning of mathematics (Fennema & Franke, 1992). “The goal of CGI teacher development has always been to help teachers develop an understanding of their own students’ mathematical thinking and its development and how their students’ thinking could form the basis for the development of more advanced mathematical ideas” (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996, p. 406). CGI does not involve a particular curriculum or specify a program of instruction (Hiebert, Carpenter, Fuson, Wearne, Murray, Olivier, & Human, 1997). Unlike the other professional development projects to be considered here, CGI does not have an explicit equity component, nor was it targeted at a particular group of students. However, it has been implemented in settings with high concentrations of students of color.

For example, Carey, Fenemma, Carpenter, and Franke (1995) offered descriptions of CGI classrooms in a predominantly African American school context. Twenty-two first-grade teachers from 11 schools in Prince George’s County, Maryland, an urban school district bordering Washington, D.C., participated in a study designed to determine the efficacy of CGI with African American students. The student populations of the classrooms in the study exceeded 70% African American. Furthermore, 7 of the 11 schools participated in Chapter 1, a federally funded program of Title 1 of the Elementary and Secondary Act, a good indicator of high concentrations of low-income students in a school. The teachers, who participated in the study attended a 2-week summer inservice program that was followed with five full-day professional development days offered during the academic year. The researchers document a change in the teachers’ implemented mathematics curriculum, with a greater focus on problem solving beyond that typically associated with the first-grade curriculum. The teachers also displayed an ability to take advantage of student thinking about important mathematical ideas, ultimately building on student understanding to establish new knowledge of school mathematics.

Villasenor and Kepner (1991) reported on the implementation of CGI in a minority context. The study was carried out with 12 treatment classes and 12 control classes in which the percentage of non-White populations ranged from 57 to 99%. The CGI group performed significantly better on a 14-item word-problem posttest, an interview on word problems, and an interview on number facts. The CGI students also used advanced strategies significantly more often than non-CGI students on both problem solving and number facts. Peterson, Fennema, and Carpenter (1991) argued that “Villasenor’s results are important because they provide concrete evidence for the effectiveness of the CGI approach with a disadvantaged population of students” (p. 78).

Fennema, Carpenter, Jacobs, Franke, and Levi (1998a) described a longitudinal study of students’ mathematical thinking. CGI students were interviewed several times in first through third grades on number facts, addition and subtraction word problems, and nonroutine problems (involving multiple steps and requiring interpretation and analysis). In third grade, the students were also given extension problems that involved money and three-digit numbers and on which students could not use paper and pencil. The researchers considered both the students’ performance and the type of strategy used to solve the problem. The results suggest that the outcomes of CGI instruction may not be the same for boys and girls. There were no significant gender differences in the number of correct solutions on the number facts, addition and subtraction word problems, or nonroutine problems. However, boys successfully solved significantly more of the extension problems than girls did. Strong gender differences were also found in strategy use across the three years. “Girls tended to use more modeling or counting strategies, while boys tended to use more
abstract strategies such as derived facts or invented algorithms. In the third-grade spring interview the girls used significantly more standard algorithms than did boys’ (Fennema et al., 1998a, p. 9). On the basis of additional analyses, the researchers suggested that there is a connection between these two findings. Students, both boys and girls, who used invented algorithms in earlier grades were better able to solve the extension problems than those who did not. Thus, girls’ lower performance on the extension problems seems to be related to their strategy use in earlier grades.

Fennema, Carpenter, Jacobs, Franke, and Levi (1998b) stated that invented algorithms are more closely linked to conceptual understanding than the standard algorithm, which can be used procedurally without understanding. Thus, “the gender differences that were reported in the study strongly suggest that more girls than boys are following a pattern of mathematical learning that is not based on understanding” (p. 20). Although the evidence from previous studies indicates that CGI students learn with more understanding than students in traditional classrooms, it seems that girls may not be benefiting from this instruction to the same extent as boys.

As a result, the authors (Fennema et al., 1998b) called into question the suggestion that teaching consistent with reform recommendations will, in and of itself, lead to greater equity. They noted that

many advocates of basing curriculum on understanding as well as most scholars who study teaching and learning believe that equity issues can be addressed by such an emphasis. There is an underlying assumption that one program based on understanding will enable all students to learn in an equitable fashion. (Fennema et al., 1998b, p. 20)

On the basis of the findings of this study, the authors suggested that this assumption may not be valid. Equity may not be a natural by-product of reform teaching. In this case, there appears to be a possible gender equity problem.

**Project IMPACT.** Project IMPACT “is a school-based teacher enhancement model for elementary (K–5) mathematics instruction designed to foster student understanding and to support teacher change in predominantly minority schools” (Campbell, 1996, p. 449). There were six schools involved in the original study (three treatment and three control). The model involved: (a) a summer inservice program, (b) an on-site mathematics specialist in each school, (c) manipulative resources for each classroom, and (d) teacher planning and instructional problem solving during a common grade-level planning period each week. The focus of the model was on instructional approaches consistent with a cognitive, constructivist perspective on learning, emphasizing interaction and collaboration rather than the typical direct instruction approach. Unlike CGI, Project IMPACT focused specifically on teaching for understanding in urban schools. Thus, content addressing “teaching mathematics in culturally diverse classrooms” was included in the program’s summer inservice. Supported by campus-based mathematics specialists, instructional change occurred in most treatment classrooms, particularly where the instructional leadership by the principal encouraged and embraced the reform process. The students in these schools were assessed in the middle and at the end of each school year. Campbell (1996) summarized the results as follows:

The influence of the IMPACT treatment on student achievement was not immediate. The students in the IMPACT treatment schools did not evidence statistically significant higher achievement, as compared to the students in the comparable-site schools, until the middle of second grade; however, once established, this mathematics differential continued through second and third grade. (p. 463)

In her dissertation White (1997) examined the nature of questioning in four third-grade classrooms both before and after the teachers went through the Project IMPACT
12. POLITICAL AND SOCIAL CONTEXT

summer inservice program. The study documented the question–response pattern, the cognitive level of the question (low or high), and the race and gender of the student who responded. White found that students’ educational experiences, as reflected in classroom questioning, differed both between and, in some cases, within classes. There were two teachers, Ms. Davis and Ms. Tyler, who were fairly equitable in their distribution of questions. “They posed questions to all students across questioning patterns and cognitive levels” (White, 1997, p. 300).

In Ms. Atkins’ class, however, the distribution was more skewed. Overall, the majority of the questions were answered by female students. Yet a look at the different cognitive levels reveals racial patterns as well. Most of the high-level questions were answered by White and Asian girls. Black and Hispanic girls were asked a relatively low number of high-level questions. According to White (1997), the origin of this disparity lies in Ms. Atkins’ perceptions of students’ academic ability and her own discomfort with mathematics. Ms. Atkins wanted to ask high-level questions, but her own lack of understanding caused her to call only on students who she thought would give the correct answer. Thus, only the students perceived to be of high ability were selected to answer high-level questions. A similar pattern of focusing only on the students who were perceived to have the greatest mathematical understanding was found in the class of the fourth teacher, Ms. Smith.

This detailed study of question and response patterns is important for at least two reasons. First, it documents a partial success story for Project IMPACT in terms of improving equity in classrooms. Two of the four teachers appeared to change their practices as a result of their participation in the initial IMPACT summer inservice and ongoing campus level assistance. Both Ms. Davis and Ms. Tyler were more equitable in their distribution of questions after the inservice than they had been before. This study is also important because it suggests the need to look closely at teachers’ explanations for their actions to more fully understand what is happening in the classroom. This is similar to the assertion made by Stanic and Reyes (1986) of the importance of considering teacher intentionality. For example, the case of Ms. Atkins indicates that teachers’ inequitable actions can originate from a variety of sources, including inadequate content knowledge.

QUASAR. QUASAR is described as “an educational reform project aimed at fostering and studying the development and implementation of enhanced mathematics instructional programs for students attending middle schools in economically disadvantaged communities” (Silver & Stein, 1996, p. 476). One purpose of the project was to help students develop a meaningful understanding of mathematical ideas through engagement with challenging mathematical tasks. The QUASAR project supported teachers and administrators in six urban middle schools. Each school site worked with a resource partner—typically mathematics educators from local universities—to improve the school’s mathematics instructional program with a focus on mathematical understanding, thinking, reasoning, and problem solving. The site teams operated independently in the design and implementation of its curriculum plan, professional development, and other features of its instructional program. There were regular interactions among representatives from all QUASAR sites. Moreover, each site-based team, benefitted from financial support, technical assistance, and advice from the QUASAR staff, housed at the Learning Research and Development Center at the University of Pittsburgh.

Silver and Stein (1996) described three analyses used to assess the effectiveness of this kind of instruction. Unlike the CGI and IMPACT studies, there was no control group in the QUASAR study; therefore one method used to determine the impact of QUASAR was the examination of changes in student performance over time. The results from the first 3 years of the project indicated that “students developed an
increased capacity for mathematical reasoning, problem solving, and communication during that time period” (Silver & Stein, 1996, p. 505). A second method of evaluation used a variety of tasks from the National Assessment of Educational Progress (NAEP) as pseudo-control groups (Silver & Lane, 1995). The QUASAR students were given items from the 1992 eighth-grade NAEP. The results were compared with those of NAEP’s national sample and disadvantaged urban sample. The findings from the analysis of student performance on the nine open-ended tasks were very informative about the effectiveness of QUASAR. QUASAR students performed at least as well as the national sample on seven of the nine tasks. Silver and Lane (1995) noted that this is an important result in light of the fact that the national sample had significantly outperformed the disadvantaged urban sample on all nine tasks. They stated that

The findings clearly suggest that the mathematics performance gap between more and less affluent students has been significantly reduced for students attending the QUASAR schools. Thus, the performance of QUASAR’s students is far greater than would have been expected, given their demographic similarity to NAEP’s disadvantaged urban sample, and one can infer that the instruction at QUASAR has a beneficial impact on students’ mathematical performance. (Silver & Lane, p. 62)

A third method of evaluation examined outcomes other than achievement, considering whether QUASAR instruction was linked to increased access and success in algebra coursework. Silver and Stein (1996) reported that students from QUASAR schools were both qualifying for and passing algebra in ninth grade at substantially higher rates than before QUASAR.

A PROBLEM IN SEARCH OF A METHOD

There are many lessons learned from the research reviewed in this chapter. We have organized our discussion of the literature reviewed in this chapter into two sections. The first section is a discussion of the findings from the various studies with a focus on understanding how access to school mathematics has been studied and the strengths and limitations of the methodology employed in the studies. The second section is a call for a theory of interpretation in the study of democratic access to school mathematics. This section builds on the literature reviewed in this chapter.

Major Findings

Two questions guide our discussion in this section: (a) What have we learned about democratic access to school mathematics in the United States as a result of this review? (b) Do these findings have implications for scholars outside the United States? We view these questions as interrelated and important to the effort to catalyze more research on democratic access across the globe.

There were several key findings from our review. First, large-scale, nationally representative survey studies indicate that poor students are disproportionately placed in lower tracks and have access to less qualified teachers of mathematics. This method of exploring democratic access across economic dimensions provided a converging set of findings. In contrast, to understand the tracking practices experienced by racial minorities required multiple methodologies to capture the complexity of the organizational practice. The large-scale survey studies provide a mixed and unclear message. By focusing the research methodology to a smaller number of sites and a wider variety of data sources, however, the findings are more clear. Racial minorities are disproportionately placed in lower tracks. Students in lower tracks have less access to high-status knowledge, highly qualified teachers, and classrooms conducive to learning. The explanations for this practice vary. One explanation for differential curriculum
opportunities is that parental involvement influences track placement. This explanation infers that certain parents—mostly White—are placing significant pressure on the system to ensure access for their children. This explanation does not appear to fully explain why students are segregated.

Another explanation is that teachers’ beliefs about students can influence track placement. Beliefs and expectations are powerful constructs related to achievement across the educational spectrum. There is growing evidence that suggests the Black-White achievement gap, in part, is associated with expectations and stereotypes (Steele, 1997). Low expectations influence how African American, Hispanic, and low-SES students are socialized into mathematical learning opportunities. For example, Smith (1996) investigated how early access to algebra (eighth grade) influenced subsequent access to advanced mathematics courses and high school mathematics achievement. She found that early access to algebra has an effect beyond increased achievement and, in fact, may socialize a student into taking more mathematics. In essence, having a credit for a year of algebra at the beginning of high school is a credential, regulating the expectations of school personnel and ultimately providing access to more advanced coursework in mathematics. Understand the logic: The credential increases both the students’ and teachers’ expectations about how much mathematics the student will take in high school, keeping students in the college-prep track longer and producing higher results.

The importance of understanding the role of tracking as described or inferred from studies by Hoffer et al. (1995), Smith (1996), and the many other tracking-related studies and legal challenges reviewed here is that this body of literature points to a single direction—course-taking matters. More important is that policy can be used to intervene. One method of intervention is to mandate a common curriculum at the eighth grade and high school. Of course, this option is not without problems. A centralized curriculum cannot solve all access issues. One key question is, “How do you help students make the transition from the elementary school curriculum to earlier access to algebra?” This is especially relevant for students who have been underserved and, as a result, are not prepared for the transition. The research conducted in CGI, Project IMPACT, and QUASAR indicates that providing elementary and middle school students with a conceptually rich, problem-centered approach where student thinking is central to instructional practice is an important element of the transition process. There is one challenge for administrators attempting reform efforts built on precepts consistent with this type of instruction. The challenge is how teachers form beliefs about what is appropriate mathematics. Many of the goals of the basic skills movement are consistent with teachers’ perceptions of students’ ability (Knapp & Woolverton, 1995; Zeichner, 1996).

Teachers’ beliefs were also a part of the explanation for the failure of schools to mediate opportunity to learn. Students in schools were profoundly affected by the culture of the school. The culture of schools is at least loosely connected to teacher’s beliefs and expectations about students. Moreover, the enactment of substantive reform can be blocked by teachers’ beliefs. This retrenchment is further complicated by routine and unquestioned mechanisms that allow for the maintenance of the status quo—and ultimately denied educational opportunities. Once the classroom door closes, many students of color and female students are subjected to learning environments that fail to capitalize on their learning potential. It appears that classrooms are divided into separate worlds for “different” students. Some students are provided access to high-level discourse and others with little or no opportunity to engage in real conversations about mathematics. Too often a student’s race, SES status, gender, or a combination of these constructs are related to access to knowledge in classrooms.

Real democratic access to mathematics was discovered in two distinct literatures. The school restructuring literature indicated that more equitable schools were smaller
with a focus on core curriculum goals. Furthermore, these schools practiced authentic instruction and teachers assumed a collective responsibility for student learning. Similarly, in her study of secondary departments, Gutierrez (1995) found that a rigorous common curriculum, commitment to students, collective enterprise, and innovations instructional practice were positively related to student performance in mathematics.

In sum, the literature in the chapter points to several factors that influence real democratic access to mathematics. Together, they provide a foundation for future research on democratic access in mathematics. We are convinced that mathematics education scholars across the world would better understand democratic access to mathematics in their respective countries if additional scholarship existed in the following areas:

- Student access to mathematics as a function of teachers’ beliefs (about mathematics and student ability) and student demographic background
- Collective perspectives about mathematics across departments and schools
- Diversity challenges and opportunities in the school and classroom learning environment
- Authentic performance and active in-depth learning across demographic groups
- Organizational impediments and support mechanisms related to opportunity to learn mathematics

Our review is an effort to understand the multiple dimensions of democratic access listed above. We have limited our analysis to the United States. We contend that the legal, social, political, and educational practices of this particular country are unique, but the areas of research are not. We believe these areas of research are relevant in many countries of the world. For example, England, Australia, Brazil, Japan, Germany, and South Africa have systems of education that do not adequately serve the needs of many student demographic groups (Allan & Hill, 1995; Hoff, 1995; Grant & Lei, 2001). One challenge for mathematics education researchers interested in studying these broadly defined areas will be method.

**Interpretation and Method**

Bishop (1992) offered the following question, “To what extent can we perceive national perspectives on research in mathematics education?” (p. 714). He reasoned that there are some methodological tendencies within certain countries; however, national characterizations overstate the case and are unproductive. Bishop described three traditions that capture the research enterprise in mathematics education across the world. He stated, “The Pedagogy was concerned with the development of new procedures and practices, the Empirical-scientists were concerned to develop standardized evidential procedures in order to explain reality, and the Scholastic-philosophers were occupied with theoretical analyses and qualities of arguments” (p. 716). Each orientation to research shares one common trait: a method-driven approach to scholarship. The method-driven approach to understanding social problems has both strengths and weaknesses. One strength of this approach is that the method-driven researcher typically has a set of methodological criteria to judge the quality of research. The criteria are focused on acceptable research methods, rather than on the quality of new knowledge gained about the problem under study. The latter is a limitation of this research approach.

We argue that there is a need for additional research in mathematics education that is problem-centered with a focus on democratic access. This means the researcher would use methods across the three traditions in mathematics education (or beyond) to provide a more cogent explanation of democratic access in mathematics education.
One excellent example of problem-centered research related to democratic access was conducted by Oakes in her role as expert witness (see People Who Care v. Rockford Board of Education). Oakes accumulated a set of evidence that gave the courts a powerful wealth of quantitative and qualitative data to illustrate how placement practices inhibited access to high-level knowledge for Black students and favored White students. In the context of a legal case, methodology is relevant; however, seeing and understanding the problem is central. As expert witness, Jeannie Oakes, was required to provide a comprehensive story of the facts at hand. Would an expert witness be effective if he or she simply provided technical jargon related to methodology and a limited set of findings? Most likely not. Instead, the expert witness interprets the multiple sets of data and evidence to make sense of the phenomena under investigation. Each piece of evidence represents a new encounter informed by all the prior data. There is what must seem like an endless process of interpretation. An essential part of hermeneutics is the effect of cultural heritage and world view on interpretation. The sociology of knowledge recognizes the influence of societal values upon all perceptions of reality. We offer four principles of interpretation for those interested in the study of democratic access in mathematics education.

1. Consciously reconstruct our preunderstanding: If we desire an honest examination of access issues, we must define carefully where we and our research tradition stand on the examination of access in mathematics. This is accomplished at three levels—individual, nationally, and internationally—across the field of mathematics education. Unless these preunderstandings are brought to the surface they will covertly dominate and skew our research, for it is natural to want our research to corroborate rather than challenge our presuppositions.

2. Engage all research related to democratic access to knowledge in schools: We must take advantage of the methodological tools that are outside of the traditional paradigmatic boundaries of mathematics education. It is particularly crucial to review research that seems to contradict the conclusions our tradition prefers. It is quite natural for a field to examine only those traditions that favor its position and to explain the others away (or simply ignore).

3. Exegete all literatures relevant to issues of access: This is difficult, for each tradition has different methods and criteria for acceptance. Thus, it is no wonder that few systematic reviews of access issues exist. Yet this would be a major step toward the development of a theory of democratic access in mathematics education.

4. Trace the developing contextualization of democratic access through the history of education: The changing viewpoints on equity exemplify the development and restatement of the construct through the differing eras and situations of schooling in the broader context of society. Moreover, we should study the history of democratic access in the development of our traditions in mathematics education.

The importance of hermeneutics for the study of democratic access in mathematics education derives from its focus on applying this interpretative method to the study of human activity, social interactions, and products (texts). We offer these four principles as a beginning step toward a theory of democratic access in mathematics education.

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12. POLITICAL AND SOCIAL CONTEXT

13.1. INTRODUCTION

The arrival of a new millennium has transformed the expectation of educational change in contemporary societies. There is a renewed interest in education; every country, in one way or another, is preparing to face the future with education as the sustaining foundations of what has been called the century of information and knowledge. Therefore, the transformation of educational systems will have to incorporate the immense scientific and technological developments of the past decades (which are in part responsible for our expectations for the new century). Every country, depending on its sociocultural and economic conditions, is preparing to face new challenges in education, as education is being understood not only as a systematic solution to most immediate social needs, but also as a means to face the unknown and the unexpected. This suggests the need to reformulate what is taught, as well as how and why it is taught. Consideration of these issues will allow us to define new requirements of mathematics education that must be identified at all levels of learning from elementary school to university. Thus, we foresee important transformations in the field of curricular design and development, as well as in the application of new learning tools.

In countries such as Mexico, in addition to teaching specific skills, we will need to teach students to think critically about the ongoing changes in the world and about how these changes can affect educational and national realities. It will be necessary for education to generate the ability to respond with a spirit of innovation when faced with the changing reality.

In Mexico, access to knowledge cannot be regarded as a politically neutral issue because there is an obvious problem of exclusion for those who are on the margin
of the educational process at any of its levels. Our inclusion in the contemporary world of globalization demands that we have the critical ability to transfuse scientific and technological developments into our educational realities. We cannot forget that a preexisting school culture has left a significant mark on the players within the educational system. This school culture requires the gradual reorientation of its practices and cognitive and epistemological assumptions to gain access to the powerful ideas of mathematics and to the development of equally fundamental skills for these ideas, such as exploring, modeling, handling of information, and the ability to systematize. This is not always possible under the traditional teaching model that has dominated education until recently.

Today in Mexico, 7 out of 10 Mexicans live in urban areas; 25% of the population is concentrated in four major cities, and a population of 10 million is dispersed in very small communities. Great ethnic diversity and high levels of poverty and illiteracy live side by side in these communities. This simultaneous concentration and dispersion is a negative factor impacting the development of education at a national level (Academia Mexicana de Ciencias, 1999). The greatest challenge is the development of an education system that will enable us to deal with this diverse social and cultural reality. Responses from the educational community must generate curricular reorganization processes around conceptual fields that promote critical thinking and an evaluation of technological environments.

The projects, presented in this chapter show different ways of incorporating educational research into curricular development. Educational research has identified powerful ideas that can be used to develop a curriculum. Nonetheless, such research cannot take place on a single front, given the social and cultural characteristics of our country. As a country, Mexico must deal with a twofold problem: the education of dispersed groups and that of large urban nuclei. From this stems the nature of the projects discussed in this chapter.

Our first project responds to the needs of the most vulnerable communities. In contrast, the second project, is aimed at a school population made up of regular students within the education system. It is possible and feasible to cultivate powerful ideas that generate different levels of mathematical thinking through the mediation of computing instruments. In countries such as Mexico, these new applied research projects are the key to assimilating scientific and technological knowledge.

From an international perspective, curriculum innovation projects reveal three essential commonalities (Black & Atkins, 1996): (a) the importance of the students’ practical work; (b) the importance of making explicit the link between different scientific fields, as well as between these fields and other knowledge domains; and (c) acknowledgement of the fact that mathematics and science are ways of knowing and explaining the world.

13.2. BACKGROUND

Here we describe some basic characteristics of the Mexican educational system. Essential changes in basic education programs are suggested by the present curriculum reforms. In particular, the structuralist approach to curriculum has been replaced by an approach based on problem solving (Secretaría de Educación Pública, 1995). Other changes include the need to recognize that learning is not an automatic consequence of teaching.

In primary school, the problem-solving approach is oriented toward concrete situations, and students are encouraged to discuss and compare their own ideas. This goes hand in hand with the fact that mathematical knowledge is constructed through successive abstractions, and in addition, it makes clear the instrumental character of mathematics. In secondary school, it is expected that students (12- to 16-year-olds)
will strengthen their problem-solving skills and be able to transfer these skills to other domains. In other words, they are expected to begin the complex process of decontextualizing their own knowledge. We understand this process as one that enables students to establish rich connections with their knowledge.

At the secondary-school level, mathematical content is grouped into five domains: arithmetic, algebra, geometry, data handling, and probability. Geometry is regaining the place it once had in programs preceding the structuralist approach. The current approach to geometry focuses on the development of reasoning skills from hypotheses constructed and tested by the students themselves. On the other hand, algebra is studied as a modeling and problem-solving approach.

It is within this general background that we will try to situate two development projects, both of which were elaborated and implemented in Mexico with the purpose of improving and eventually transforming mathematics education. These projects are compatible with the development of the educational and cultural conditions necessary to make possible democratic access to powerful mathematics.

At this point, we begin elucidating our notion of access to powerful mathematics through school culture: It mainly has to do with providing the student with the opportunity to (a) experience the construction of mathematical knowledge within school according to her or his level of development; (b) develop her or his creativity through exploration of the different approaches that emerge from discussing questions posed within the classroom; and (c) developing her or his own computing techniques or procedures.

We can assume that students will, sooner rather than later, start to bring calculators to school (if this has not already happened). The access to powerful ideas in the context of teaching and learning with calculators can mean a change in the way we work with decimal numbers, for example, reinforcing the powerful idea of approximation. To a great extent, this is made possible by the positional meaning of digits in decimal expressions. This is undoubtedly one of the central features of the representation system. Unfortunately, the uncritical use of calculators leads to the idea that all decimal expressions are finite. If we want students to reach a higher conceptual level (which is the same as saying that they have taken on a powerful idea), we must propose teaching that goes into depth regarding the idea of approximation and removes the false idea that all numerical decimal expressions are finite. This is achieved through suitable teaching models, such as the systematic study of the change of units of measure in calculations of length, area, and volume. These activities bring out another important idea, that of "better approximation." The synthesis of the ideas of approximation and better approximation constitute an example of a powerful idea that it is worth developing at different levels of the education system and with various technological resources (NCTM, 2000).

Other powerful ideas will come with numerical calculation. It will not be necessary to always insist on the accuracy of numerical results. The development of calculating skills combined with the numeric control of hand calculators might be an invaluable resource for students outside the classroom. In the particular case of geometry, the possibilities of dynamic tools such as Cabri might lead to a change in the current conception of school geometry as a pedagogical model, which closely follows the axiomatic organization of geometric knowledge. Nonetheless, we should not forget that large portions of the Mexican population live in dispersed communities, particularly in the country. Hence, the educational system will also need to provide answers that favor the integration and development of these communities. Our project "Dialogue and Discovery" faces these challenges. This project has minimal technological requirements (if the resource to the technology of writing can be understood that way). But the growing complexity of our world requires people whose education has trained them to develop their activities with increasingly greater levels of systematicity. Mathematical education can contribute in a significant way to
the achievement of these goals as long as teaching models allow the appropriation of conceptual tools to model and formalize situations, for example, to introduce a higher level of predictability and therefore better control over the consequences of social action.

13.3. DIALOGUE AND DISCOVERY: A CURRICULAR DEVELOPMENT PROPOSAL FOR SMALL RURAL SCHOOLS

In this section, we will deal with the question of how to introduce powerful mathematical ideas into rural schools where young instructors, graduates from 9th grade and aged between 16 and 21 years old, have to work with groups of children from 1st to 6th grade simultaneously, all in a single classroom.

Thus, we provide and discuss some of the arguments that emerged during the process of elaborating a curricular development program, the “Community Courses,” aimed at schools with the characteristics mentioned above. Many of the decisions that were taken in the particular case of mathematics are part of a more general strategy that oriented the program’s global design and which includes the areas of Spanish and science. Hence, we will refer to this general strategy by illuminating the specific case of mathematics.

13.3.2. The Community Courses

The Community Courses program represents an alternative to representative school organization, providing primary education to children living in very small rural communities (less than 100 people) dispersed throughout the country, where it is not cost-effective to install regular schools. The program was instituted 25 years ago by a decentralized government agency, the National Council for the Promotion of Education (CONAFE). Today, there are around 15,000 courses serving approximately 140,000 children. The program works as a contract between the community, the instructor, and the CONAFE. The community is in charge of providing a physical space to run the courses, food and shelter for instructors, and for supervising the adequate operation of the courses. The instructors are usually young people from the rural community, between 16 and 21 years old, who have graduated from secondary school (9th grade) or, less frequently, from high school (12th grade); they work only for a period of 1 or 2 years within the community, receiving a small payment from CONAFE. Once their working period is finished, CONAFE grants them a scholarship to continue with their own studies.

One instructor deals simultaneously with between 10 and 30 pupils from the six different grades at the primary level. They receive a 2-month intensive training during the summer and subsequent timely supervisions during the year. As the reader may have guessed, we are talking about the most modest schools in the country.

In 1975, CONAFE asked a group of education researchers from the Education Research Department at Cinvestav (the Center for Research and Advanced Studies) to develop both a pedagogic model and a set of supporting materials for the program. These were elaborated between 1975 and 1978 and were used for 15 years. In 1989, the revision and actualization of these materials was considered and assigned again to Cinvestav. During the next 4 years we elaborated a new set of materials, nine textbooks in total that comprise the Dialogue and Discovery, series which will be discussed here. These books have been used in the Community Courses program since 1994.

In the process of creating this series, we used contributions from two different approaches to the problem of classroom teaching: (a) research on the didactics of
specific disciplines in this case mathematics, and (b) ethnographic research on school culture and teaching practices. We are particularly interested here in showing the way in which these two approaches were integrated to answer the specific problems that the development of curricula entails.

To grasp the meaning of “powerful ideas” in mathematics learning, we begin by examining existing research on the didactics of mathematics.

13.3.3. Powerful Ideas in Mathematics Learning in Primary School

Today the greatest challenge in mathematics teaching in primary school is still the same one that the greatest curriculum reformers described almost 40 years ago: overcoming a tendency to reduce the discipline into the teaching of mechanisms that are supplied with few meanings.

Beyond the differences among contemporary approaches, there is a consensus among the research community that we must contextualize the knowledge taught in basic education to let construction of meaning in mathematical situations to take place instead of a premature formalization.

For almost twenty years, we have studied the didactic conditions that favor the learning process of mathematical knowledge in the primary school classroom. Mathematical knowledge is considered a mediating tool to deal with specific problems. Learning is considered an outcome from an interaction between the individual and the environment that is full of interference. Our efforts focus on studying the specific milieu that will favor the construction of mathematical knowledge in the classroom (Brousseau, 1987).

Exploration within the classroom is the main methodological tool we have used in our studies. The design of teaching strategies relative to a tangible idea is carried out from an analysis of the problems in which that idea works as a solving tool. Moreover, to support the development of solution strategies, the situation should allow students to see for themselves how they were able to solve a problem, or to what extent they solved it. The latter is a form of programmed verification that is destined to be replaced progressively by a form of semantic validation through argumentation (Block, 1991).

From this perspective, introducing powerful mathematical ideas into the primary school classroom gives students studying arithmetic and geometry the chance to experience personal, and therefore significant, construction of mathematical knowledge. This provides them with appropriate tools, according to their age and cognitive abilities, to perform creative work by elaborating and testing their own conjectures and constructing or refining their own calculating techniques.

13.3.4. Research and Curricular Development

Ethnographic studies about teaching practices have also contributed significantly to eradicating the myth of the apparent transparency in the relation between pedagogic models and real teaching practices. They have helped us to understand that teachers construct their practice from their own experience and, in specific school conditions, with the cultural resources at hand (Rockwell & Mercado, 1988). Thus, when designing materials for the Dialogue and Discovery project, we assumed that their effective use does not depend only on the instructor’s adaptation to the materials but primarily on how the proposal can be adapted to the instructor’s working habits. In addition, we assumed that there is a creative relationship between the instructor and the textbooks, a relation in which “textbooks are interpreted and proposals are both reformulated and selectively integrated into a practice that is constructed every day within the classroom” (Rockwell, Block, & et al., 1993). The main challenges and interests of
this project emerge precisely from the extreme conditions of austerity in which the proposal should work, paired with the purpose of developing a high-quality program that is able to recover existing knowledge on teaching and learning processes.

13.3.5. On Methodology

Before producing the materials, we visited diverse communities to learn the general conditions in which the instructors had to work, as well as to observe their working practices. In addition, we met with members of technical teams from different states in the country and learned about the main experimental difficulties experienced by the instructors, as well as about their opinions on the characteristics that the proposal should have. From this preliminary data, we made important decisions regarding the proposal’s general design.

During the design stage, we systematically tested the textbooks in diverse communities throughout the country. The main purpose of these tests was to confirm that the situations were adequately designed both for the students and for the instructors (i.e., that they were sufficiently clear and feasible in terms of their organization). After the first version of the proposal was available, we conducted an evaluation in several communities during a single school year; at the end of the year, we received the materials used with numerous comments from the instructors and students.

13.3.5.1. Selection and Reorganization of Contents

The current national curriculum was employed as a framework to produce the Dialogue and Discovery project. Nevertheless, because there was a need to adapt the program to actual available time which were substantially shorter than the official standard teaching time, the contents of the four areas were submitted to a careful selection and reorganization. We knew that, at this level, children could profit from dealing with certain unit partition experiences. The teacher’s expectation of obtaining precise results hinder the formative value of these experiences, however. This is why we decided to introduce fractions in the 3rd grade. In the same way, we decided to exclude certain aspects from this topic, such as fraction multiplication and division, that are not fundamental at a basic level and that usually generate significant teaching and learning difficulties.

13.3.5.2. Separating the Primary Curriculum into Three Levels Instead of Six

The organization of the children’s and the instructor’s work was intended both to make it possible for the instructor to simultaneously pay attention to the students of all six grades and to make the effects of the rural conditions result in a positive outcome. First, the curriculum was separated into three levels, with two grades per level. In this way, the instructor is able to plan the tasks for three groups of children instead of six, and during the class he or she could distribute attention between the three groups. This means that the children works twice, once per year, with each level’s tasks. This form of organization is supported by the understanding of learning as a cyclic process in which a single problem set can be dealt with repeatedly throughout time, each time in more depth. Hence, the intention is not to “repeat” tasks, but to deal with the same situations in a more systematic in depth way, including new concepts. We also expected that the collective work of children with diverse degrees of knowledge would be beneficial to the whole group, for those who know more and for those who know less.
This consideration implies an additional reason to privilege the design of reusable or recyclable activities during the course. A situation consisting of drawing the number of objects corresponding to a given number can be a useful in assessing or revising situation; however, it is not a useful learning situation because only those who already know the number’s representation can successfully solve it. Now let us consider the following situation: the instructor puts 10 white and 10 black cards on the table. Student A leaves the classroom while student B takes a certain number of white cards, puts them in a bag, and writes on the blackboard the amount of cards he or she has put away. Student A comes back, looks at what is written on the board, and takes the same amount of black cards. They then directly compare quantities, and if they are the same, they win.

Some of the children’s typical solutions are

- Student B draws each white card introduced in the bag, or a line for each card, on the blackboard.
- Student B writes down small numbers to represent greater quantities, for example, for a collection of five cards, 2 2 1.
- Student B tries to express with one sole number the cardinal of the collection. Occasionally Student B or A makes mistakes because he or she does not know the written numerical series very well or because mistakes in counting.

Unlike the previous problem, students who are in the process of acquiring knowledge of the initial numerical series can participate in this problem. It can inspire various solutions that respond to different levels of knowledge and allow for the systematic verification of the attempts. In a similar way, applying a definite algorithm, for example division, is something that can only be done in one way, and only if one is already familiar with the concept. In contrast, solving a problem that consists of finding how many marbles each of three friends will get if they share 24 marbles equally can be approached by distributing the marbles one by one, by using additions to compute and verify, or by using multiplication to compute and verify.

We suggested making some situations more complex by manipulating numeric variables or by introducing certain constraints, so that they can still be a challenge for those who already possess a solution for the original version. One of the hardest aspects of the design is transmitting the meaning of the activities to the instructor. Let us look at an example in which the instructor’s subtle modification changes the meaning of the activity. The original situation is concerning addition and subtraction and sets out the following problem: The instructor gives a group of students a set of 20 objects that they have to count. One student leaves the classroom while the others aggregate or take away objects from the set. The student comes back and tries to figure out what the others did to the set. The aim is to encourage the student to bring into play his or her own resources to determine if any objects were added or subtracted and, if so, how many.

For example, if seven objects were taken from the set, the student can verify that objects were taken away by counting those 13 that remain. To determine how many were subtracted, he or she can add the objects that are missing to reach 20, separating them from the others and then counting them, or continue counting from 14 to 20, counting a second time on his or her hands, in the units that are missing (15 is 1, 16 is 2, etc.). When objects are added to the set, the solution is much simpler; one need only separate 20 and count the rest. In the first instance, the children think there is no way to know with certainty what they are being asked and tend to give estimated quantities: “they took about three away”. This situation is one of the first experiences children have in which a calculation allows them to anticipate quantitative events.
In one of the observed sessions, the instructor performed the situation several times in the following way:

- Each child counted the objects before going outside the classroom.
- In all the cases, the other students added objects.
- Each time a student came back, he or she was asked to separate 20 objects from the set.
- Afterwards, he or she was asked to count the rest.
- When errors occurred, the student repeated the operation.

The task was so well controlled that it ended up being a simple counting exercise. This kind of difficulty led us make as explicit as possible the tasks’ purpose, as well as the possible answers, mistakes, and procedures that the children were likely to have made within the narrow limits of space. We will return to this point later while discussing the materials’ design.

Among the tasks we are currently proposing is that of game playing. We have developed a set of 20 games, each with four variants in degree of difficulty, but all related with one or several contents from the mathematics primary curriculum. Some of the games simply represent an opportunity to exercise algorithmic thinking through the introduction of certain variants in tasks that allow the enrichment of the students’ work. Certain type of riddles, some designed *ex profeso*, appear to be well suited to this purpose, for instance, building “magic squares.” In contrast, other games entail the construction of a winning strategy. Students with little knowledge can play them giving rise to autonomous decision making, formulation of hypotheses, and pragmatic verification. An example of these games is the classic “race to 20,” in which two players participate. The first player writes down a number 1 or 2; the second player adds to that number a 1 or 2, and so on. The player who reaches 20 first wins. Once the child has learned how to win (and this takes time), the goal changes. For example, the goal can be to reach 21 or 22, or the size of the steps may change (for example, one can add 1, 2, and 3). These variations help the search for more general strategies.

The fact that specific learning contents are not made explicit in these tasks will presumably help the instructor allow the children to work more freely.

### 13.3.5.3. Supporting Texts

We produced two handbooks for instructors containing the design of class development for the three levels, as well as a game book and a manual on “the experience of being an instructor” with multiple practical recommendations and testimonies from other instructors. We also provided the national textbook, produced by the Ministry of Education. For second level students, we produced an activity card kit, and for level three we produced four workbooks, taking into account that at this level students have better writing skills and study a considerable amount of time without the instructor’s supervision.

We will discuss only some features of the instructor’s guide, those representing a part of the target user’s adaptation process in a project of curricular development. These guides contain the curricular development proposal in detail, class by class, as requested by instructors and former instructors. They argued that programming the activities was one of the most difficult tasks to carry out. As we wrote the guides, we assumed that there should not only be a working guide for the instructors, but also a guide for them to learn more mathematics and how it can be taught.

Finally, we included at the end of each chapter the assessment guides. We considered this necessary because, as is well known, the tools used for evaluating student knowledge significantly influence what is considered relevant for teaching. The
activities that we propose are basically problem situations, written or formulated orally, with eventual support from concrete materials. We also included information about possible answers, mistakes, or solving strategies, providing several criteria to help appreciate the students’ progress, as well as some suggestions in case difficulties or failures to complete the tasks are identified.

Experimental work led to considerations related to form. The number of pages assigned to the development of each topic was restricted from the start, considering the time available for the instructor to read them, as well as the real class time (for instance, 46 pages for level-one mathematics); the need to be careful about the writing style was also taken into account, for example, using brief sentences and paragraphs. In addition, it was necessary to include as many figures and photographs as possible to support written materials.

13.3.6. Additional Comments on the Dialogue and Discovery Project

The development of the Dialogue and Discovery project, was an attempt to produce not only an original proposal that would contribute to the learning of meaningful knowledge, but also a feasible proposal adapted to the needs of the particular target users. In this sense, we believe the methodology that was used to develop the project, as well as the features that shaped it, constitute a contribution to the field of curricular design. The last national-scale reform for primary school (between 1992 and 1994), made by the Mexican government, used many ideas generated from this project in a substantial way. Nonetheless, an essential part of our study is still missing: An assessment of its efficiency when compared with the entire educational system operating in regular conditions. It is appropriate to mention the results of a recent study: A preliminary report from “The First National Standards Evaluation in Primary Education” (Secretaria de Educación Pública, 1995), shows that the percentage of students from the Community Courses that successfully meet established standards in mathematics is similar to the national average. Considering the working conditions of these schools, this is a significant achievement.

As Rockwell (1994) pointed out, a sophisticated research design capable of integrating and controlling variables—such as training, supervision, and real use of the materials—is needed. The latter represents one of the most urgent tasks pending in the field of curricular development. Finally, observing the children while they play the “race to 20” game, helped to alleviate our skepticism about the possibility of introducing “powerful mathematical ideas” into the most disadvantaged regions of the country.

13.4. THE TECHNOLOGY PROJECT

The burst of new technologies in education has frequently produced uncritical optimism about the possibility of transforming the foundation of educational systems. For this reason, it became necessary to challenge paradigms supporting the belief that it is through the mechanical use of these new tools that the great majority of individuals would be able to access complex and powerful mathematical notions. The access to powerful ideas has to take into account, from the start, the mediation between the technological tools and the sociocultural environment surrounding individuals and schools. The different strategies that students, teachers, and schools establish to incorporate technology depend to a great extent on the interpretative resources developed within this environment. Although we are not going to study in depth the relationship between culture and epistemology, it is an essential issue
in education. According to Balacheff and Kaput (1996), the main impact of information technology on educational systems is epistemological and cognitive because it has contributed to the production of a new form of realism in mathematical objects. This new form of realism depends on the interpretative resources provided by the sociocultural environment. At the same time, however, the existence and use of this technology can transform the initial interpretations derived from the sociocultural environment.

Thus, technology has the power to become a sociocultural and educational agent for change but this process of change is complex.

In the proceedings of a world conference on higher education, *Higher Education in the XXI Century* (chapter 12), held in Paris in October 1998, attendees concluded that it is necessary to encourage research through the construction of networks that allow democratic access to knowledge and through the adaptation of new communication technologies to national requirements. Resources of productivity lie in the technology of knowledge generation, information processing, and symbol communication (Castells, 1996).

13.4.2. The Project

In Mexico, as in many other countries, the incorporation of technology into the educational system is driven by a policy of primary importance. A number of educational plans exist that incorporate technology into classrooms. One of these is our national project, “Incorporating New Technologies into School Culture: The Teaching of Mathematics in Secondary School” funded by the Ministry of Education and the National Council for Science and Technology in Mexico (CONACYT, Project 526338S). This project is aimed at

1. Gradually incorporating various pieces of technology into the mathematics and science curricula at the secondary school level
2. Implementing the use of technology supported by a pedagogic model that allows the construction of learning environments oriented toward the improvement of mathematical education
3. Encouraging the design and use of computing environments that can improve the traditional teaching and learning methods (i.e., with paper and pencil).

The general objectives of this program are to raise education standards, to train teachers in the use of technology, and to broaden the students’ opportunities of education.

We plan to continue the program well beyond the year 2000. Initially it covered 15 States (out of 32) including both rural and urban sites. The software selected for the project includes Cabri-Géomètre, spreadsheets, and algebraic calculators (TI-92). To achieve the aims mentioned above, we have had to investigate the impact of technology on the teaching and learning processes. This has several implications for the assessment and implementation of the program. For instance, *usability* problems can affect the student’s achievement of educational goals—the raising of educational standards or the advantages that students gain from introducing technology into the classrooms. The reported results of this project take into account the progress made concerning the global goals we set for ourselves at the start of the project. Among the proposed goals is that of exploring the effects on the cognition of the students, of the insertion of computational instruments into the teaching model. We will report on those results mainly from the perspective of geometry and algebra (see Rojano’s chapter of this book) because in the development of the project, it has become apparent that these disciplines are the most promising in the context of our work, and may allow students to develop powerful ideas in the mathematical education field.
13.4.3. Some Results from Fieldwork

During the development of the project, we conducted interviews and written tests to evaluate the students' mathematical learning as mediated by computing tools. These are resources to obtain information that can be used as feedback for the general management and assessment of the entire project. Following are the results obtained in interviews and written tests while working with Cabri Geometry. This work is done in secondary schools (12- to 16-year-olds). One of the problems that influenced the students' completion of tasks in Cabri was learning to draw with the mouse. When the teacher asked the participants to draw geometric figures, they tended to use the mouse as a pencil metaphor. For instance, they experienced difficulties drawing segments because they moved the mouse from left to right as if they were using a pencil. Similarly, when asked to construct a triangle they proceeded by drawing three different segments instead of selecting the triangle option from the menu bar. The figure they drew looked like a triangle but did not have its properties. For instance, students were amazed when they tried to assign an area to those triangles and the environment was not responding in this respect. As a result, the users experienced difficulties in understanding relationships between the triangles they had drawn and the ones they could draw using the corresponding Cabri menu.

We also became aware of other problems, such as the difficulty in measuring angles in the Cabri environment. Some children were not able to see an 89.98° angle as a 90° angle. Our school mathematical culture is one that still demands "exactness." This is one of the obstructions we must take into account when working with numerical domains with calculators and computers. We have to be careful with considerations of usability especially when the technology was not originally designed for the cultural context in which it is being incorporated.

The above findings become central in evaluating implementation outcomes. If usability problems, as culturally determined, are not adequately taken into account, the introduction of the technology into schools might fail to achieve its original goals. In other words, although technological tools might be adequately designed to meet specific educational goals, if students cannot use them because it is culturally inadequate, the implementation will fail. If assessments do not take into account such usability issues and contemplate exclusively educational indicators, such as student achievement, then these assessments will probably end up recommending inadequate prescriptions in many cases.

Other problems that influenced the user-task interaction was that the technology easily shifted from being an educational tool to being an educational goal. At the end of the sessions, we asked students what they have learned, and the majority answered, "to use the computer." The latter can be due to both usability and to the novelty of using computers. Nonetheless, this is an important issue because if the software is not being used as expected, then the initial educational goal will not be achieved.

According to Balacheff and Kaput (1996), design can aid the development of fluency between diverse mathematical representations, but it can also lead to the construction of misconceptions and misunderstandings. Therefore the interface can no longer be considered as a mere superficial layer because what is involved is not mere perception but interpretation (Balacheff & Kaput, 1996, p. 475). They provide a well-documented review of existing computational technology in mathematics and describe how differences in design can affect the student's mathematical experience.

13.4.4. An Interview: The Voice of the Students

In this section, we introduce a task aimed at examining the mediating role of the calculator in the coherent oral expression of knowledge that students constructed during geometric activities (Manouchehri et al., 1998, p. 437). Geometry software
for calculators has usability limitations because a calculator’s screen resolution and speed can complicate the construction of objects in that environment. Nevertheless, it might be useful to enhance the cognitive activity of its users. We also have explored ways in which object manipulation and dragging, help students discover the invariant properties of a geometric object. We now present some of this work in relation to the central angle theorem.

The tasks we are about to present (part of a larger set of activities for secondary school pupils), were aimed at documenting how the tools provided by the calculator mediate the students’ activity of expressing a mathematical proposition. We suggest that the students’ expressions of coherent and meaningful mathematical propositions prove that they have constructed the relevant relationships or, in other words, the structural links that constitute a figure.

Looking at Fig. 13.1, the teacher asks the students to do the following:

**Teacher:** Drag point B to the right and then to the left along the arc, but look carefully and try to answer these questions:

Does angle B change when you move point B along the arc?

1. Take point A and drag it to the left and to the right. Does angle B change?
2. Take point C and drag it to the left and to the right. Does angle B change?

Before the students started to complete the task, the instructor questioned them about their previous knowledge on the subject. Only one student knew that the angle B, in Fig. 13.1, remains constant as long as one does not move points A or C. Interestingly, the students did not know how to explain this behavior. None of the participants knew the central angle theorem or that a triangle inscribed in a semicircle is always right. The following are some of the participants’ answers taken from the task sessions:

**Felipe (F):** It looks as if the angle doesn’t change, even though point B is moving!

**Manuel (M):** Let me see, I can’t see . . . maybe . . . Felipe and Manuel are talking about Fig. 13.1, while dragging point B to the left and to the right in their calculator screen.

**Teacher (T):** Do you think that the angle will change?

They both answered yes, and this is exactly what the rest of the group was expecting.
(T) (addressing the entire group): Observe what is changing and what is not changing, and try to keep on doing the task. If you find something interesting for you don’t hesitate to tell me.

The participants worked for 20 minutes on this task. Then the teacher proposed the following construction and asked the corresponding questions: Draw the segments from A and C to the center O of the circle (in Fig. 13.1). Drag point A to the left and to the right; observe angle B as well as the angle formed by the segments that connect points A and C to the center O of the circle (central angle). Repeat the operation with point C.

Move point A or point C until they are collinear with O, the center of the circle. How is angle B changed?

Felipe and Manuel, moved point A until it was collinear with point C and with point O. Finally, they moved point B, showing the teacher what they were doing at every moment. The rest of the participants observed what Felipe and Manuel had found.

F: It looks like when the angle in the middle is 180°, angle B is 90°!
T: Why are you saying so?
M: We have already tried it, and it seems that way. Look!

After 20 more minutes, nobody could further expand the argument about the rightness of the angle B. Then the teacher proposed the students measure and label the central angle and angle B. After 15 minutes, the students called the teacher and showed him a table in a notebook, one column showing the values for angle B and the other showing the values for the central angle.

F: One column is almost twice as large as the other!
T: How can you express what you found?
F: The center angle is two times greater than the other.
T: Just like that?
F: Ah . . . Within a circle the center angle is two times greater than the other.

Two out of six teams continued to complete the tasks, but only one team (Felipe and Manuel) had made additional drawings and started labeling the angles. They argued that these additional drawings and labels were intended to “help them discover.” The rest of the teams did not know what to do and did not propose any additional drawings.

What is the aim of these practical sessions? To study how students express and construct their arguments while trying to “prove” a theorem from the exploration of the links that exist between the different elements in the figures provided. Of course, exploration and expression are possible, in enhanced ways, because of the dragging capability of the software. The central angle theorem is an attraction pole, a means to link circles, rays, radii, and tangents, to create a local organization (Moreno, 1996) of a fragment of geometric knowledge. Because two important general objectives of this project are to broaden students’ educational opportunities and, concurrently, to train teachers in the use of technology, we find it compulsory to articulate a reflection on the use of these computational tools and the environment wherein they are explored.

13.4.5. Reflections on Computational Tools and Environments

The ideas that we present in this section include a considerable part of the theoretical framework of the technology and of the research project. We list references to relevant works, articles, and theses derived from the project at the end of the chapter.
Working with the virtual versions of mathematical objects promotes the constructive activity of students. Indeed, these virtual versions produce the sensation of material existence, given the possibility of changing them where they exist, that is, on the screen. Students’ growing familiarization with computational tools allows these tools to be transformed into mathematical *instruments* (Guin & Trouche, 1999; Rabardel, 1995) in the sense that computational resources are gradually incorporated into the student’s activity. We suggest, then, that exploring with computational tools eventually allows students to realize how the mediational role of these tools helps them reorganize their problem-solving strategies. For example, when secondary school students are asked to explore the relationships between the inscribed angle in an arc and the corresponding central angle, we see two behaviors in the classroom: students remain immobilized by the question (we think this is because they are not able to mobilize their expressive resources) or, when they have computational resources at their disposal (for example, calculator TI-92), they are led to draw up comparative tables between angles and to eventually realize that that the central angle is “nearly double” the inscribed angle in the same arc. The students’ strategy, taking the inscribed angle from the central angle is possible thanks to the expressive power the students acquire through the computational tools. In the absence of these, as we have already mentioned, it is not feasible for students to carry out the numerical comparison between the angles and to establish a conjecture, nor are they capable of producing a formulation associated with their explorations and express it in the language of the computational medium in which they are working. The computing environment is an *abstraction domain* (Noss & Hoyles, 1996), which can be understood as a scenario in which students can make it possible for their informal ideas to begin coordinating with their more formalized ideas on a subject. An abstraction domain supplies the tools so that exploration may be linked to formalization. In the example of dynamic geometry, we can put it this way: The exploration of drawings and of their properties gives rise to the recognition of a system of geometric relationships, which in the final analysis constitute the “geometric object.” This abstract object that rises out of such exploration is still “linked” to the environment: The student can talk of its general properties but use the language, the means of expression, supplied by the environment.

One of the aims of research in this field is to understand how technology implementation should be conducted. We know that the first stage could entail working within the framework of a preestablished curriculum. Successful innovations should be able to “erode” traditional curricula, however. At that point, it becomes fundamental to understand the nature of knowledge of students that emerges from their interactions with those mediating tools. Working with computational tools in school media leads us to face the work from two different angles (Berger, 1998): as *amplifying* tools and as *cognitive reconceptualizing* tools. These amplification and reconceptualization processes can be illustrated in the following way: The amplification process is similar to the function of a magnifying glass. Through this lens, we can enlarge objects visible at first sight. Magnification does not change the structure of the objects that are being observed, however, on the other hand, the reorganization process can be compared to the act of seeing through a microscope. The microscope allows us to observe what is not visible at first sight and, therefore, to enter a new plane of reality. In this way, the possibility of studying something new and of accessing new knowledge arises.

Computing environments provide a window for studying the evolving conceptions of students and teachers because they use the tools provided by that environment. Our students refer to *mathematics* as a set of symbolic expressions. Accordingly, knowledge of mathematics means being able to use procedures to transform a symbolic expression into another symbolic expression. Graphing tools produce a shift of attention from symbolic expressions to graphic representations. Representations are tools for understanding and mediating the way in which knowledge is constructed.
Our didactic work with computational tools led us to consider the phenomenology one can observe on the screens of calculators and computers. The screen is a space controlled from the keyboard, but that control is one of action at a distance. The desire to interact with virtual objects living on the screen provides a motivation for struggling with the complexities of a computational environment (Pimm, 1995).

Computational representations are executable representations, and there is an attribute of executable representations on which we want to cast light: They serve to externalize certain cognitive functions that formerly were executed only by people. That is the case, for instance, with the graphing of functions. During the time that passes while the graph is being drawn on the screen, the student observes the characteristics of the function that are reflected in its construction. We propose, therefore, that the student has the opportunity to transform the graph into an object of knowledge. This is similar to what the Greeks did with writing. They used the writing system not only as an external memory but also as a device to produce texts on which to reflect. As Donald (1993, p. 342) has suggested, the Greek’s critical innovation consisted of “externalizing the process of oral commentary on events.”

Explorations within an abstraction domain facilitate the understanding of the character situated in the propositions and the situatedness of its proofs (Moreno & Sacristán, 1998, 2000). Situated proofs refer to the understanding and articulation of processes within the context in which they have been explored. Let us explain: At first, students might make some observations situated within the computational environment they are exploring, and they could be able to express their observations by means of the tools and activities devised in that environment. That is the case, for instance, when the students try to invalidate (e.g., by dragging) a property of a geometric figure and they are unable to do so. That property becomes a theorem expressed via the tools and facilitated by the environment.

A situated proof is the result of a systematic exploration within an (computational) environment. It could be used to build a bridge between situated knowledge and some kind of formalization. Students purposely exploited the tools provided by the computing environment to explore mathematical relationships and to “prove” theorems (in the sense of situated proofs). Let us illustrate this point with the description of the case of a situated exploration in a classroom: In this section, we allow the description of the experience to speak for itself.

After the students acquired a certain amount of skill in the graphing of polynomial and rational functions, it was clear to them that the effect of zooming in on the graph of a function results in straightening the graph in a small interval. In other words, we can say that applying the zoom can be seen as taking the derivative in the graphical register of the function (Duval, 1995 & Tall, 1996, p. 310).

At this point, we considered the possibility that the didactic virtues of a cognitive conflict could promote the students’ levels of conceptualization (i.e., they could generate a powerful idea) with regard to the graphing of functions through the resources of the TI-92 calculator, for example. This manifested itself when we presented the students with the function (when graphed in the window \((-1, 1) \times (-1, 1)\)

\[ Y = \frac{(\sin(100x))}{100} \]

Zooming in on any point of the graph of this function causes unexpected behavior: the new function graph reveals an oscillatory behavior that was hidden in the first graph.

There are many things that become clear through this exploration. First, this is not possible without the help of the computational resources at our disposal. Second, it allows the relationships between the graphing and the screen’s resolution to be systematized, in this case, for the TI-92 calculator. This is equal to the achievement of
a powerful idea, which includes the understanding of the screen as a representational space. Later, we have introduced the task of exploration of the function:

$$\sum (2/3)^n \cos(9^n \pi x), n \geq 0.$$  

This is known as the Weierstrass function. Historically, this function marked a milestone in the development of mathematical analysis, because it is a continuous but non-differentiable function. We do not try to give our students a formal demonstration of these mathematical characteristics, of course; rather, the idea is to use the abstraction domain supplied by the computational environment to explore whatever the student observes when graphing the polynomial approximations that correspond to the series defined by the function. We were interested in seeing what kinds of statements were put forward by the students in discussions on the process of graphing.

While observing the drawing process of the polynomial approximations of the Weierstrass function, students noticed the randomness “hidden” in such a function. They realized this characteristic of randomness because of the dynamics supported by the executable representation of the function (Lupiáñez-Moreno, 2001). Once again, the idea of considering the effects of the screen resolution on the graph turned out to be powerful. Through the instrumentalization of this idea, students could discover the degree of complexity of the function, perhaps only from a visual-dynamical point of view, but even this objective is worthwhile because it opens a window into a
mathematical world with the potential to enhance understanding beyond the curriculum. The tool is used here as a microscope, not as a magnifying glass. There are different cognitive demands in drawing a figure by hand (Roth & McGinn, 1997) and using a computing device to accomplish that task. If we draw a circle using the border of a circular object as a guide, we obtain some valuable information on the control we have to practice with our hand. So obtained, the information can be understood to result from the mediational role of the tool (the border of the circular object we used). Drawing within a computational environment set forth a different cognitive demand from students. The nature of the mediational tools applied in each case support this assertion. There is a considerable amount of research on this topic. Recently, Chasapis (1999) discussed the mediation of tools in the development of the concept of a circle. He suggested that human action and thought is different when students work with a compass than when they work with tracers and templates. From the first moment students access the tools as instruments to enhance their expressive power, after considerable work with the mediating help from teachers, they might enter the higher level of reconceptualization. In principle, considerable familiarity with the tools is needed to be able to produce this reconceptualization a process (Guin & Trouche, 1999).

With computer explorations, we can associate the notion of a “situated theorem,” when the tools employed become visible as part of the expression. As Noss and Hoyles (1996) explained, students can generate and articulate relationships that are general to the computational environment in which they are working. This means students can develop an ability to state general propositions in the language of the environment. We can say that these computational environments derive their educational power from their ability to manipulate and externalize abstract ideas.

Now we will describe and explain some key aspects of the communal and cultural aspects of the projects described in this chapter.

13.4.6. Final Remarks on the Projects

We have described two different projects, developed within the Mexican educational system. We have also described the implementation of technology that, in the near future, might impact school practices and that can lead to new educational developments. Finally, we have tried to exhibit the type of mathematical understanding achieved by students during the implementation of these new approaches, for instance, those mediated by algebraic calculators and computers. Let us recall our description of the idea of access to powerful mathematics through school culture. It mainly
has to do with providing students with the opportunity of experiencing the construction of mathematical knowledge at school according to their level of development; with developing students' creativity through exploration and discussion of the different approaches that emerge from discussing questions posed within the classroom; and with developing students' individual computing techniques or procedures.

It must be emphasized that these projects have been conceived to respond to the real educational problems that are seen in our particular society.

We believe that in the decades ahead, these problems and the whole sociocultural environment from which the result will continue in our country as in Latin American countries in general. In particular, we will continue to face the need to take care of considerable populations with high levels of dispersion, and, at the same time, we will have to respond to the problems created by the addition of new technologies to our school systems.

The characteristics of sociocultural and economic development, which we have described in this chapter, are widely shared among Latin America. It is from this perspective that we see great potential in the educational proposals, found in our projects. These are viable projects, that go beyond the particular conditions of our country. Of course, the incorporation of information technology into school systems must be gradual, adopting a systematic approach. By this we mean that it is not just a matter of installing equipment in the absence of an educational and social project, which may lend importance to the social acceptance of these technologies. Broad social support is indispensable, and may better the quality of education and generate conditions in which new conceptual frameworks (powerful ideas) may spread to the largest possible number of schools within a country.

We want now to consider issues related to the development of the technology project from the viewpoint of the community and school culture. Researchers are aware that educational software is not culturally neutral (Crawford, 1990). For instance, the design of educational software incorporates the values and priorities of the designer. The designer’s sociocultural environment will play a role—which could be an implicit role—while producing a piece of educational software. This is closely related to usability issues that we have discussed in a previous section of this chapter, and this issue is central to our project because we are using several software environments that underlie a series of powerful mathematical ideas. Many of these ideas are closely bonded to the curriculum, but others are not. The latter convey an opportunity to explore future changes that might be incorporated into the already-mentioned curriculum. Computational environments enhance students access to powerful ideas. The feasibility of dealing with general mathematical ideas within a computational environment highlights an important feature of these tools and environments: the access to systematization, a true powerful idea.

The technology project has demanded a global and local level of assessment. The global level focuses on understanding the educational system as a complex one: the interactions of students, teachers, parents, and administrators all within an educational environment. The goal of this level of assessment is to regulate the educational processes taking place at school. This includes taking care of teachers’ evolving conceptions and administrators’ and parents’ new attitudes toward technology. On the other hand, the local level concentrates mainly on case studies. The latter is sought to provide useful feedback for improving dissemination and implementation, as well as to produce auditable trails of documentation that can reveal the nature of achievements.

Data from the local level of assessment such as filmed interviews with students were used to analyze the evolution of skills and specific knowledge according to the mathematics curriculum. Tasks were designed with a model of collaborative work in the classroom in mind. These tasks were implemented according to evolving lines in
the different curriculum contents—for instance, from intuitive to exploratory dynamic geometry.

Different pieces of software (Excel, Cabri, SimCalc) have been used at different sites, and the calculator is being used at every site. Pupils collaborate in pairs and small groups of three when working in front of the computer. When using the calculator, they work individually but also in groups. Teachers have noted that when students share a calculator, they often seemed to have an advantage. In addition, when two students each have a calculator but work together, they generate a variety of approaches and also discuss their work. The teacher’s role consisted of (a) giving support to students as they worked out the activities described in the worksheets and (b) organizing collective discussions to enhance individual experiences and problem-solving abilities. In addition to being a mediator during the classroom activities, the teacher is also a mediator between students and the tools as the students appropriate these tools.

The focus of this work is to cast light on the role of computing tools as shapers of school mathematical culture. The results discussed here provide evidence of the impact of learning environments on the ways in which children express their mathematical thinking. This is in part due to the close interaction occurring between the students and the tools. For instance, while working with the calculator, students can enter a formula and observe results of the calculations that are carried out with that formula. The student becomes aware of a broad generality expressed by the formula instead of looking only at the symbolic manipulation. In this fashion, he or she is introduced to a powerful mathematical idea. Many researchers participating in this project have observed this trend during the development and implementation of diverse activities.

We can add some remarks from the global assessment perspective:

1. Parents value technology because it brings better career opportunities to their children.
2. Teachers point out that technology helps build a new learning milieu within the classroom in which new strategies for problem solving and new ways of introducing teaching materials can emerge.

Also from a global perspective, the project tries to answer questions such as the following:

1. What new insights are productive teachers developing?
2. Are the teachers’ and parents’ expectations evolving together with the project?
3. Is the evolution of values manifested accordingly to regional cultures?

Teachers clearly do not want their involvement in the project “to be determined by the whims of elected political representatives”; they want to ensure a continuing participation in it.

When the teachers were asked about their perceptions of the quality of students’ learning, they said that they were pleased that their students were more interested in mathematics. Students were learning to reason and had become more sensitive to the introduction of mathematical ideas before they dealt with them in the normal classroom.

Teachers play a central role in helping students assimilate what they know. As professor Lesh has told us, teachers seem very comfortable with technology now and seem to be more worried about other issues within the project, such as student assessment and student commitment. Now is the time to provide teachers with the tools to consider and promote new ways of learning, not only as an internal process but also as a social event.
ACKNOWLEDGMENTS

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REFERENCES


The relationship between the learning and uses of mathematics in school and out of school has been discussed in recent decades from several distinct perspectives. Investigators in Brazil have made an outstanding contribution to this debate. The peculiarities of Brazilian society have enabled them to ground their theoretical perspectives firmly in the realities of people’s lives. This country faces the challenge of an active working-class force that has emerged largely independent of formal schooling. For researchers this posed an important question: how could the same population experience failure in school and be quite skilled in jobs that often involved mathematical skills? Two disciplines that have approached the problem from a sociocultural stance are ethnomathematics education (D’Ambrosio, 1985) and developmental psychology (Nunes, Schliemann, & Carraher, 1993). At first glance, they appear to be quite distinct—ethnomathematics educators are concerned with historical and anthropological analysis of the mathematics of different sociocultural groups, whereas the developmental psychologists study the psychological processes involved in the learning and using of mathematics in specific sociocultural contexts. A systematic comparison of the two approaches (see Table 14.1) suggests, however that in fact the approaches appear to be concerned with the same phenomena at different levels of analysis, the former at a sociogenetic and the latter at an ontogenetic level.

Reflecting on the Brazilian experience one can see that both ethnomathematics educators and psychologists agreed on one critical issue: both demanded that the forms of knowledge associated with out-of-school practices should be considered legitimate. They diverged on the focus of their analysis, however. Nunes et al. (1993) stressed the cognitive aspects. Based on the findings from their program of research in “street mathematics” (studies with groups that have learned and use mathematics in their daily outside-school activities), they concluded that
TABLE 14.1
Two Perspectives on Outside School Mathematics

<table>
<thead>
<tr>
<th>Ethnomathematics Education</th>
<th>Developmental Psychology</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D’Ambrosio, 1985)</td>
<td>(Nunes et al., 1993)</td>
</tr>
<tr>
<td><strong>Level of analysis</strong></td>
<td><strong>Ontogenetic level:</strong> Analysis of the individual’s psychological processes involved in learning and using mathematics in specific sociocultural contexts</td>
</tr>
<tr>
<td>Sociogenetic level: Historical and</td>
<td></td>
</tr>
<tr>
<td>anthropological analysis of the</td>
<td></td>
</tr>
<tr>
<td>mathematics of different sociocultural groups</td>
<td></td>
</tr>
<tr>
<td><strong>Focus of analysis</strong></td>
<td><strong>Focus of analysis</strong></td>
</tr>
<tr>
<td>Relationship between social–political order and individual learning: how the value social groups attribute to certain forms of mathematics mediates its “transmission” and “appropriation”</td>
<td>Relationships between culture and cognition: How specific cultural tools mediate mathematics cognition</td>
</tr>
</tbody>
</table>

Street mathematics has often been treated in the literature as “lesser mathematics” involving idiosyncratic, intuitive, childlike procedures—techniques that did not allow for generalisation and should thus be eliminated in the classroom through carefully designed instruction. We were able to document the fact that street mathematics is not learning of particular procedures repeated in automatic, unthinking way, but involves the development of mathematical concepts and processes. (p. 153)

Studies of the uses of mathematics by street vendors, fisherman, carpenters, foremen, farmers, and students enabled Nunes and her colleagues to provide a detailed analysis of how specific cultural tools mediate mathematics cognition. D’Ambrosio’s (1985) reading of the history of mathematics led him to emphasize connections between what counts as legitimate knowledge and political and power relationships in society. In his view, the relationship between social–political order and individual mathematical learning needed to be considered. He argued that treating school mathematics education as value-free [as a value-neutral activity] could lead to “disruptive social effects” and that this could be observed in countries like Brazil, where the way unschooled people used mathematics outside of school was disparaged and repressed. According to D’Ambrosio, a hidden assumption in school mathematics was that other forms were not worth knowing because they did not play a part in modern life. These “hidden values,” which were ignored, played a part not only at the level of organization of the school curriculum, but at the level of individuals in their self-esteem, constraining what was learned or repressed.

D’Ambrosio’s research agenda addressing the sociogenetic level has been expanded by recent work in the field of critical mathematical education (Knijnick, 1996; Skovsmose, 1994). It is also necessary to have an understanding of the impact of political and social order on ontogenetic development in relation to mathematics knowledge. A psychological framework integrating the complementary views of Nunes et al. and D’Ambrosio may shed light on current conceptualizations of mathematics learning. This is the way my own empirical work has evolved: from a perspective that considers the mediating role of cultural tools to one that also accounts for the mediating role of social valorization (Abreu, 1995b, 1999; Abreu & Cline, 1998). Much of this development was initially data driven, or grounded in ethnographic-type descriptions. However, current developments in the field of cultural psychology seem to provide a sound theoretical basis for incorporating the emerging empirical findings and to redefine the research agenda. The next section will briefly outline the main focus of cultural psychology and how it will be used as a framework to review
past research in out-of-school mathematics and inform discussion around unresolved issues.

CULTURAL PSYCHOLOGY: A FRAMEWORK TO REFLECT ON PAST RESEARCH AND FRAME THE AGENDA FOR THE FUTURE

Within the discipline of psychology the renewed interest in cultural psychology is linked to a shift in the focus of the studies from products to processes. Studies on mathematics learning have played a central role in this development (Lave, 1990). Efforts to understand the relationship between mathematical thinking and culture can be traced to the interest of Western science in testing the universality of cognitive development and to the expansion of Western style of schooling in other societies (Cole, Gay, & Glick, 1968). The latter included mathematics as one of the key subjects in the elementary curriculum. Cole (1977, 1995) recounted his first task in (cross)-cultural research as being “to figure out why Liberian children seemed to experience so much difficulty learning mathematics” (Cole, 1995, p. 23). He also noted that his “graduate training was in the tradition of American mathematical learning theory, which at that time entailed the use of algebra and probability theory to provide a foundation for discovery of presumably universal laws of learning.” This happened about 35 years ago, and as he confessed, he “knew almost nothing about the teaching of mathematics, and even less about Liberia” (Cole, 1995, p. 24). As now seems obvious, Cole and his group soon grew sceptical about their own knowledge and methods. The contradiction between the logic of thinking that Cole and his colleagues were trying to investigate and the logic used by the Liberians was illustrated in a well-cited anecdotal episode (Lucariello, 1995). In the course of a classification study, which required participants to sort objects, it was observed that they tended to use functional groupings. The participants justified their sorting on the basis of their understanding of how “smart” people would undertake the task. The researchers then asked them how they thought a “less smart” person would undertake the task. Following this request the objects were grouped in taxonomic categories. Obviously, there were distinct types of logic in operation: the logic of taxonomy in the mind of the researcher and the logic of functionality in the mind of the participants.

Cole’s example at first glance exposes the inadequacy of the research procedure used. However, as cautioned by Rogoff (1984), this does not allow us to jump to the conclusion that variation between cultural groups could be explained solely in terms of inadequacies of research procedures, either in tests and tasks or in the difficulties with communication between researchers and participants. These explanations would reduce the differences in performance to differences in “display” (Shweder, 1990) and presuppose that in ideal circumstances of testing the differences should disappear. Still implicit in such a view is the idea of cognition as a process located in an autonomous individual mind. Cole and his colleagues also observed that differences could be linked to the tools used as mediators. To their surprise, the differences were not only linked to a tool being available, but to a more complex organization of the tool itself. Hand spans and foot lengths were tools used both in the United States and by the Kpelle of Liberia. When Cole and his colleagues tested both Americans and the Kpelle in tasks that required estimating length with hand spans and foot lengths, the Americans performed better despite the familiarity of both groups with the tools. Further analysis revealed that although both groups were familiar with these length measures, the Kpelle did not relate different measures in a system, thus making the task more difficult to accomplish. In contrast, the Americans were familiar with well-articulated systems, such as inches, feet, and yards, that enabled them to translate
between the measures. Observations of this kind revealed the need for a shift from studying products to studying processes. This shift is at the heart of the movement from cross-cultural to cultural psychology (Cole, 1995). It marked the start of a new era in psychological approaches to mathematics learning outside of school. The revised theoretical and methodological foundations led to the emergence of new research program, such as the series of “street mathematics” studies carried out by Nunes and her colleagues in Brazil in the 1980s (Carraher, Schliemann, & Carraher, 1988; Nunes et al., 1993). Research in “everyday cognition,” “situated cognition,” and “mathematics learning in and out of school” flourished in the 1980s, with seminal research works being published (see, for instance, Lave, 1988; Rogoff & Lave, 1984; Saxe, 1991).

A basic claim shared by the principal proponents of cultural psychology (see Stiegler, Shweder, & Herdt, 1990; Wertsch, 1991) is that the adoption of such a perspective offers the possibility of following an alternative agenda that focuses on accounting for diversity in human psychological functioning. Of course, understanding variations between groups and individuals has been at the core of psychology, but the mainstream approach searches for its sources in the universality of the mind and the presence or absence of capacities, properties, traits, and so forth. Cultural psychology searches for the sources of diversity in socioculturally specific experiences. As Engeström noted, “the challenge is to go beyond traditional psychological conceptions which place motives inside the individual” (1999, p. 255). This standpoint does not deny the existence of universals linked to the biological makeup of humans (Bruner, 1996; Cole, 1996; Nunes & Bryant, 1996). Instead, it attempts to pay more attention to issues that appear to have been neglected since psychology was established as a science by Wundt (Cole, 1995).

Proponents of cultural psychology approaches also share the view that understanding diversity requires attention to the interplay between the individual, society and culture. For analytical purposes Much (1995) suggested that the person, the society, and the culture can be seen as “the three systems of cultural psychology.” She argued that each can be seen as a system in terms of its organisation and dynamics. The first system “is a person, with a distinctive biological make-up and unique history of experience.” The second system “is a ‘society,’ more precisely, the local social structures (for example, the family and other institutions) of a society or culture.” And, the third system “is culture in its symbolic sense, culture as a representational system, the collective symbol systems and institutionalised meanings for interpretation and organisation of experience and action in local social contexts” (Much, 1995, p. 100). Although each can be seen as a system, they are not independent of each other. Instead, as Much stressed, they are “mutually constitutive” or “co-create each other.” For instance, there is content overlap among the three systems. A personal identity, for example, may entail the mastering of certain representational systems as part of one’s culture and also positions in social structures, such as roles and status.

Rogoff’s (1995) three planes model to account for human development in sociocultural contexts is also based on the idea that interplay between different parts of a system can be studied separately but are mutually constitutive. Her three planes of analysis focus on the personal, the interpersonal, and the community. These planes bear resemblance to Much’s three systems. To Rogoff, the community is the institutional-cultural plane of the activity; the interpersonal includes interaction with others directly or through the social organization of cultural activities; the personal is the plane of the individual’s change (in “which individuals transform their understanding of and responsibility for activities through their own participation,” Rogoff, 1995, p. 150). In her view, however, these planes are part of a whole (activity or event), and each plane can be taken as the main focus of analysis (the foreground) while the others remain in the background. Recent reviews of sociocultural approaches to mathematics learning led me to believe that an analysis addressing relationships between different systems or planes of activity can shed new light on ways of interpreting and studying
critical issues. For instance, it might help to clarify the emergence of individual diversity among mathematical learners and in particular whether this diversity originates from the interplay between sociocultural and person systems (Gauvain, 1998).

If mathematics learning is examined from a cultural psychology perspective, the mutually constitutive systems or planes must be addressed. In particular, it will be useful to clarify the ways in which the interplay between the systems is studied. This is the approach I follow in my review of the literature. I use Much’s three systems, combined with the notion that cultural psychology “emphasises mediated action in a context” (Cole, 1996, p. 104), as a framework to present the review and my own thinking on mathematics learning in out-of-school contexts. The next section of the chapter reviews research that has focused on the mediating aspects of (a) the cultural system (the role of cultural tools, both mental and physical); (b) the social system (the role of social interactions and other types of social processes); (c) the person (the role of the individual agency in the reconstruction of the cultural and the social at the level of the person system). The decision to follow a descending order from Much’s third system to the first is adopted to reflect the movement in the cultural psychology of mathematics education from an understanding of mathematics of particular cultural and social groups to an understanding of the person as a participant in sociocultural practices.

**FOCUS ON THE CULTURAL SYSTEM**

Approaches in the study of the impact of a cultural system on the learning and uses of knowledge vary, depending on the way “culture” is conceptualized. Valsiner (1989) raised this issue by asking the question “What is ‘culture’ in the minds of psychologists?” (p. 502). He attempted to answer this question by pointing out how the underlying view of culture in mainstream cross-cultural studies has led to its treatment as an “independent” variable. That is, culture has been understood as “something that is, in its essence, shared in a qualitatively similar manner by all (or almost all) members of the given ‘culture’ (as a population, society, or an ethnic group)” (p. 503). For Valsiner, this treatment of culture overlooks its historical–developmental dimension. Cultural psychologists have certainly moved beyond the notion of culture as an independent variable, but this does not mean they have shared a definition of culture. Cole (1995) suggested caution in the use of the term culture and referred to the search for a “generally accepted” definition as a “hopeless enterprise” (p. 31).

Most studies that have explored issues related to mathematics learning in out-of-school contexts, at the level of a cultural system, have emphasized the historical, practical, and socially organized nature of the activity. These include studies by anthropologists such as Lave (1988) and Brenner (1983, 1998a, 1991), mathematics educators (Bishop, 1988a, 1988b; D’Ambrosio, 1985; Knijnick, 1993, 1996) and developmental psychologists (Cole et al., 1968; Nunes et al., 1993; Saxe, 1991). A major influence on psychologists working in this area has been Vygotsky’s notion of cultural mediation (van der Veer, 1996). A central focus of the psychological studies has been on understanding what are the tools or artifacts used in specific social practices and their role in mediating mathematical action and thinking. As noted by Resnick, Pontecorvo, and Säljö (1997) in situated theories of cognition,

The concept of tool is expanded...beyond the conventional view of a tool as a physical artifact. Not only physical artifacts but also concepts, structures of reasoning, and the forms of discourse that constrain and enable interactions within communities qualify as tools. Vygotsky...originally distinguished tools from signs, or language. However, subsequent influential developers of theories of socially situated cognition...have suggested that many kinds of thinking, as well as physical actions, are carried out by means of tools. (p. 3)
According to Cole, the relevance of “thinking about culture as a medium constituted of historically cumulated artifacts which are organised to accomplish human growth must be demonstrated by its ability to help us understand the processes of learning and development.” (1995, p. 35). He seemed to have no doubts that he achieved these demonstrations in his empirical investigations. There were two distinct phases in the investigation of cultural influences on mathematics learning in out-of-school contexts. In the first phase, scholars traveled to other countries to investigate the mathematical thinking and learning of “foreign” people. Although not always made explicit this traveling was often politically motivated. The sponsoring of this type of research was linked to the intention of Western countries to introduce their style of schooling in “nonschooled” cultures (see, for instance, Cole et al., 1968). The second phase involves the researchers’ return to their home countries. This also has a politically motivated dimension, coming from the researchers themselves, who realized that the lessons learned abroad could throw light on the learning of diverse groups in their own country. It was also linked to the need in industrialized countries to increase the level of mathematical education of the whole population (Zaslavsky, 1990). Key studies from these two phases are reviewed below.

Cross-Cultural Studies

Some of the pioneer studies of mathematics in out-of-school contexts were carried out on non-Western cultures (Brenner, 1983; Gay & Cole, 1967; Lave, 1977; Pettito & Ginsburg, 1982; Saxe, 1982; Saxe & Posner, 1983). These studies can now be considered a landmark in view of the realization of psychologists that cultural differences are not necessarily associated with deep cognitive differences. They also persuaded psychologists to rethink their research methodology. Instead of taking school knowledge as the reference and formulating tasks from this perspective to see whether individuals transfer to other settings, the researchers engaged in ethnographic observations in outside school contexts. In these studies one can see, along with some traditional cross-cultural concepts, the emergence of new constructs such as apprenticeship, distinct arithmetic systems, and strategies that mark a new era of research in out-of-school mathematics. One also can see how unexpected findings have challenged views about key constructs that have been used to explain mathematical thinking and learning in and out of school. Among the key constructs (linked to a universalistic, culture-free, view of mathematical cognition and learning) challenged were concrete versus abstract dichotomy and the notion of transfer.

Challenging the Concrete versus Abstract Dichotomy. About three decades ago Piaget (1966) wrote “it is quite possible (and it is the impression given by the known ethnographic literature) that in numerous cultures adult thinking does not proceed beyond the level of concrete operations, and does not reach that of propositional operations, elaborated between 12 and 15 years of age in our culture” (p. 309). Abstract thinking for Piaget was characterized by the independence of the form (abstract) from the content. Ability to reason abstractly in this perspective presupposes the existence of specific cognitive structures. Basic mechanisms that will allow individuals to construct these cognitive structures were considered endogenous to human nature. Thus, in theory individuals will be able to achieve a level of abstract thinking independent of the culture they live in.

Vygotsky also distinguished knowledge between different cultural groups in terms of the concrete versus abstract. His view of the origins of these forms of thinking was distinct from that of Piaget, however. He held that abstract thinking demanded a higher cognitive ability and as such was acquired through sociocultural mediation. Thus, the ability of individuals to reason in concrete and in abstract terms would be
a reflection of the historical development of their sociocultural groups. Collaboration between Vygotsky and Luria led to a series of studies in Uzbekistan, where this view was articulated. In the foreword to Luria’s (1976) book, Cole stressed that

His [Luria] general purpose was to show the sociohistorical roots of all basic cognitive processes; the structure of thought depends upon the structure of the dominant types of activity in different cultures. From this set of assumptions, it follows that practical thinking will predominate in societies that are characterised by practical manipulations of objects, and more “abstract” forms of “theoretical” activity in technological societies will induce more abstract, theoretical thinking. (p. xiv–xv)

Thus, it should not come as a surprise that until recently the concrete versus abstract dichotomy was used as an important criterion to distinguish between mathematics learning in and out-of-school. This notion was challenged when the studies consistently failed to identify the presence of any formal cognitive structures in individuals living in non-Western societies, who nevertheless demonstrated high competence when dealing with their local practices. Piaget (1972) attempted to revise his theory to accommodate these findings. New insights emerged when researchers stopped trying to establish the existence of particular cognitive structures, through the use of Piagetian-type tests, to investigate the strategies actually used in the solution of practical arithmetical problems.

The focus on strategies rather than structures led to new emphasis on differences in the use of mathematics in and out of school. According to Nunes (1992b), the differences were not along a concrete–abstract dimension, but were to do with the existence of multiple arithmetic systems in a single culture. Initial support for this assumption was based on Reed and Lave’s (1981) observations of differences in strategies and types of errors depending on whether tailors used the system linked to tailoring or to school. Reed and Lave argued that the use of tailoring-based tools (use of counters, such as fingers, pebbles, marks on paper) was associated with a manipulation-of-quantity strategy. Errors in this case tend to be of small magnitude. On the other hand, use of school tools was associated with a manipulation-of-symbols strategy. Similar findings were observed in other studies (e.g., Grando, 1988; Nunes et al., 1993). The link between strategy and the systems available in particular cultures is in line with Vygotsky’s notion of cultural mediation. Vygotsky, however, tended to conceptualize out-of-school forms of knowledge as more concrete compared with the scientific and abstract forms linked with school, a distinction that became questionable when strategies were analyzed.

Challenging the Construct of “Transfer.” The findings in research on strategies used to deal with mathematics problems outside school also led to a questioning of the relationship with school mathematics. For it has long been assumed that the secret of school mathematics lay in its “power of transfer.” This assumption, however, needed to be rethought. In 1987, Lauren Resnick wrote that

Schooling is coming to look increasingly isolated from the rest of what we do. . . . part of the reason for this isolation may be that schools aim to teach general, widely usable skills and theoretical principles. That is their raison d’etre. Indeed, the major justification offered for formal instruction is—usually—its generality and power of transfer. (p. 15).

Out-of-school practices in non-Western cultures provided an environment suitable for empirical testing of the “power of transfer” of school-related skills. In these settings, researchers could easily find people with different degrees of exposure to schooling from none to advanced levels. This situation also meant that the learning of mathematical
skills required for specific crafts and professions were often embedded in the apprenticeship. Within the same group it was possible to find people who were going to vary both in terms of levels of skill in the profession and in terms of their levels of schooling. Various researchers took advantage of this naturally occurring situation to disentangle the effects of schooling on cognitive development (Greenfield & Lave, 1982).

A classic example is the work of Jean Lave among the tailors of Monrovia, Liberia. As an anthropologist, she spent several months observing the work of masters and apprentice tailors. This enabled her to gain access to the arithmetic tailors used, such as estimating size, in inches, of the waistbands of pairs of trousers. Lave then used this knowledge to develop a strategy to study the impact of schooling and tailoring on mathematical skills. Her strategy consisted of devising arithmetical tasks that varied according to their degree of familiarity with tailoring or schooling practices. She then applied the tasks to tailors who also varied in two dimensions: (a) none to 10 years of schooling and (b) a few months to 25 years of tailoring experience. Lave observed specific effects. Schooling contributed more to the performance in school-oriented tasks and tailoring to the tailoring-oriented tasks. On the basis of these findings, Lave concluded that “It appears that neither schooling nor tailoring skills generalise very far beyond the circumstances in which they are ordinarily applied” (Greenfield & Lave, 1982, p. 199). This study was only a starting point for a challenge of the view that schooling has general cognitive effects, which would transfer and generalize across practices (see Lave, 1988). Evidence from other studies supported Lave’s ideas of the context-specific nature of cognition (for a review see, Laboratory of Comparative Human Cognition [LCHC], 1983). Two decades after research in out-of-school contexts began to cast doubts on the existence of “transfer” and “generalization” as mechanisms supposedly located in the mind, a theory of how learning is carried out and applied across practices is not yet available (Engeström, 1999), although some interesting alternative conceptions have started to emerge (Noss, Pozzi, & Hoyles, 1999).

Research that followed an ethnographic approach not only challenged the notion of transfer but also demonstrated that inclusion of cultural practices in analyzing mathematical cognition was complex. The influence of cultural practices in cognition could not be explained in terms of “general cultural effects.” This became apparent in Gay and Cole’s observation of the contrasts in performances of the same individuals in the same “content domain” (e.g., measurement). For instance, adult Kpelle did better in tasks estimating the measurement of volume than in those measuring length. The Kpelle had not acquired a general ability to carry out estimations of measurement. Instead, they had acquired specific skills closely related to the nature of the thinking and mediating tools involved in the practices. The same was found to be true of the performance of poorly educated American adults. That is, their ability to estimate was not general, independent of what they were required to estimate (e.g., amount of rice), nor was it independent of the measuring tool used or of the situation.

Saxe (1982, 1991) also illustrated the specificity of the impact of cultural practices in cognition by studying the introduction of Western-style currency in the Oksapmin community (Papua New Guinea). He found that people with little participation in commercial activities continued to use traditional counting systems based on body parts, whereas the others who actively participated in economic exchanges had adopted a hybrid counting system combining body parts and numerical representation. In short, developmental cross-cultural research that follows an ethnographic approach has led to a dismantling of established ideas on the nature of human mathematical cognition. This movement in the field of mathematics learning has been paralleled in other areas of human development (Eckensberger, 1995; Woodhead, 1999).
Everyday Mathematics in the Context of Western Societies

The insights gained into non-Western cultures have encouraged outside school research in the researchers’ own cultures (Brenner, 1998b; Carraher, Carraher, & Schliemann, 1982; de la Rocha, 1986; Lave, 1988; Lave, Murtaugh, & de la Rocha, 1984; Masinglia, 1994; Masinglia, Davidenko, & Prus-Winniowsaka, 1996; Murtaugh, 1985; Scribner, 1984). This shift in emphasis is acknowledged as a change in strategy. Cole (1995) made this explicit when he stated that “instead of engaging in cross-cultural research, we began to focus on children in our own society.” In the 1980s, this research was conceptualized in terms of “everyday cognition” (Rogoff & Lave, 1984). “Everyday” was initially adopted as the contrasting term to “laboratory” or “test situations.” Some quotations from Rogoff (1984, p. 2) that reflect this contrast are

“Subjects who perform poorly on logic or communication problems in a test situation often reason precisely and communicate persuasively in more familiar contexts.”

“Observations that children’s capacities appear quite different in their familiar environments than in the laboratory.”

“Young children routinely have difficulty in referential communication tasks, yet in everyday situations they adjust their communication.”

“Laboratory skills seem rather separate from thinking outside the laboratory may lead to an assumption that only in natural environments.”

A careful reading of Rogoff’s chapter suggests that her goal was not to create a theory of differences in psychological functioning between laboratory and everyday settings but to demonstrate that cognition is not context-free. “Everyday” was (and still is) used to differentiate not only research laboratories from other settings but also school versus outside school settings (Brenner, 1998a; Civil, 1995; Nunes, 1992b; Schliemann, 1995; Zack, 1998). According to Lave (1998), the key issue is the interpretation of “everyday.” She distinguished between a functionalist view in which “the label ‘everyday’ is heavy with negative connotations emanating from its definition in contrast to scientific thought” (p. 14) and a practice theory view, in which “the everyday world is just that: what people do in daily, weekly, monthly cycles of activity” (p. 15). Thus, Lave argued that “a schoolteacher and pupils in the classroom are engaged in ‘everyday activity’ in the same sense as a person shopping for groceries in the supermarket after work and a scientist in the laboratory” (p. 15).

Lave’s conceptualization of everyday contexts and how they have contributed to the structuring of learning and uses of knowledge has been supported by findings from the Adult Maths Project, conducted in the United States. She started this project in 1978, apparently immediately after the end of her investigations among the tailors in Monrovia, which took place between 1973 and 1978 (Lave, 1990). The Adult Maths Project included a supermarket shopping study designed to investigate uses of “well-learned and routine” arithmetic (de la Rocha, 1986; Murtaugh, 1985). Examining the mathematical performance of a group of adults shoppers in three settings—routine supermarket shopping, best-buy simulation experiment, and school-arithmetic tests—the findings showed a gap between the performance in the schoollike tests (average 59% correct answers) and in the supermarket (98%) and best-buy experiment (93%). Analysis of relations between performance in the three situations and schooling showed that years of schooling was a good predictor of performance in the schoollike tests but bore no statistical relationship to performance in the other two situations. Lave concluded that these results were against the “logic of learning transfer,” which is based on the generality and power of transfer of the school procedures to other situations. She used these findings to expand the notion, first put forward in her studies in Liberia, of the inadequacy of the mechanism of “transfer”
to account for the relationship between knowledge acquired in different contexts. As an alternative, she proposed that variations in the performance of the same person across contexts could be explored in terms of the following:

- Interrelations between situations, occasions and activities, which might shape arithmetic practice: For instance, use of particular measurement systems in supermarkets might require the person to compare and convert measures.
- Relations between the problem solver and the problems: An important dimension of this relationship seems to be the degree the practice “allows” the person to be in control. Experiencing control can be subjective, but is also shaped by the way activities are structured. Lave observed that at the supermarket, one has the option of abandoning the arithmetic for other types of solutions. The same option might not be available in a mathematics classroom.
- The prominence of the mathematics in the activity that unfolds in different settings: In mathematics classrooms and numerical tests the prominence of using mathematical tools is built into the situation. This might not be the case in supermarket shopping. For instance, some shoppers can establish priorities in terms of a “quality,” such as “organic food.” Giving prominence to quality, comparing prices might not be important, thus no calculations are carried out.

The above observations led Lave to conclude that success and failure in mathematics might best be understood in terms of relations between persons, their activities, and contexts rather than solely in terms of cognitive strategies. Lave’s analysis offered some understanding of variations in the uses of the mathematics by adults that apparently have already “acquired” competence.

Replacing the Concrete versus Abstract Dichotomy by the Notion of “Tool Kit.” Without entering into a discussion of the concrete versus abstract nature of mathematical knowledge (for a reconceptualization, see Hoyles, Noss, & Pozzi, 2001), what I want to emphasize here is that this dichotomy seems inappropriate for distinguishing between out-of-school and school mathematics (Nunes, 1992b; Schliemann, 1995). Historically, mental calculations used by unschooled people were viewed as an example of concrete (practical) thinking. Analysis of strategies linked to oral and written arithmetic challenged this view (Nunes et al., 1993). Using a repeated measures design, Nunes and her colleagues asked working-class Brazilian children to solve mathematical problems in three situations: simulated store problems, word problems, and computation exercises. As predicted from previous studies (Carraher et al., 1982) the children were more likely to choose oral strategies in the simulated store situation and written strategies in the computation exercise. Accuracy also varied as a function of the type of sum and strategy. For addition, the difference in correct answers between oral and written strategies was small, but for the other three operations—subtraction, multiplication, and division—the difference was quite marked. For instance, accuracy in oral subtraction was 62% compared with 17% in written strategies. How is it that the same children offered two different performances in problems apparently similar?

Nunes (1992a) suggested that part of the answer can be linked to properties of the two systems that then mediate problem solving in different ways. She analyzed the solution of the computation 252 – 57. Adelson, who solved the problem orally, provided the following account: “57 minus 52 equals 5. 200 take away 5 equals 195.” Angela wrote it down in the traditional vertical column format and explained: “12 minus 7, 12 minus 7, let me see how much, 7, 8, 9, 10, 11, 12. It’s 5” (writes down 5). Five minus five equals nothing. Two minus nothing equals two.” According to Nunes (1992a), the strategies of Adelson and Angela illustrate the following differences between oral and written arithmetic:
Oral arithmetic
- Preserves the relative value of number
- Proceeds in the order we speak, from large to small numbers
- Allows different types of manipulation, such as dealing with different values and then adjusting
- Preserves the meaning of the situation

Written arithmetic
- Sets the relative value of number aside
- Usually follows the opposite order in which we speak
- Given numbers are strictly adhered to in problem solving
- Sets the meaning of the situation aside

Nunes’s analysis highlighted the fact that common errors in both written and oral calculations were not linked only to the functioning of the mind, but inherently to specific organizations of systems of representations—to the tool that mediated the person’s mathematical action. Cultural practices provide tools that both expand and impose limits on the operations the person can carry out (Nunes, 1992b; Nunes & Bryant, 1996; Schliemann, 1995). In this sense, systems of representations, such as mental tools, can be seen as sharing some properties with tools used to operate in the physical environment. For instance, using a bike as a means of transport compared with using a car shapes the action in different ways. Indeed, the analogy between mental and physical tools was used by Vygotsky to illustrate his claim that human cognition is mediated by tools and signs (Scribner & Cole, 1981; Vygotsky, 1978).

Findings such as the above suggested that to understand human performance it is necessary to look at the interaction between the agent (person carrying out a mental or physical action) and the cultural tool. As Wertsch (1998) noted, the way we attribute competence often obscures the role played by tools. He gave the example of asking someone to do a multiplication with large numbers and afterward asking the same person how the solution was obtained. A common reply would be, “I just multiplied . . . ,” and the person demonstrates writing the algorithm. But who solved the problem, the person alone? Certainly not; he or she used a cultural tool—an algorithm for multiplication—to mediate the solution of the problem. Thus, for Wertsch it is more appropriate to say, “I and the cultural tool I employed” solved the problem. The same applies to Nunes’s examples. Adelson used an oral arithmetic procedure as a tool and Angela used a written algorithm. One cannot guarantee that Adelson would have solved the same problem if asked to use the written tool. He would have needed to have it available in his “tool kit.”

In Wertsch’s (1991) view, a “tool kit” approach offers a more appropriate way of explaining variations in performances between groups. He argued that it enables one to replace the “metaphor of possession” (p. 14), which looks for differences in terms of “having” or “not having” mental capacities, such as for higher order abstract thinking, by a “tool kit metaphor.” In his perspective, “a tool kit approach allows groups and contextual differences in mediated action to be understood in terms of the array of mediational means to which people have access and the patterns of choice they manifest in selecting a particular means for a particular occasion” (p. 94). He goes further, claiming that

as long as the metaphor of possession shapes the debate, a basic issue—the different uses or functions of a tool—escapes the attention of those involved, and they often find themselves in the somewhat ridiculous position of claiming that there are no differences between groups that are obviously different. If the argument is formulated in terms of tool kit analogy, however, with the understanding that different groups may employ similar tools in different ways, much of this confusion can be avoided. (p. 95)
There are various aspects in Wertsch’s concept of the “tool kit” that need to be considered. The first is the properties inherent to specific tools. The second is the reasons that inform a particular selection of tools. Finally the processes that explain different uses of similar tools needs consideration. The first aspect has been widely researched as illustrated above, but the other two need attention. In my opinion, they are linked to gaps or unresolved issues in a culturally sensitive approach to learning that I attempt to address below.

Unresolved Issues on the Way Cultural Tools Mediate Cognition

The Use of the Same Tool in Different Social Practices: Socially Supported Developmental Process? Recontextualizing? Evidence that what one person learns in one practice does not always translate to other practices makes it possible to question traditional explanations of transfer in terms of an automatic process (Bliss & Säljö, 1999). It is also the case, however that people move between practices and that some overlapping between uses of knowledge has been observed. Some researchers have argued that difficulty explaining these movements could be linked to lack of a developmental dimension in the investigations (Abreu, 1998a; Saxe, 1991; Van Oers, 1998b). For instance, Saxe (1991) noted that the investigations of the 1980s on how adults address mathematical problems linked to their everyday activities do not treat cognition from a developmental perspective, a perspective in which cognitive forms are understood as evolving in a complex psychogenetic process, shifting in function over the course of their evolution. For instance, we rarely observe individuals sampled at different ages or at different points in their acquisition of a trade. (p. 12)

Saxe (1991) addressed developmental issues in his investigations among Brazilian candy sellers (boys aged between 6 and 15 years). His study involved ethnographic observations in which the specific mathematics of candy selling was described and demonstrated through structured interviews. One example examined was how children decided the price of candies for retail sale. Saxe observed that younger children were more likely to rely on other people (wholesale store clerks, parents or colleagues) while the older ones did the calculations themselves. Among the children who did the calculations he observed that they predominantly made use of practice-linked strategies. He also observed, however, that some children used school-linked strategies, which in fact involved the appropriation of both practice and school-based mathematics. This shift from the practice-linked to the school-linked strategies was linked to the extent of the children’s schooling. Saxe also analyzed the influence of selling experience on school mathematics. He compared the performance of sellers and nonsellers in schoollike arithmetical problems. He found that sellers obtained more correct solutions than nonsellers did, and that the source of success was linked to their specialized strategies.

Saxe’s developmental approach provided two new insights into the way transfer may be conceptualized. The first is to regard transfer as a constructive developmental process taking shape over time through a progressive appropriation and specialization of forms of knowledge. The second is to regard transfer as a socially supported process. Viewing the interplay between school and out-of-school practices from this perspective, I pose new questions: If the construction is a socially guided process, will the appropriation and specialization outside school follow the same pattern as that occurring in school? Differences in social constraints, or indeed in the way learners and teachers make sense and negotiate perceived constraints, may lead to different
kinds of appropriation. It is not enough to have the tool in the kit; one must figure out whether it is appropriate to use the tool in particular communities of practice. Through which processes do children develop an understanding of which tools should be used and which should be left in the tool kit?

Cooper’s recent analysis of social class differences in children’s responses to national curriculum mathematics testing in England also raises some questions regarding the transfer of knowledge (Cooper, 1994, 1998; Cooper & Dunne, 1998). Following Bernstein’s sociological approach, he argued that the boundaries between everyday and school mathematics are laid down by socialization practices. Cooper (1994) noted that in England, “it appears to be the case that success in school, as measured by national testing in mathematics, will depend on the child’s capacity or willingness to approach tasks with a particular orientation to meaning, one which brackets out everyday, common-sense knowledge as a resource” (p. 162). In this perspective, children who can read the rules of what counts as legitimate school mathematics are more likely to succeed. This reading, however, seems to be influenced by the social origins of the child. Cooper and Dunne found “that working and intermediate class children seem to be more predisposed than service class children, at age 11, to employ initially their everyday knowledge in answering mathematics tests items and that this can lead to the under-estimation of their actual capacities as it is currently defined” (1998, p. 119). The social class differences certainly suggest that their social origins predisposed them to certain interpretations. In addition, the fact that they used information acquired in their home context in a testing situation at school suggests their knowledge was carried out across contexts. For those that take a broad definition, this can be defined as “transfer”: “the use of information or skills characteristic of one domain or context in some new domain or context” (Robins, 1996, p. 1).

Children’s approach to realistic test items based on ways they would cope with the situation outside school can also be characterized as recontextualizing: Children attributed meaning to the test items by applying what they have learned in their everyday contexts (Van Oers, 1998a, 1998b). Although transfer is usually seen as a mechanism located inside the head, (re-)contextualizing was suggested by Van Oers to encapsulate the idea that intellectual activity is embedded in sociocultural activities. The advantage of this concept over transfer is that contextualizing takes into account the social structuring of the practice. Indeed, Cooper and Dunne’s qualitative analysis of interview data shows that the children’s difficulties emerged from confusion about what knowledge was required in the specific context. For instance, in a “tennis item,” children needed to understand that knowledge that was appropriate in a sport context was not appropriate (or legitimate) for solving the school mathematical test. These difficulties were not observed in other items that did not prompt the children to bring knowledge from the outside world.

We now need to look back at the predominant pattern of recontextualizing of the working-class children, in the light of Saxe’s contention that the response is also a reflection of social support. We then need to look at the way solutions to realistic problems were handled in classrooms. To what extent did learning situations provide opportunities for these children to externalize and negotiate conflicts related to defining boundaries, when applying the same knowledge tools in different contexts? This question can also be formulated in terms of how social support can contribute to making visible the tools one selects and the way one makes use of particular tools. Gorgorió and her colleagues (Gorgorió, Planas, Vilella, & Fontdevila, 1998; Gorgorió, Planas, & Vilella, 2001) explored this dimension in their research with immigrant children in Catalonian schools. They also observed that if given a problem that bears some resemblance to out-of-school experiences, children contextualize it within different frames. The researchers used “realistic” problems to structure teaching situations that stimulated children to externalize their frames of interpretation. In doing this,
their aim was to give immigrant children the opportunity to externalize and negotiate conflicts related to what counts as appropriate tools and the way they are used in their home culture or in Catalonian mathematics classrooms. Although their research is still at an early stage, it seems to be going in a direction worth exploring further. As Noss et al. (1999) suggested, it is more likely that “pieces of knowledge,” rather than a whole concept, can be used across practices. These pieces will then be connected through a “webbing mechanism,” which will enable the learner to put the pieces of knowledge together and create new webs of meaning. Thus, it seems that what Gorgoriô’s group is doing in the classroom is, in fact, developing strategies to provide support for learners to make visible the pieces of knowledge they bring in and to negotiate ways of integrating (recontextualizing) these within the context of school mathematical practices.

The complexities of this type of research and classroom practice are well illustrated in recent studies attempting to map the challenges faced by teachers who believe that it is worth actively helping children to bridge the gap between their in school and out-of-school mathematics (Adler, 1997; Atweth, Bleicher, & Cooper, 1998; Civil & Andrade, 2001; Fraivillig, Murphy, & Fuson, 1999). Future research needs to continue exploring the developmental dimension. It is crucial to gain more understanding of patterns of development in learning and uses of mathematics followed by both individuals and groups and how these patterns can be related to the social and cultural structuring of the practices.

**Conventional versus Unique Appropriation of Cultural Tools.** Most of the theoretical advances mentioned above have resulted from studies conducted within traditional out-of-school practices. There was also a tendency in these studies to describe patterns that applied to the whole group. This means that there is still a need to explain how individual diversity can emerge. I explored this field by studying a traditional practice. Between 1995 and 1998, I carried out an investigation into the use and understanding of mathematics by sugarcane farmers in the northeast of Brazil (Abreu, 1998b, 1999). Sugarcane farming was introduced to the area in the 16th century and has since played an important role in the local economy. My motive to engage in this particular project was to gain an understanding of why the farmers have difficulties appropriating modern technology. Both theoretically and methodologically, this study was informed by the so-called everyday cognition approach (Rogoff & Lave, 1984) and by a Vygotskian view of mathematics learning as the internalisation of sociocultural tools.

Focusing my analysis on farmers’ traditional practices enabled me to document the existence of mathematical tools specific to sugarcane farming. The ethnographic approach led me to describe the particular mathematics, used by the farmers, which differed from school mathematics (which is also the basis of modern technology). For instance, they had specific length and area measures, formulas to calculate areas, and a variety of “oral” strategies to solve sums, involving both additive and multiplicative structures. In addition, the findings from the interviews about the strategies used by the farmers were revealing about the way their experiences with the use of specific tools mediated their cognition. For instance, in problems related to the amount of fertilizer applied by area they were sensitive both to the units of measurement and to the numerical relations. When solving problems that involved the use of mathematics, farmers chose as mediators those forms of representation that were closely linked to their practices. These tools enabled the user to function efficiently and perform meaningful cognitive operations.

When I look back at my data, it seems obvious that my research was a good example of an approach that was looking for diversity between groups within the wider Brazilian culture but was still marked by a homogeneous bias of the “within-group”
type of analysis. By concentrating on the similarities of tool use among farmers, I overlooked evidence that showed unique appropriation of the tools and did not explore the mechanisms behind this (Wertsch, 1995). I did not carry out any analysis that addressed development at the level of the individual or, in Rogoff’s (1995) terms, the individual’s participatory appropriation. Nevertheless, this latter type of analysis may well clarify the origins of diversity among individuals in similar cultural practices.

Looking retrospectively at the data, I can see two distinct patterns of the farmers’ reconstruction of cultural knowledge. For instance, on reexamining their procedures for calculating the areas of quadrilateral and triangular plots of land, I can see more than one pattern. In the most prevalent pattern, farmers followed the convention. For example, the area of a triangular plot of land was found by multiplying the average of the two opposite sides by one half of the length of the remaining side \( \text{Area} = \left( \frac{(a + b)}{2} \right) \times \left( \frac{c}{2} \right) \). For those following this pattern, it is as if personal knowledge copied the conventional cultural knowledge, a truly Vygotskian account of a reconstruction of the social at a psychological level. A less common pattern seems to indicate a more complex process. For instance, instead of following the conventional procedure, one of the farmers multiplied one of the sides by half the length of the smaller side of the triangle \( \text{Area} = b \times \left( \frac{a}{2} \right) \). When I asked why he had not averaged the “opposite sides,” he said that could not be done and went on to explain that the largest side was discounted because it can be seen as equivalent to the diagonal of a quadrilateral. Thus, using it in the formulas would result in overestimating the area of the triangle (Abreu, 1998b). In this pattern, there was an indication of uniqueness: Personal knowledge was grounded in cultural knowledge, but it was not a copy of it. The reference to the diagonal of a triangle also suggests that the unique solution could be a hybrid form that combines pieces of knowledge from farming practices with pieces from school practices. If this is the case, a crucial requirement in understanding the problem of diversity is to try to gain some insight into what motivates certain individuals to produce these new forms of knowledge: Cognitive understanding? Valorization of knowledge? A combination of cognitive and social understandings? In addition, the evidence that different patterns coexist suggests that it is necessary to explore developmental processes. What type of experiences can lead some people to follow one pattern and others not to follow it? Next, I review research that has focused on the social system and then return to research on the diversity in patterns of reconstruction of knowledge when discussing the person system.

FOCUS ON THE SOCIAL SYSTEM

As described above studies on out-of-school mathematics arose from cross-cultural psychology, and this seems to have shaped the way the impact of the social system was initially explored. First, the studies carried out on particular types of apprenticeship pointed to links between the uses of mathematical tools and sociocultural and institutional contexts. Second, the notion that the tools one uses to think are cultural generated interest in a particular type of asymmetric social interaction: between someone who is more knowledgeable and someone who is less knowledgeable, such as parent–child, teacher–pupil, or master–apprentice. For instance, adults structure and guide young people in a way that facilitates the reconstruction on a personal (psychological) plane of knowledge that pre existed on a social plane (Rogoff & Morelli, 1989; Rogoff, 1990). Kirshner and Whitson (1997) referred to these two foci of research as the situated cognitionists’ ways of breaking out with individual accounts of learning. The first focus could be linked to Lave’s critical anthropology agenda, informed by “syntetic social theory” rooted in the ideas of Marx, Bordieu, and Giddens (see Lave, 1988). The second reflects an agenda informed by Vygotsky and aimed at exploring
interactions in the zone of proximal development, “an interactive system within which people work on a problem which at least one of them could not, alone, work on effectively” (Newman, Griffin, & Cole, 1989). I first review some key studies under these two focuses and then explore some emerging issues.

The Mediating Role of Social Institutions

In the sequence of out-of-school studies that suggested the use of tools was linked to the context of the practices, researchers explored the applicability of these ideas to formal institutional contexts. Although research on cultural systems has addressed context-specific cognition related to activities (e.g., tailoring, schooling, selling), research on the social system has addressed issues related to social institutions, where these activities took place. Examples of such studies are Säljö and Wyndhamn (1993) and Schubauer-Leoni (1990). Both investigated the applicability of the ideas to different contexts within school, by comparing performance inside and outside the mathematics classroom. Säljö and Whyndhamn found that Swedish students, when asked to solve postage problems, called on different strategies according to the context in which the task was presented. In the context of a mathematics lesson, 57% of the students attempted some type of calculation. In a social studies lesson, however, only 29% used calculations. In this context most of the students found the solution by reading the postage table. Schubauer-Leoni (1990; Schubauer-Leoni, Perret-Clermont, & Grossen, 1992) found that 8- and 9-year-old Swiss children used different solutions to addition problems according to the context in which they were tested. Only 3 out of 34 pupils used conventional arithmetical notation when tested outside the classroom, compared with 17 out of 39 pupils when tested in the classroom.

Because the cultural artifacts available were similar, one can hypothesize that differences in the solution were related to what Minick, Stone, and Forman (1993) referred to as a “multitude of genres.” Mathematics-like language provides different uses that emerge as function of the ways tasks are interpreted. The institutional place, where the activities were presented to the children, provided different frames for the interpretation of the task and allowed them a choice of tools. But what are the mechanisms responsible for this type of dynamic? Walkerdine (1988) used the framework of “discursive practice” to analyze the relationship between signifier and signified in both the home and school mathematical practices of young children. In her view, the person’s construction of school mathematical knowledge is regulated in the discourse of the practice—specifically, in the mathematics classroom. The same reasoning could be applied to the relations of signification that children construct in the mathematics lesson as compared with other school practices (e.g., social science lessons, playtime outside the classroom, etc.). According to Walkerdine “transfer of learning” is not a question of “central processors” inside the human mind, but a result of the production of new relations of signification from practice to practice. Her ideas support the framework that has emerged in studies focusing on the cultural system, such as the view that continuities and discontinuities in the use of knowledge across practices cannot be explained by transfer. In addition, she added new elements by suggesting that discursive practices regulate forms of recontextualization, and consequently, new patterns of development.

The Mediating Role of Social Interactions

Although the influence of social interactions in learning is one of the main lines of investigation in sociocultural theory, most of the empirical research in mathematics learning has been conducted either in controlled experimental situations or in classrooms (Cobb, 1995; Forman, Larrreamendy-Joerns, Stein, & Brown, 1998; Schubauer-Leoni & Perret-Clermont, 1997; Wood, 1999). Thus, analyses of social interactions
specific to the transmission of mathematical knowledge in out-of-school practices are rare. Saxe (1996) acknowledged this gap in his own research. In a retrospective analysis, he wrote, “While in the Oksapmin fieldwork I had spent time observing social interactions in which arithmetical problems were generated and resolved in the trade store, I had not attempted systematic social interactional analyses” (p. 288). In subsequent research in the United States (Saxe, Guberman, & Gearhart, 1987) and in Brazil (Saxe, 1991), Saxe and his colleagues introduced detailed analysis of social interactions.

He did this by researching number practices in White families living in Brooklyn, New York. He sampled families from both working- and middle-class groups with young children aged 2.5 to 4 years. Previous research has revealed that children of this age have developed some basic mathematical knowledge, but the enculturating processes through which these children acquire this knowledge had not been addressed. By conducting in-depth interviews with the mothers, Saxe et al.’s research revealed the social structuring of early number practices at home. These practices had been organized taking into account age and class differences. For instance, the younger children tended to be engaged in less complex activities, while the older children engaged in more complex activities. Furthermore, analyses of mother–child videotaped interactions revealed the dialectical nature of the process: “Mothers were adjusting their goal-related directives to their children’s understandings and task-related accomplishments and children were adjusting their goal-directed activities to their mothers’ efforts to organise the task” (Saxe, 1996, p. 292).

Further support that families play a part in the structuring of practices that contribute to the child’s development of mathematical cognition outside school was obtained in Guberman’s (1996) study with Brazilian families. Children from working-class families in Brazil often engage in commercial transactions. From an early age, children commonly buy food from local shops. (Previous research shows that engagement with commercial activities is linked to the development of mathematical skills; Abreu, 1995b; Abreu, Bishop, & Pompeu, 1997; Saxe, 1982). Taking this into consideration Guberman observed and interviewed parents of Brazilian children aged 4 to 14 years. He explored how parents structured their children’s learning of mathematics by varying the degree of responsibility assigned to the child in purchasing tasks. His findings revealed four levels of engagement in purchase tasks that were related to increasing arithmetical complexity. For instance, at Level 1, the child was given the exact amount of money needed for the purchase. At Level 2, the child is not given an exact amount and is told to wait for the change. At Level 3, the child is required to check the change. At Level 4, the child is expected to calculate the cost of the purchase. Identification of currency and the requirement to make calculations varied in complexity from Levels 1 to 4. Guberman concluded that “even in activities distal from direct verbal interaction with parents, children learning often is regulated by parents” (p. 1621).

The types of mathematical competencies studied by Saxe and Guberman have often been referred to as “informal mathematics.” Furthermore, some authors assert that “much of this informal mathematics can develop in the absence of adult instruction; indeed, many adults are quite surprised to learn how much their young children or students know in this area.” (Ginsburg, Choi, Lopez, Netley, & Chao-Yuan, 1997, p. 165). Saxe’s and Guberman’s findings do, however, suggest that the absence of what groups believe is formal instruction in mathematical concepts cannot be confused with lack of socially organized activities that support a child’s learning. In fact, in the same way that adults would not often define their use of mathematics in outside-of-school activities as “mathematical,” they would likewise not define their children’s practices as “mathematical.” The mediating role of adults and more expert peers in structuring and supporting the new generations in their local practices, however, is revealed when explored in systematic research.
Unresolved Issues in Social Mediation: The Impact of Changes Imposed by Macrosocial Structures

Sociocultural approaches to mathematics learning—in particular, those based on initial interpretations of Vygotsky and apprenticeships models—are now recognized as providing a limited account of the impact of the social system on learning (Duveen, 1997; Forman, Minick, & Stone, 1993; Goodnow, 1990, 1993; Goodnow & Warton, 1992). Study of interactions such as those described above were criticized for providing “descriptions of perfectly orchestrated dyads” (Litowitz, 1993, p. 187) and for lack of consideration of the influences of the wider social structure (Stone, 1993). To consider the impact of macrosocial structures on our current understanding of mathematical learning in outside school contexts, I analyze a second scenario from my research with sugarcane farmers. This scenario illustrates the farmers’ struggle to cope with changes imposed by the macrosocial structures that involved varying degrees of exposure to new mathematical tools.

At the time of my fieldwork in the farming community, the farmers were in the midst of coping with one of these external demands. Changes in the Brazilian economy at a macrolevel led the government to impose new criteria for the payment of sugarcane: Since 1984, the system of payment for sugar-cane according to quality has been imposed by law. In the past, the criterion had only been a function of the weight of the sugarcane produced, independent of quality. This was the only system the farmers had ever experienced, and mathematically it was quite simple. Following the old system, farmers could calculate how much they would receive from the sugar mill by multiplying the amount (tons of sugarcane) they had produced by the price per ton. With the new system, they need to deal with the variables that define the quality of the sugarcane. In the end, the price could be found by multiplying the index of quality, times the amount produced, times a fixed price per ton. The new system posed some difficulties for the farmers, such as (a) the index of quality was calculated by highly sophisticated computing and laboratory technology located in sugar mills, and (b) the index of quality did not apply to the total quantity of sugarcane a farmer produced, but the tests were carried out on each specific delivery. The consequence was that the price per ton was variable. For farmers to find out an average price, they needed to go through complex calculations. (These were not straightforward because the information was presented in sophisticated forms with values for the different factors, and farmers had to read complex tables.)

The farmers’ ways of dealing with the new system could be analyzed from a Vygotskian-based perspective focusing on the limits on cognition imposed by not mastering the new mediating tools. This would emphasize a particular type of relationship between the cultural system and the person’s system. Such an orientation, as seen above, dominated my thinking at that stage and can be noted in the guidelines for the following “questions” to be covered when interviewing the farmers (Abreu, 1998b):

1. What do you think about the payment for sugarcane by amount of “sacarose” in it?
2. Do you know what agio means?
3. Do you know what desagio means?
4. Which system do you think is better for you and why?
5. How did you calculate your income before the new system? And how do you calculate it now?
6. In cases where you have an agio of x% (5% and 10%), what will it represent in gain?
7. In cases where you have a desagio of x% (5% and 10%), what will it represent in loss?
Apart from the first and fourth question, which left some room for the farmer to articulate his experience in general, the other questions were limited by mathematical understanding. Research, however, as a human enterprise, is not a unilateral process shaped by the researcher (Grossen, 1997). My mathematical focus did not prevent farmers from reinterpreting the questions and articulating answers revealing different facets of their experiences with the change.

For the individual, changes in macrosocial structures can have different types of impact, which can be linked both to mathematical knowledge itself and to the way knowledge influences identities. The first impact of the change was that it required a type of mathematical knowledge most of farmers did not have. Traditional mathematics (Abreu, 1999) enabled the farmers to grasp some understanding of the new system, such as when comparing whether they were making or losing money, but it was limited when they had to read and interpret tables combining different variables and when they were required to understand concepts such as percentages, decimals, and positive and negative numbers. The second type of impact was on the farmers’ identities. The changes made them experience loss of control; it brought uncertainty and threatened their standing. They were not sure where they stood, whether they could contract services, whether they could borrow money from the bank, and, perhaps more important, whether they could survive in business. The third impact was that exposure to technological innovation and modern institutions (e.g., schools; banks) over time raised the farmers’ awareness that some forms of knowledge are more powerful than others and also that some are more “accepted” than others. For instance, a contract in a bank could either be signed or stamped by a fingerprint. The farmer who signed might be functionally as illiterate as the one who stamps his fingerprint, and both may be unable to read the contract. The first method, however, enabled the person to feel part of the literate community (inclusion); the second assigned the person to the category of “illiterate” (exclusion from a high-status group). The ability to sign was then highly valued by the group, whereas use of the fingerprint a reason to be ashamed. The same applied to the whole of traditional farming mathematics when compared with school mathematics. I referred to this phenomenon as a group’s valorization of knowledge. This seemed to reflect the status of the practices in the wider social structure.

This second scenario shows some of the shortcomings of the explanations about the learning of mathematics in and out of school that are based on studies conducted in traditional practices (Duveen, 1997). In current societies, it is more likely that both adults and children will frequently experience exposure to change and coexisting practices (development at the plane of the cultural and social system). In these circumstances their type of experiences might be closer to those of the farmers when new technologies are introduced, that is, knowledge will be experienced both in terms of mastery and in terms of the identity by those in the role of transmitter (e.g., teachers, experts, parents) and those in the role of learner.

**THE PERSON SYSTEM: MOVING THE FOCUS FROM COGNITION TO IDENTITY**

The need for a better account of the interplay between the cultural, social, and person systems was shown in the previous sections. For instance, there were issues related to the uniqueness of the individual and patterns of development in the reconstruction of the cultural tools at the person level and also issues related to the social valorization of knowledge, changes in social structures, and the person’s and the group’s sense of identity. Cultural psychologists suggest that these issues should be addressed by referring to the person, instead of referring to cognition. Thus, they reveal their attempt
to bring the concept of self, human agency or identity into the psychology of human learning and development (Lucariello, 1995; Shweder, 1990, 1995). Bruner proposed that one of the tenets of a “psychocultural” approach to education is to take into account the existence of perspectives that reflect both the shared culture and individual histories. To him, “nothing is ‘culture free,’ but neither are individuals mirrors of their culture” (Bruner, 1996, p. 14).

The understanding of this interplay between sociocultural factors and the individual is still an issue (Damon, 1991). As Gauvain (1998) mentioned in her review of research in mathematics learning in sociocultural contexts, the individual part of the process has been largely overlooked not only in the field of mathematics learning but in accounts of “intellectual development from a sociocultural vantage.” Furthermore, she argued,

Although many may argue that individual analysis is anathema to a sociocultural approach, and this may be true in a philosophical or theoretical sense, a conceptualisation that allows the investigator who holds a sociocultural view to locate individual thinking and individual contributions within social processes is sorely needed. In practical terms, such linkages are imperative for evaluating the worth of this overall approach to classroom learning. After all, individual children are experiencing these social learning contexts and individual children will be held accountable for and be provided with or excluded from opportunities stemming from these experiences. (p. 564–565)

According to Engeström (1999), advances in sociocultural explanations of learning will require asking “carefully-focused and theoretically-grounded questions.” He distinguished agendas of research in situated learning in terms of weak and strong versions. The basis for this distinction was that the “the former speaks of contexts, the latter speaks of practices, and of participation in communities of practice” (p. 250). In his view, the minimal requirements for a strong version are “(a) that the study is focused on some relative durable socially-important collective practice, and (b) that the researcher presents some sort of ethnographic description and analysis of the collective practice in which learning is embedded” (p. 210). The research I described above with the Brazilian farmers falls in this category. This approach enabled me to obtain insight into the farmers’ (a) specific mathematics tools, (b) mediation of cognition by their specific mathematical tools, (c) awareness of the limitations of traditional mathematical tools to cope with innovation, and (d) perceived relative power of their indigenous mathematics (Abreu, 1999). Continuing with the line that explores mathematical learning in terms of participation in communities of practice, which are both cultural and social structures, I explore how the insights obtained from the studies on farmers produced new research questions related to the relationship between the cultural, the social, and the person system. The challenge is to develop ways of understanding the emergence of diversity between individuals who come from groups that share cultural and social systems, without attributing these differences to motives exclusively located inside the individual (see also Engeström, 1999).

Family Background and Reconstruction of Out-of-School Mathematics by Children

The first issue that emerged from the study with the farmers was the nature of the child’s participation in family practices and associated mathematical knowledge. In the apprenticeship approach, the enculturation of the children in their community home practices was often taken for granted. One can ask, however, why the farmers would want to transmit to their children knowledge they knew to be marked as low status. This raises questions about how parents structure the participation of their children in their own mathematical practices. Consequently, questions arose about the
within-group homogeneity of children from similar backgrounds. I started exploring these issues by interviewing schoolchildren in rural areas of Portugal (Abreu et al., 1997) and Brazil (Abreu, 1995a, 1995b) about their participation in home practices and the support they received from significant members of the family. Then I extended this research to children and their parents in multiethnic primary schools in England (Abreu & Cline, 1998; Abreu, Cline & Shamsi, 2001b).

**Findings from Studies with Children.** Observations from the studies with schoolchildren (Abreu, 1995a, 1995b; Abreu et al., 1997) revealed a new angle on how they participated in outside-of-school mathematical practices. Instead of a common pattern of enculturation into home practices, evidence of heterogeneity emerged. Shopping activities related to the everyday life of the family—for example, fetching bread, fruit, or vegetables—were common practices among the schoolchildren in my studies. Interviews with children engaged in similar practices revealed differences in what the child was in charge of: (a) some of the children just did the shopping, and adults took responsibility for economic exchanges (that is, they fully accepted the vendor’s sums and change or their parents took responsibility for that); (b) some of the children shared the responsibility with their parents or with other adults; and (c) some of the children described themselves as being in charge of the economic decisions involved. The manner in which the children experienced this situation seemed to influence the extent to which they used mathematics in the home practices. At a first glance, these categories of engagement can be seen as developmental. Indeed, it is not difficult to see some resemblance with Guberman’s (1996) levels of engagement. However, in-depth comparative case study analysis suggested that the diversity could be linked to the way parents’ supported their children’s engagement in out-of-school practices, but also that they might take into account other factors than the children’s developmental level. For instance, children’s accounts suggested that not all the families engaged their children in the use of traditional home mathematics. These observations motivated our inclusion of parents in subsequent research (Abreu & Cline, 1998). It seemed likely that parents were playing a key role in the diversity among children in their reconstruction of the cultural systems of knowledge of the home practices. But the dynamics through which their influences operated were unclear. Were their own valorizations of the coexisting mathematical practices the crucial factor? Or was their influence shaped in interactions (and perhaps negotiations) with the children themselves (i.e., a joint construction)? What type of active role did the children play in their own development? To explore these issues, the research strategy adopted with the Brazilian schoolchildren (Abreu, 1995a, 1995b) was expanded to include a parental perspective. The new studies were conducted in multiethnic primary schools in England (Abreu & Cline, 1998; Abreu, Cline, & Shamsi, 1999).

**Findings from Studies with Parents.** For these studies, schools with representative numbers of children from Bangladeshi (Abreu & Cline, 1998) and Pakistani families living in England (Abreu et al., 1999) were selected. The patterns of achievement of children in these multiethnic schools had some similarities to the schools in Brazil. Children of Bangladeshi and Pakistani origin, on average, still underachieve at English schools (Ofsted, 1999). Nonetheless, as in Brazil, within any single-year group, there was variation in performance among the children from the same home group, which include both high and low achievers. It was also likely that there were differences between children’s home and school mathematics because their parents experienced a different culture and a different school system through having gone to school in their country of origin. Therefore, it was possible to follow the original question: “Do the children who succeed [in school mathematics] establish a different relationship with their home knowledge than the ones who fail?” (Abreu,
1995b, p. 124), by incorporating a parental perspective. That is, if children establish a different relationship do parents play a role and, if so, what does that role involves?

For each child selected as case study, data collection included several methods, consisting of classroom observations, interviews with the child, interviews with the classroom teacher, and one interview with parents. One aspect of the findings from these studies worth mentioning before moving to the parents is, “What is home mathematics for the child?” In the communities studied in Brazil and Madeira, questions about what the child usually did after school easily led to a conversation about out-of-school mathematical practices. Their engagement in buying in the local shops or after-school paid work made this link easy to follow. With the schoolchildren in England, this was not the case. When asked to recount their after-school activities, the White children emphasized leisure activities such as watching television, sports, hobbies, and music. The Pakistani children’s accounts emphasized a further structured process of education at home or at the Mosque. Although the schools varied in the amount of homework assigned, the children did not have any difficulty in recounting instances when their parents or other relatives (siblings, aunts) helped them with school mathematics. Therefore, relevance of home mathematics to the children emerged in practices where parents and relatives supported them with school mathematics.

The same pattern of findings was obtained from interviewing parents. The more vivid accounts of their engagement in helping the children with mathematics at home were related to school mathematics. It was also in these accounts that both children and parents made explicit the differences between the way mathematics was tackled in the child’s school and at home (Abreu et al., 2001b). Accounts from the parents showed that differences between home and school mathematics could be experienced in terms of

- the content of school mathematics and in the strategies used for calculations (examples included differences in algorithms for subtraction and division),
- the methods of teaching and the tools used in teaching (for example, methods for learning times tables, use of calculators), and
- the language in which they learned and felt confident doing mathematics.

Apart from the differences related to language, the first two differences applied to both groups. They seem to be linked to the parents’ experience of a different school system (immigrant parents) and or to changes in the curriculum over time in England. This meant that in both ethnic groups, if parents (relatives) were to support the child at school properly, they had to figure out the necessary process of transition between their own mathematical practices and the ones the child was experiencing (Abreu, Bishop, & Presmeg, 2001a).

Differences within the same ethnic group emerged when we focused the analysis on (a) the influences of the parents’ positions in the way they tackled the differences, and (b) a comparison between information obtained from the parent and their child regarding representations, experiences, and negotiations of differences between home and school mathematics. There was some evidence that the way parents structured their support to their children was likely to be colored by their own positions about which form of knowledge they valued more. For instance, they took positions regarding the language by which mathematics was communicated, or regarding the importance of knowing the times tables by heart. Understanding the basis on which parents construct valorizations of mathematical knowledge is an area for further study.

The patterns of interaction between the child and the parent’s experiences offered further insight into the emergence of within group diversity. For example, comparative case study analysis highlighted some possible links between patterns of interaction and children’s performances. Comparison of two Pakistani children, both in Year 2
and in the same mathematics class, showed that for the low-achieving child, there was a divergence between the child’s preferences and the way the parents were trying to support him in learning school mathematics. His parents were not aware that the transition exposed the child to differences between the way they taught mathematics to him at home and the way he was learning at school. The child’s accounts revealed that he believed the teacher knew better than his mother did and that he preferred the English language. At home, however, his parents were teaching him in Urdu and did not show any awareness that the change in language at school could cause him difficulties. This seemed a case in which the parents’ representations of primary school mathematics were still associated with their own schooling. Difficulties in communication with the school might have reinforced their representation. By contrast, in the case of the high achiever, there was a convergence in the way differences in methods, language, and identities were negotiated. In this case, the parents—in particular, the mother—had developed representations of home and school learning, which included a theory of how her child might experience the transition. For this child the differences, between the mother’s and his teacher’s mathematics did not mean that home practices had “inferior status.” Like the low achiever, he also preferred the English language, but in this case the mother was prepared to help the child bridge the gap between the two languages.

A basic distinction between the low achiever and high achiever was seen in the parents’ awareness of the existence of differences in their own ways of doing mathematics and those that their child was being taught at school. The parent of the high achiever also showed more awareness of the child’s preferences. The success was then achieved through sensitive interactions, in which the mother learned from the child and then adjusted her strategies to fit with his needs and preferences. A case study with a White parent, however, showed that being aware of the difference was not enough to support a child. In this case, the mother had developed an acute sense of the differences between her methods and those of the child but was experiencing it as a “burden.” Therefore, interactions conducive to success seemed to require awareness of differences, flexible adjustment, and specialized mathematical knowledge. In sum, the fact of including the perspective of parents revealed a common facet in the intersection of the three systems. It showed that parents, as significant actors in the social system, did not just recreate their own mathematical background for their children. They deliberately selected some forms of knowledge from those available in their cultural system as appropriate to transmit to their children and rejected or hid other forms. Finally, the actions of some parents seemed to take into account the active role of the child. Again, this is an area that needs further investigation. We need to ask which dynamics explain how and why some parents grasp an awareness of their child’s needs and preferred modes of learning, whereas others do not. Could particular dynamics be related to parents’ valorization of their home practices in comparison with the child’s school mathematical practices? How can valorizations held by parents and children create “resistance” (Duveen, in press) in parent–child interactions?

Valorization of Mathematical Practices and Social Identities

I have argued that the interaction between parents and children in situations in which they must negotiate home and school mathematics is not unidirectional. Of course, one of the aspects that can contribute to bidirection is a different type of understanding of a mathematical tool. At this point I focus on another aspect, however, which is related to the association between practices and identities. At the base of my argument are the assumptions that (a) children develop an understanding of the valorization of coexisting practices and associated social identities, and (b) they develop personal positions, which make them active agents in the way they participate in the practices.
These assumptions in fact reflect a well-established view on the way people gain an understanding of the social world around them. The notion of understanding social valorization can be seen in terms of Tajfel’s (1978) complementary processes of social categorization and social comparison. Both processes are considered to be constructed as part of socialisation. Social categorization relates to “the ordering of the social environment in terms of groupings of persons in a manner which makes sense to individuals” (Tajfel, 1978, p. 61). Social comparison will emerge from a human tendency to evaluate the ordering within the system of values of a society (Moscovici & Paicheler, 1978, p. 253). These processes will then contribute to an individual’s development of social identity: how a membership in particular community of practice contributes to an understanding of who they are and where they stand.

The findings obtained in the studies, in which my colleagues and I explored children’s understanding of social practices in which mathematics is used, pointed to the development of similar processes. It is important to bear in mind that all the studies involved small samples (20 to 24 children). They were, however, conducted in different countries: Portugal (Abreu et al., 1997), Brazil (Abreu, 1995a), and England (Abreu & Cline, 1998; Abreu et al., 1999).

First, the findings from these studies indicate that the children had developed ways of categorizing some practices of their community as involving uses of mathematics and others as not involving them. For instance, in the farming community in Brazil, it was more likely that children would categorize working in an office as a practice in which people use mathematics than working on a sugarcane farm. In England, few children categorized driving a taxi as a practice that requires the use of mathematics, but they could see shopping as one that does.

Second, the children had developed a basis for the comparison of coexisting practices. Justifications for categorization of a practice as being mathematical or nonmathematical included references to the presence or absence of tools (calculators, cashiers), the nature of the job, and the different status of communities of practice. They also associated performances in school mathematics with given social identities. For instance, for tasks in which they had to choose adults who might have been good or bad at school, the children chose more adults in white-collar professions (office workers) as possibly having been the best in school mathematics; conversely those in blue-collar professions and other lower social status practices as possibly the worst. These results have now been replicated in Spain (Gomez-Chacon, 1997; Planas, 1998).

Third, the children had developed positions that could be indicators of the way they were constructing membership of specific social groups or their own social identities. Again, comparative case study analysis has provided insights into the development of positioning. Children who shared knowledge of the social valorization of sugarcane farming activities could assume different positions. For instance, most children interviewed in the Brazilian study shared the view that sugarcane farming was a low-status practice. On this basis, some developed the position that people engaged in this work did not know mathematics, despite realizing that they were proficient in their indigenous form of mathematics. Some of the children who denied the existence of mathematical knowledge in sugarcane farmers were attributing a low self-identity to themselves because they were already actively engaged in that community.

The understanding of the relationship between learning and identity is an area that requires the development of conceptual and methodological tools; however, the findings from the study with the school children in England (Abreu et al., 2001b) helped to elaborate some ideas of the contributory factors in the ways children develop their positions. For instance, when explaining their preferences for doing mathematics at home or at school, they took into account (a) the mathematics that should be learned, (b) the mediating role of other people, and (c) the mediating role of language. Interestingly, for all the three categories, the children mentioned justifications that
covered both cognitive understanding and evaluative judgments. For instance, in talking about their preferences for the mathematical tools they encountered at school or at home, some children referred to cognitive factors, such as facility in using a particular strategy. Alternatively, some children mentioned social comparison, in terms of the valorization of the tool. Children who mentioned the mediating role of others analyzed this role in cognitive terms (e.g., commenting on whose explanations the pupil understood more), in affective terms (e.g., commenting on with whom the pupil felt more at ease in asking for help), or in social comparison terms (e.g., commenting on who was seen as having knowledge or competence). Finally, the mediating role of language was treated by some Pakistani children at a cognitive level (e.g., commenting on whether they grasped mathematical concepts more easily in one language or the other). For others, however, language preference had an affective–comparative basis (e.g., defined in terms of the language in which they felt more confident). In sum, the salient factors that the children chose to justify their positions varied. Some gave primacy to their feelings of competence with specific tools; others put the emphasis on the quality of the interpersonal relationships (namely, those whom they felt to be less threatening and more patient in providing support); others seemed to be guided by the status of the practice in the social structure. Of course, some children mentioned more than one reason. What was common to all justifications was that they involved some type of identification with the form of knowledge (mathematical tools), with the social actors involved in a practice, or with the practices of a certain social group. Thus, we may see the importance of continued research focusing on the person and incorporating the notion of identity construction with that of the cognitive construction by mapping it into the cultural and social systems.

CONCLUSION

Research since the 1970s on out-of-school mathematics has followed the same direction as that of Vygotsky’s theory on the impact of culture in the mind. According to Minick et al. (1993) Vygotskian research of the late 1970s and early 1980s tested the plausibility of the theoretical framework. It was one-dimensional and focused on a discussion around the relationship between cognition and cognitive tools. Following this initial period, Minick et al. suggested that during the 1980s the framework was broadened to pay attention to the following: (a) the way that institutions (e.g., schools, families, commercial and financial institutions, etc.) structure contexts both in terms of styles of interactions between people and of the cultural artifacts made available (e.g., books, computers, calculators); (b) “language” as a multitude of distinct speech genres and semiotic devices in opposition to a generalized or abstract semiotic system (this multitude of genres is linked with participation in specific social institutions and specific practices); (c) real people in opposition to “abstract bearers of cognitive structures,” for whom appropriation of knowledge passes through identification with communities of practice.

The cultural psychology approach I adopted in this chapter is intended to pay attention to the areas mentioned by Minick et al. As far as empirical studies are concerned, research on out-of-school mathematics scarcely investigates the last two areas. The attention to social institutions and social interactions became apparent when research in out-of-school mathematics shifted from cross-cultural comparisons to social practices within the Western societies. This highlighted the need to understand the constitutive role of the social system on learning. As illustrated by this review, however, the study of issues related to how distinct mathematical discourses are articulated, produced, and reproduced and also about participation and identification are merely emerging as foci of research in out-of-school mathematics. One can speculate this is linked to
particular traditions of research in the field. Studies on out-of-school learning have drawn attention to the situated nature of mathematical practices, which in its turn pinpointed the need to explore classroom learning from this perspective. The reconceptualization of mathematics classrooms as situated communities of practice have been a first step, but as yet that have not offered a satisfactory account of the interplay between the individual and the sociocultural. The fact that we can hardly produce explanations at this level could be linked to ways of conceptualizing and studying learning and development in situated practices (Damon, 1991). This may be due to an association between situated practices and “sociophysical space” (Kirshner & Whitson, 1997) that overlooked the fact that individuals move between practices. Attention to this movement will involve issues that can be located at the interface of the social and person system, such as that of identification as related to communities of practice, which are part of wider social structures rather than isolated units, and also at the interface of the cultural and person system, such as the construction of hybridized tools and discourses in the intersection of practices. The relevance of continuing research along on these lines has been recognized by those concerned with issues of diversity and equity in mathematics education (Secada, 1988, 1992). This will require substantial theoretical and methodological investment.

In theoretical terms, frameworks, such as the one illustrated in this chapter, will require further development to provide specifications of key constructs that would be part of each system or plane of analysis. This is far from an easy task. For instance, authors vary in their definitions of culture and cultural systems. The same could be said about the social system, not to mention the person system and the conceptualization of its interplay with the other two systems. If we emphasize the importance of identity in mathematics learning, as illustrated in my review, this will raise a series of new questions. For instance, what are the dynamics of particular constructions of identities? Types of membership? Sense of belonging? Resistance to participation? How can the construction of identities be related to the development of mathematical knowledge?

In methodological terms, it seems there is still much to be learned by focusing on specific cultural groups or in specific socially important collective mathematical practices in out-of-school contexts. These foci could possibly clarify the uses of out-of-school mathematics (Hoyles et al., 2001) but need to be expanded to consider transitions between communities of practice (Abreu et al., 2001a). The use of ethnographic strategies also is necessary to place the individual and the communities of practice within the cultural and social systems. The study by Hoyles et al. clearly illustrated how these ethnographic strategies can radically change what counts as mathematical competence at work. Systematic accounts of development of mathematical understanding and uses of mathematical tools require more than that, however. The person plane of analysis requires methodological tools to study and analyze specific aspects. There is now some understanding of the design of tools for investigating the understanding of mathematical concepts and strategies, but this does not apply to tools to investigating the associated processes of construction of identities. Social identity construction traditionally has been studied as part of individual social development, such as gender identification. We need, to be careful however, with transpositions by questioning the extent to which the same principles should be applied to identities in mathematical practices.

Finally, as far as educational policy is concerned, research in out-of-school contexts suggests that in situations where home backgrounds differ markedly from school backgrounds, children might benefit from an approach that helps them bridge gaps and cross boundaries. This is far from being the way curricular reforms are implemented. For instance, the numeracy framework in England emphasizes the need for
parents and communities to be involved in their children’s mathematics education to ensure achievement (Brown, Askew, Baker, Denvir, & Millet, 1998; Department for Education and Employment, 1998). The parents’ involvement, however, is not seen in terms of helping the children to integrate home and school numeracies (Baker, Street, & Tomlin, 2000). It is instead portrayed in unidirectional terms in which the parents are expected to support school numeracies, but no attention is given to home numeracy practices. Although these policies are in line with representations of teachers and parents who view the school mathematics as the relevant one for the child’s success in today’s society, they might fail to take into account the actual experiences of the developing child. The social and cultural basis of representations from teachers, parents, and curriculum planners need to be understood along with an examination of their impact on the way learners experience transitions between their school and out-of-school mathematical practices.

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CHAPTER 15

Research, Reform, and Times of Change

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Reform movements in mathematics education are hardly new phenomena, yet the scale and frequency of such movements in recent years make them legitimate and, indeed, pressing objects for educational research. The main goal of this chapter is to spell out some of the characteristics and difficulties of research directed specifically toward reform and reform movements. The general thesis that emerges is that research into reform must reflect the complex and global nature of reform itself and therefore must be of a different kind than other traditional forms of educational research.

The discussion comprises two main sections. The first section looks at some aspects of reform movements in mathematics education, especially those of this century. No attempt is made to provide an exhaustive survey; the goal is only to obtain some idea of the nature of reform by examining several reform components and their realization in actual reform movements. Indeed, the wide range of efforts referred to as reform movements makes the basic question of what reform is difficult to answer. It may be that only an intimation of a definition is truly possible. We emphasize, however, that whatever else it may be, a reform movement is a phenomenon that necessarily involves the whole complex of students, teachers, researchers, parents, and politicians; it is a phenomenon very much connected with values.

The second section considers the main question, namely, What should characterize research on reform? The discussion begins with the problem of methodology moves on to the problem of how research and practice interact, and finally considers questions researchers must ask about change. The methodology of research on reform is conditioned by the necessity of research being itself part of the reform process. There are some models for such self-referential research, and it may be that these have their clearest application in reform movement research. The sometimes uneasy relationship between research and practice is clearly something that the researcher on reform must both confront and study. Obviously, this is not a concern that belongs exclusively to research on reform, but it is one that must be at the very center of this brand of research. Finally, because reform has everything to do with change, researchers on reform must consider the question of change: They must consider how reform effects change and how it responds to change. The latter is particularly important because
recent reforms have been conceived with the rapid changes in society, knowledge, and technology in mind.

Our chapter concludes with some thoughts as to why questions of content must be among the concerns of the reform movement researcher. This is so for the simple reason that the researcher, in wishing to bring about a change and needing to explain and justify a change, must give firm attention to the definition of ends. Moreover, because, as discussed in the first section, a reform movement is so wide in scope, defining ends must take place in a wide forum involving the views not only of other mathematics education researchers but also of mathematicians, scientists, philosophers, and historians. The chapter ends, therefore, with a call for cooperation among these different intellectual disciplines.

REFORM IN MATHEMATICS EDUCATION

Reform in mathematics education, as with any reform, has its motives, its initiators and movers, its implementation, and its assessment. Naturally, these components of reform are more easily separated on paper than they are in fact. Indeed, it is not by accident that often those who first suggest the need for reform are also those who initiate the actual reform, who take charge of its implementation, and who assess its results. Nevertheless, for the sake of exposition, we consider each of these four components individually.

Motives for Reform

It is wrong to think that reform aims only to abolish old institutions and practices that have become effete, that is, that its principal motivation is negative, mere dissatisfaction. Yet it must be said that most reforms do begin with some kind of dissatisfaction—and usually very basic kinds: students’ leaving school with only minimal mathematical knowledge and skills, failing to pursue the least mathematically oriented careers, and, of course, performing poorly on standard examinations.

Poor performance on standard examinations has been a motive for change almost from the moment the idea of a standard examination came to be. In fact, the development of such examinations and of reform movements go hand in hand. As an early example, one can point to the use of a citywide examination to test the accomplishments of the Boston school system in 1845. The original intention of the examination was neither to reform nor even to carry out disinterested assessment, but to prove to the secretary of the Massachusetts Board of Education, Horace Mann, that Boston schools deserved the economic support they were receiving. In the end, the poor results on the mathematical parts of the examination, among others, prompted a review of instructional practices and school organization, and fueled Mann’s interest in such tests as a tool for the improvement of education (see Kilpatrick, 1992). Standardized examinations have not ceased to have this double role of both assessing an existing educational system or previous reform and prompting a new reform. Thus, the well-known report *A Nation at Risk* (1983), which gave impetus to the National Council of Teachers of Mathematics (NCTM) *Standards*, also cited low performance on international assessments as one sign that a change was in order. We return to the question of standardized tests when we consider assessment of reforms.

Modernization: University versus School Mathematics and Science

A more positive motive of reform has been what one might call “the necessity of modernization.” This arises with the recognition that what is being taught in school is somehow out of step with what mathematicians really do and with what people
who use mathematics really need. The great mathematician Felix Klein, who was also, incidentally, president of the Commission on the Teaching of Mathematics until his death in 1925, referred to a “double discontinuity” whereby secondary school teachers teach without regard to university mathematics and university students destined to become a mathematics teachers study their subject as if irrelevant to school mathematics. Thus, he said,

There is a movement to abolish this double discontinuity, helpful neither to the school nor to the university. On the one hand, there is an effort to impregnate the material which the schools teach with new ideas derived from modern developments of science and in accord with modern culture. . . . On the other hand, the attempt is made to take into account, in university instruction, the needs of the school teacher. (Klein, 1908/1939)

One of the changes Klein particularly wanted to see was the introduction of the function concept into the secondary schools. However, Klein’s desire to find common ground between the mathematics and science of the university and the mathematics of the schools was not restricted to a single subject or concept; Klein wanted to see a general unification of programs within the university system itself and within secondary school system. He wanted to see, for example, a weakening of the division between the three traditional types of German schools, the Gymnasium, Real-Gymnasium, and Oberrealschule (Kilpatrick, 1992; Schubring, 1994). Indeed, the motive of modernization, although it may focus ostensibly on a particular concept, typically has a more general view of mathematics and its place in the schools.

No doubt, the clearest examples of the modernization motive can be seen in the “new math” movements throughout the 1950s and 1960s. Of course, one must speak of movements because there were many. On the European continent, the most prominent practical efforts were those of G. Papy in Belgium and André Lichnerowicz in France in the mid to late 1960s (Legrand, 1989). The spirit of the European movement was expressed by the mathematician Jean Dieudonné at the Organization for European Economic Cooperation seminar held at Royaumont in 1959 (see Griffiths & Howson, 1974, p. 141). Dieudonné put his case as follows:

My specific task today is to examine, from the point of view of the present curriculum in mathematics in universities and engineering schools:

a) What mathematical background professors in these institutions would like to find in the students at the end of their secondary school years.
b) What they actually get
c) How it would be possible to improve the existing situation . . .

In the last 50 years, mathematicians have been led to introduce not only new concepts but a new language, a language which grew empirically from the needs of mathematical research and whose ability to express mathematical statements concisely and precisely has repeatedly been tested and has won universal approval.

But until now the introduction of this new terminology has (at least in France) been steadfastly resisted by the secondary schools, which desperately cling to an obsolete and inadequate language. And so when a student enters the university, he will most probably never have heard such common mathematical words as set, mapping, group, vector space, etc. No wonder he is baffled and discouraged by his contact with higher mathematics. (quoted in Howson, Keitel, & Kilpatrick, 1981, p. 102)

Dieudonné was one of the moving forces of Bourbaki, and the axiomatic style of Bourbaki mathematics was the model for the European new math movement and, to a great extent, for the American movement as well. The words Dieudonné mentioned—set, mapping, group, vector space—were important for this movement not merely because they were modern but mostly because, for the university mathematician, these were the unifying concepts of modern mathematics (Kilpatrick, 1997).
The American movement began around 1950, also with an acute sense of the discontinuity between university and school mathematics. Some attempts were made at the end of the 1940s, notably by the College Entrance Examination Board (CEEB), to help university-bound students make a smoother transition from high school to university mathematics (Wooton, 1965, pp. 6–7). The CEEB program fell short of being a genuine reform, however, because it was directed only toward a small number of students and contained no general view of what mathematics education should be. By contrast, “The work that began in the 1950s,” as Wooton (1965, p. 7) put it, “was of a totally different nature.” The organization leading the way in the early 1950s was the University of Illinois Committee on School Mathematics (UICSM). The UICSM’s experimental textbooks and teacher manuals “presented the world with a startling new and bold conception of the way [the UICSM] felt mathematics should be presented to high school students” (Wooton, 1965, p. 7). The work begun by the UICSM was, of course, continued on an unprecedented scale by the School Mathematics Study Group (SMSG), founded in 1958.

The SMSG was led by Edward G. Begle of Yale University, and in general it was characterized by a close working relationship between university research mathematicians and school mathematics teachers (Pollak, 1985, p. 235; Wooton, 1965, pp. 17–18). It was an honest attempt to find some middle ground between university and school mathematics; nevertheless, the results seem to have favored the former. The pressure from the university science and mathematics community to bring the school programs up-to-date is hard to resist, especially when the requirements and the fruits of scientific research are so great. Indeed, even though the SMSG is no longer active, the modernization motive for reform that it represented is still very much alive. One still hears the call of university mathematicians to bring school programs in line with the demands of the university. For example, the Berkeley mathematician H. Wu, (1997, p. 950) reacting to the reforms proposed by the NCTM Standards, wrote, “The most obvious reason why school mathematics education should matter to university professors is that a continuing influx of mathematically incompetent students would decimate the university mathematics curriculum.”

**Societal Needs**

The needs of the university bring us to the second important motive for recent reform movements, namely, the needs and demands of society. The two are close, and perhaps really should not be distinguished at all. It is often thought, for example, that the new math movement was a direct reaction to the Soviet’s launching of Sputnik in the 1957. Strictly speaking, this cannot be true because the new math movement began long before 1957, as discussed earlier; however, it is undeniable that the movement would never have been quite so extensive or influential without Sputnik. Certainly, the SMSG owed much to all the national interests and concerns that were wrapped up in that small, round, beeping satellite. But even without Sputnik, the necessity to modernize was, for the founders of the SMSG, inseparable from needs of society. This is clear from the following statement in first SMSG newsletter:

The world of today demands more mathematical knowledge on the part of more people than the world of yesterday, and the world of tomorrow will make still greater demands. Our society leans more and more heavily on science and technology. The number of our citizens skilled in mathematics must be greatly increased; an understanding of the role of mathematics in our society is now a prerequisite for intelligent citizenship. Since no one can predict with certainty his future profession, much less foretell which mathematical skills will be required in the future by a given profession, it is important that mathematics be so taught that students will be able in later life to learn the new mathematical skills which the future will surely demand of many of them.
To achieve this objective in the teaching of school mathematics three things are required. First, we need an improved curriculum which will offer students not only the basic mathematical skills but also a deeper understanding of the basic concepts and structure of mathematics. (quoted in Wooton, 1965)

In the NCTM Standards (1989), the social motive for reform is explicit and immediate, political and economic. The Standards present two sets of goals as the basis for reform in school mathematics. These are, in order, the societal goals and the student goals. The societal goals, which are preserved and no less emphasized in the newer NCTM Principles and Standards (2000), comprise (a) “mathematically literate workers,” (b) “lifelong learning,” (c) “opportunity for all,” and (d) “an informed electorate” (NCTM, 1989, p. 3). All but the last concern the specifically economic interests of society. The first and second are directed toward what the NCTM sees as the nature of the modern workplace. Mathematical literacy is necessary, according to the Standards, because modern work demands the routine use of complex technology and the ability to assimilate new and complex ideas; “Traditional notions of basic mathematical competence have been outstripped by ever-higher expectations of the skills and knowledge of workers; new methods of productions demand a technologically competent work force” (NCTM, 1989, p. 3). Lifelong learning is necessary because of the rate at which job requirements change and because of the rate at which the modern worker is likely to change jobs. The third goal, “opportunity for all,” arises from the recognition that, on the one hand, “Mathematics has become a critical filter for employment and full participation in our society,” and that, on other hand, minorities and women are underrepresented “in careers using science and technology” (NCTM, 1989, p. 4). The kind of reform that the Standards envisions, therefore, would create a workforce that is productive, creative, flexible, and open to all.

Although obviously set in terms appropriate for the modern workplace, the goals expressed in the NCTM Standards and Principles and Standards are much in the spirit of past reforms governed by a motive of social utility. In the United States in the first years of the 20th century, Edward W. Stitt compared in a deliberate way what was taught in schools with what was actually used in the business world, and he was one of the first to do so. Others, such as F. M. McMurry, W. S. Monroe, and G. M. Wilson, also showed concern that there was too great a distance between school mathematics and real-life mathematics. From this concern, the social utility movement quickly became a movement that sought to eliminate subjects from the school program deemed to be useless in the business world, subjects such as cube roots and even fractions with large denominators! For this reason the movement was later called “reductionist” by G. T. Buswell (see Kilpatrick, 1992, pp. 17–18). Already in 1930, the absurdity of this literal approach to social utility was pointed out by Burdette R. Buckingham, who wrote: “Shall we say that 60 percent of the teaching of the schools in spelling and language should be devoted to the one hundred words of most frequent occurrence—to the, and, but, to, be, etc?” (quoted in Kilpatrick, 1992, p. 18).

To its credit, the Standards approach shows how it is possible for social utility to motivate a reform without the reform movement becoming a movement of “reductionism,” for rather than taking existing business practices as given and the measure of what must taught in schools, the Standards tries to define what skills are truly needed to fulfill the demands of business and of society generally. To a great degree, this entails defining what those demands are in the first place. The Standards approach thus not only avoids the absurdities of “reductionism” but also adopts a motive of social utility that is highly positive; it does not say only what ought to be eliminated, it also suggests what should be done.

Regardless of whether the Standards actually succeeds in defining societal needs, the attempt to do so is central to its project. Indeed, as discussed at the beginning of
this section, the motive of modernization also involved confronting the question what are society’s needs, and it could easily be argued that that question was as much a focus for the reform movements of the 1950s and 1960s as it has been for the more recent Standards movement. Therefore, these two central motives for reform in mathematics education show modern reform to be actively engaged in weighing mathematics education within a greater social context. The lack of this more comprehensive view of mathematics education, for example, is why the first attempt by the CEEB to smooth the transition between university mathematics and school mathematics in the late 1940s can be viewed as hinting at a reform movement but not itself the true beginning of a reform movement. Because these motives are such that the reformer looks outward beyond the immediate concerns of the classroom, they might be termed external motives. We must now look briefly at what should be termed internal motivation.

**Internal Motivation: Research in Mathematics Education**

The internal motivation for reform is that arising directly from thought about mathematics education itself. Thus, internal motivation for reform is that associated with research in mathematics education. Research here should be understood liberally as “disciplined inquiry into the teaching and learning of mathematics” (Kilpatrick, 1992; see also Cronbach & Suppes, 1969). To what extent has research in this broad sense been a motivation for reform? It is obvious that research has had a place in reform, as we discuss later, but has research given rise to reform? It is difficult to answer this unambiguously because it is difficult to disentangle research as a justification and a guide for a reform from research as a motivation. For example, many of the arguments for and against Euclid’s Elements (which was, indeed, a focus of reform movements in England from the beginning of the 20th century) in the English educational system (Carson & Rowlands, 2000) were informed by views of how students learn and what motivates students to learn. Nevertheless, it is far from clear that such views alone were behind the proposed reforms.

The case is not entirely clear even in the case of the NCTM Standards. There is no doubt, of course, that the Standards’ program for reform is closely allied with research in mathematics education, in particular, with one of its principal theories, the constructivist theory of mathematics learning (see Ernest, 1993). In describing curriculum Standards, the NCTM (1989, p. 6) states:

> When a set of curricular standards is specified for school mathematics, it should be understood that the standards are value judgments based on a broad, coherent vision of schooling derived from several factors: societal goals, student goals, research on teaching and learning, and professional experience.

Research on teaching and learning is prominent in the part of the Standards concerning student activities. The constructivist point of view is inherent in the two principles said to guide the descriptions of these activities, namely, that the activities “should grow out of problem situation” and that “learning occurs through active as well as passive involvement with mathematics” (p. 8); it is also inherent in an earlier claim that “knowing” mathematics is “doing” mathematics (p. 6). Eventually, however, the constructivist point of view is explicitly given as the basis for the way mathematics ought to be taught:

Research findings from psychology indicate that learning does not occur by passive absorption alone. Instead, in many situations individuals approach a new task with prior knowledge, assimilate new information, and construct their own meanings. For example, before young children are taught addition and subtraction, they can already solve most addition and subtraction problems using such routines as “counting on” and
“counting back”. As instruction proceeds, children often continue to use these routines in spite of being taught more formal problem-solving procedures. They will accept new ideas only when their old ideas do not work or are inefficient. Furthermore, ideas are not isolated in memory but are organized and associated with the natural language that one uses and the situations one has encountered in the past. This constructive, active view of the learning process must be reflected in the way much of mathematics is taught. (NCTM, 1989, p. 9)

The importance of constructivism in the Standards notwithstanding, one must be careful not to overstate the case. Indeed, this can happen equally among those who do not fully understand the research on which the Standards depend and those who do understand and zealously defend it. M. T. Battista (1999), for instance, recognized the place of constructivism in the Standards, but thought they do not go far enough in adopting the theory. This, in his opinion, is not entirely by design but merely because not all known about constructivism today was known at the inception of the Standards. In his words,

At this time, I know of no commercially available mathematics curricula that are systematically and completely based on scientific constructivism. Even NCTM’s Curriculum and Evaluation Standards for School Mathematics is not completely consistent with scientific constructivism, embracing its general tenets but ignoring many of its particulars. (This should not be surprising, since the standards were developed before many of the details of the theory had been worked out.)

Nevertheless, the curricula that come closest to implementing scientific constructivism are those that were developed, with support from the National Science Foundation, specifically to implement the NCTM Standards. (Battista, 1999, p. 13)

At times, Battista put his case in a way that almost identifies the constructivist theories that inform the NCTM program for reform with the reform itself. For example, where he (rightly) attacked school districts for adopting the recommendations of the Standards in a merely superficial way, he said, “their implementations often distort the tenets of reform so greatly and are so far removed from the scientific research on mathematics learning that the efforts cannot truly be considered reform mathematics at all [emphasis added]” (Battista, 1999, p. 13). The reforms outlined in the Standards, however, have as much if not more, to do, with the societal needs discussed above as they do with research.

The main reason it is easier to find the motivation of reform in societal needs or in the necessity of modernization than in research is evident from the NCTM’s own definition of standards as value judgments. Motivation is a function of values or, equivalently, of ends—and these scientific research cannot supply. As Hiebert (2000) put it, “Research can be a powerful tool for making informed decisions in mathematics education, but it can never answer questions that have more to do with values and priorities than with the likelihood of effects” (p. 436). Kilpatrick (1992), in speaking about reform and the school mathematics curriculum, also made a point of saying that “curriculum questions are ultimately questions of purpose and value; they concern what ought to be rather than what is,“ and further remarked that “many researchers continued to believe, however, that scientific research could resolve these questions” (p. 22). In fact, there is a well-known fallacy in the philosophy of ethics that concerns just this kind of situation. It is called the naturalistic fallacy and it is precisely one in which normative statements are identified with statements of fact, such as those coming from the natural sciences. G. E. Moore (1903) first discussed the fallacy in objecting to Herbert Spencer’s use of “more evolved” to mean “better” or “higher” and his identifying “pleasure” with “goodness” (Fieser, 1993). In education, the fallacy is made if we assert that because children learn in a certain way, for example, by their actively constructing knowledge within themselves, they should be taught in a certain way.
The naturalistic fallacy, therefore, makes research as a motivation for reform, in principle, always somewhat problematic. That said, when deliberating about the values and ends of reform, researchers cannot remain completely aloof, but what their place should be rests on how we understand research into reform; this we consider later.

The question of motivation is crucial when looking at reform because it contains, as we have discussed, the values and ends of the whole reform program. The observation to be stressed is that even when the motivations ostensibly refer only to mathematics, they in fact refer to a larger context—the nature of mathematics, the place in mathematics in schools and universities, and the relationship between school mathematics and societal needs. These are all broad issues and demand a broad view. We now turn to the people involved in reform. Although on the face of it, this brings us to the more concrete side of reform, the wide issues that characterize the motivation of reform are no less present in this discussion.

**Initiators and Movers**

The questions in this section are Who urges a reform? and Who takes charge of a reform? That is, who actually sets out its goals, writes the classroom material, structures the implementation? There are several possible populations to which one can point—academicians (including both educational researchers and mathematicians), teachers in the field, politicians, parents, even students, but one group or another is hardly ever the exclusive initiator or mover of a reform movement. For this reason, writers such as Griffiths and Howson (1979) preferred to speak of projects being “teacher” dominated or “university” dominated. Moreover, it can happen, and usually does happen, that different groups are dominant to different degrees at different times and in different areas.

**University-Dominated and Teacher-Dominated New Math.** The “new math” movement is generally considered a mathematician-dominated reform. The initiators certainly came from the university community, whose dissatisfaction with the preparedness of high school students for advanced work gave the reform its initial urgency. It is fair to say, moreover, that as the movement got under way, the lead of university mathematicians remained strong. This was not surprising, given the overall goal of providing a bridge, more or less one way, from school mathematics to university mathematics. A figure such as Dieudonné, who had much to do with the character of modern mathematics, particularly modern French mathematics, could be expected to have great authority in deciding what mathematics was important to learn and for what mathematics French schools should be preparing their students.

The heavy university influence in the American “new math” movement can be seen in the representation of universities and secondary schools on the various advisory committees associated with the movement. For example, on one of the early commissions set up by the CEEB in 1955 to study the “mathematics needs of today’s American youth,” only 5 of the 14 members were associated with high schools (the list of members of this and other commissions and committees are given in Wooton, 1965). In truth, it was precisely this involvement of university mathematics that was the distinguishing mark of the “new math” movement and its point of pride. Referring to the appointment by the president of the American Mathematical Society (AMS) of a committee of eight leading mathematicians to aid the process of school reform, Wooton (1965, pp. 12–13) said:

It is difficult to overstate the importance of this move on the part of the Council and the President of the Society [AMS]. For more than thirty years, the AMS had held itself aloof from the elementary and secondary school level of mathematics and had contributed...
very little to the teaching of it. With the appointment of the Committee of Eight, it
officially expressed an interest in the mathematics curriculum of the schools, and the
approval of the Society made it possible for a large number of distinguished college
teachers and research mathematicians to enter wholeheartedly into cooperation with
high school teachers in a concerted effort to improve the quality and presentation
of school mathematics. . . . The act of appointment gave clear indication that the long
estrangement between research mathematicians and teachers of mathematics had been
breached.

on the other hand, it would be unfair to say that the concerns of teachers in the
field were not at all represented in the U.S. movement. As mentioned above, the
SMSG under Edward Begle’s leadership actively encouraged the cooperation of school
teachers and university mathematicians. In the first writing session in 1958, pains
were taken not only to have equal representation of school and university people on
the actual writing teams but also to have much informal contact between the two
(Wooton, 1965, pp. 17ff.). The influence of the teachers from the field has been noted
by the mathematician Henry Pollak, who was actively involved in the SMSG:

One of the people at that meeting [the initial writing session in 1958] who had a great
influence on all of us was Martha Hildebrandt. She was head of the department at
Proviso High School at Maywood, Illinois, and had been President of NCTM. The first
thing she said to me was: “Now remember, Pollak, you can’t teach anything after April.”
She was letting me in on one of the truths of what actually happens in the classroom.
Spring fever comes, and the kids just don’t listen anymore. I learned a tremendous
amount from her. I found, as did many other mathematicians who came to that meeting,
that the questions of what to teach, how to teach it, in what order, how to say it, and how
to combine skills and understanding were tremendously interesting, difficult issues.
(Pollak, 1985, p. 235)

One of the main British projects in the new math reform period was the School
Mathematics Project (SMP). The SMP was founded a little after the SMSG and was
influenced by the SMSG; it was also centered at a university and was advised by
university mathematicians, yet by contrast with the American project, it was from the
start a teacher-dominated project: “its activities were directed, and its texts written, by
practising schoolteachers” (Griffiths & Howson, 1974, p. 142). The first SMP textbooks
were characterized by generous verbal explanations and emphasis on application
(see Howson et al., 1981, pp. 173–176), and this was probably a direct result of teachers,
rather than university professors, being the movers of the project. It should be said,
however, that the emphasis on concrete applications, although this may well have
had its source in teachers’ classroom experience, more likely stemmed from a general
British penchant in this direction, and, more important, from the funding of the project,
which came largely from industry.

**Funding and Government Involvement.** The question of funding is important
in the present discussion, for funding is rarely blind and those who are close to the
sources of funding are therefore in a very real sense also initiators and movers of
reform. The success of a given project in a reform movement is often dependent
on its director’s ability to negotiate between these and the more immediate movers
of the reform. As Howson et al. (1981, p. 80) said, “Getting and keeping financial
support for a project is a nontrivial part of the director’s role, and occasionally this and
public relations work have occupied most of the director’s time.” This is particularly
true where the funding is public. Indeed, although funding can come from private
foundations, such as the Nuffield Foundation, which supported the British Nuffield
Mathematics Project, or the Holzer and Sloan Foundations, which helped support the
Madison project in the United States, no doubt the most important funding comes
from public sources. This immediately introduces the question of government bodies and politicians as movers of reform. It is here that negotiating between these more distant and the more immediate movers of reform becomes ever present and often full of tension.

Federal support of the SMSG via the National Science Foundation (NSF), acting in accordance with the U.S. Congress' National Defense Education Act in 1958, represented, perhaps, government funding at its best. First of all, it was generous, reaching $4 million by 1961 (Wooton, 1965, p. 107). And second, government interests were more or less in line with those of the active members of the project. The successful working arrangement between reformers and government in the late 1950s and early 1960s apparently led educators such as Francis Chase, who referred explicitly to the NSF funding patterns, to be sanguine about the general desirability of government intervention:

It is time to take leave of our irrational fears of federal control and to recognize that when the people have set for themselves goals of national scope, they may choose the means appropriate thereto, including the level of government or the combination of government levels that can contribute best to accomplishment of the desired ends. (Chase, 1964, p. 138)

Chase was not completely naive; he was well aware both of the importance of other sources of funding and of the objections specifically to government support. His positive view of government intervention, on the other hand, was not completely baseless. The scale of funding needed to support reform, if the reform is to be effective, is the kind that governments command. Moreover, the fact that societal needs are a major motivation for reform seems to make government, whose main interest, in theory, is general good of the people, the natural source of support.

The problem, of course, with government funding is that it all too often becomes mixed with crass political interests. Philip M. Smith, for example, refers to the phenomenon of "earmarking" in the funding of scientific research. According to Smith (2000, p. 37) this is:

the contemporary university practice of appealing directly to Congress for funding for research facilities—and, in so doing, bypassing all the normal review procedures of the executive-branch agencies and congressional committees. An earmark—the colorful term comes from the farmer's method of indicating livestock ownership—is an appropriation of funds for a specific purpose, almost always to the benefit of an institution within a congressperson's home district. (The funds themselves get an equally colorful label, also drawn from the language of animal husbandry: "academic pork.")

Other Movers: Reform as a Cooperative Effort. The three movers of reform we have discussed—mathematicians, teachers, and politicians—have clearly been the principal ones during the last 50 years. Researchers in mathematics education, of course, have had a important role in the most recent movements for reform such as that represented by the NCTM Standards. Although it must be said that this is largely a consequence only of the fact that there is a more distinct and coherent community of mathematics education researchers now than there was 50 years ago. Indeed, the question of whether mathematics education researchers were or were not present in older reform efforts is, to some degree, a matter of definition. Someone like Godfrey at the beginning of the century was in his own time a mathematician deeply interested in educational issues; today, he would no doubt be in a department of mathematics education!

Besides mathematicians, educational researchers, teachers, and politicians, there is also room for parents and students as active movers of reform. C. G. Hass (1964) stressed this with regard to curriculum planning in general. Referring to students, he
said they are “the major untapped resource in curriculum planning. Students are in the best position to explain many of the advantages and deficiencies of the present curriculum” (Hass, 1964, p. 146). Hass also argued convincingly that sociologists and anthropologists ought to participate in the shaping of curriculum, and, by implication, in movements for reform. He emphasizes, as we emphasize later, that reform is a cooperative effort.

To some extent, Hass’s view that participation in bringing about a reform (in his case, specifically, the planning of a curriculum) should be broad and cooperative is realized in the wide range of groups that, one way or another, have been associated with the NCTM Standards. The Standards divides these groups into three categories: endorsers, supporters, and allies. The endorsers comprise 15 organizations, mostly mathematical and scientific—organizations such as the American Mathematical Society (AMS) and the Mathematical Association of America (MAA; both also were associated with the earlier “new math” reforms), but the endorsers also include an organization such as the Institute of Management Sciences. The supporters comprise 25 organizations and represent interests that are not immediately connected with mathematics or mathematics teaching, including such organizations as the American Federation of Teachers, the International Reading Association, the National Council of Social Studies, the National Council of Parents and Teachers, and the National School Boards Association. The allies, comprising 20 organizations, represent quite diverse interests; included are the American Association of Retired Persons, the Children’s Television Workshop, Consumers Union, Junior Achievement, the National Federation of Business and Professional Women’s Clubs, to name a few.

We have treated the motivations of reform and the movers of reform at some length because these more than anything else are directed toward the foundations of reform. What we have seen is that these foundations inevitably reach far beyond the walls of the classroom. In considering why a reform is desirable one looks at the needs and nature of mathematics, the needs and values of society, and the relationship between them; in considering who brings a reform about, one discovers that a balanced participation of mathematicians, teachers, politicians, parents, and students is necessary and that all must somehow work together and come to understand their different motivations and ways of thinking.

**Implementation: Implementation as Communication**

Naturally, the discussion of the movers of reform quickly leads to the subject of the implementation of reforms; in fact, it is not always clear where one ends and the other begins. The difference, at a certain level, has only to do with size: Initiating and moving a reform may involve particular mathematicians and teachers, particular politicians and parents, but implementation involves all teachers and parents and students whom the reform touches. Viewed somewhat abstractly, the implementation of a reform is a matter of communication between initiators and movers and this greater population.

Such communication can be indirect or direct. The production of textbooks and other learning materials is an example of indirect communication in the sense that the writer of the material is not immediately present to the teachers or students using it. Inservice training, especially when it is long term and set in teachers centers dedicated to the reform program, provides a context in which communication can occur in its fullest sense and most direct form.

In neither case is communication necessarily unidirectional from the top down. Teachers and students who use learning material are usually encouraged to offer a weighed opinion as to the material’s worth and suggestions toward its improvement. Because it is in the nature of textbooks to be presented as if they were the final word, however, no real dialogue between writer and user can ensue from the readers’
reactions. Inservice activities, by contrast, are far more able and likely to become a genuine dialogue in which those directly involved in the reform are as much listeners as they are talkers.

The Kidumatica project in Israel is a good example of this kind of inservice activity. The project was established at a time of reform of science and mathematics education in Israel, and it was a direct response to the critical 1992 Harary report on science and technology education. Among other things, the Harary report, calls for the deepening of students’ and teachers’ mathematical knowledge and for the increased use of technology in the classroom. The emphasis on technology throughout the entire report cannot be taken as a separate issue from that of deepening mathematical knowledge. This is because it is never enough just to use technology; technology must be used thoughtfully (Roitman, 1997), and this means having more than a superficial understanding both of the mathematics behind the technology and of the mathematics to which one is applying the technology. Accordingly, from its inception in 1995, the Kidumatica project has aimed to cover and integrate all aspects of mathematics teaching in the secondary schools—not only educational applications of technology, but also pure mathematics, history and philosophy of mathematics, and didactic issues connected with the teachers’ own classroom experience. Such a wide scope is unusual for an inservice course, but the Kidumatica project is also distinct in three other ways more pertinent to the present discussion.

First, the program is set up so that a given group of about 20 to 25 teachers will participate in daylong, bimonthly workshops over the course of 3 years. Accordingly, the program allows for extensive contact with project staff, the importance of which has been noted by Griffiths and Howson (1979, p. 149). Second, the staff of teacher–tutors is permanent, and, although the teacher–tutors all possess advanced degrees in mathematics or mathematics education, they are also themselves school mathematics teachers or have been school mathematics teachers. Third, and most important, although each teacher–tutor is responsible for a specific range of subjects, all are present and participate in every workshop. Together, these aspects allow the teachers and the teacher–tutors to develop a particularly close relationship and create a sense of cooperation not often found in other inservice programs. This high degree of interaction results in great mutual influence, and, for the staff, provides continual feedback (see Amit, Louzun, Fried, Kapulnick, Weitsman, Satianov, Zelster, Ceasushu, & Perry, 1999). The hope of the project is that each group of teachers participating in the 3-year course will become a source of change in their own schools.

The Kidumatica program illustrates how implementation can be a process of communication, not a mere act of communication. An act of communication is something like a decree: One simply tells teachers what they are to teach, what texts they are to use, what theory of learning they are to believe—a formula for failure. A process of communication, on the other hand, involves give and take, persuading and being persuaded, talking and listening. The fact that the Kidumatica project was set up so that teachers and the teacher–tutors could influence one another, and that they have truly done so is a sign of its success in sustaining a genuine process of communication. The point is that the implementation of a reform is not just its being carried out, as if by a decree, but also its being continually refined and perhaps even redefined.

Assessment of Reform

The last remark in the previous section brings us to the final component of reform: assessment. The result of assessment must be more than a simple affirmation of success or failure; it must also aim to improve and clarify the whole reform effort. In this respect, assessment must be viewed as an ongoing activity deeply linked to implementation, rather than an activity that only follows implementation.
Recalling the SMP in Britain where “the schoolteachers involved remained in their schools, so that they could try out the new materials as they were written” (Griffiths & Howson, 1979, p. 142), one might say that a “teacher-dominated” reform, such as the SMP or the Scottish Fife Mathematics Project in the 1970s (see Howson et al., 1981, pp. 36–38, 217–220), would present the best opportunity for true assessment of the reform because it would indeed be ongoing, “formative” (to use Scriven’s [1967] term), and clearly in tune with the values and ends of the reform. It is unclear, however, whether a great enough number of teachers could possess a sufficient background in theoretical educational issues or in assessment techniques to take full advantage of their unique position. In fact, this is the same weakness from which “teacher-dominated” curriculum reform suffers in general: As Griffiths and Howson (1979, p. 143) said,

“Teacher-dominated” projects automatically prepare more practicable materials, although a striving for academic respectability often results in their setting themselves over-ambitious targets; but teachers are apt to lack the overall view of the subject which a university don possesses, and are unlikely to be as aware of modern developments. Thus undue emphasis is sometimes placed on interesting but relatively unimportant facets of the subject.

In this respect, the teacher–tutors in the Kidumatica program are perhaps better equipped to assess the progress of the reform in their work with teachers and their own work as teachers in the classroom.

Assessment taking place continuously and by teachers in the classroom or by teacher–tutors in long-term inservice programs achieves an intimacy that seems necessary to form a true judgment of what or whom one is trying to assess. Indeed, as Wiggins (1993, p. 14) reminded us, “Assess is a form of the Latin verb assidere, to ‘sit with.’ In an assessment, one ‘sits with’ the learner. It is something we do with and for the student, not something we do to the students.” To assess a reform is to assess something of a very large scale that affects a greater number of students, teachers, and parents than one could ever hope to “sit with,” however. It may also appear that a more standardized, less intimate form of assessment is necessary for the sake of “objectivity,” although this position, as we shall see, has its own serious difficulties.

Both concerns were answered in a grand way by one of the most ambitious assessment efforts ever carried out—The National Longitudinal Study of Mathematical Abilities (NLSMA) set up by the American SMSG program in 1961. This 5-year study reached 112,000 students from 1,500 schools and 40 states. Starting in 1962, three populations of students X, Y, and Z, were tested each fall and spring. Over the course of the 5 years, students in the X population were tested in the 4th through 8th grades; students in the Y population were tested in the 7th through 11th grades; students in the Z population were tested in the 10th through 12th grades and were given questionnaires after graduation. In this way, a given group of students could be studied at different times, a given grade at different times, and different grades at a given time (see Wilson, Cahen, & Begle, 1968). The view of mathematical achievement adopted by the NLSMA was embodied by a matrix model in which “categories of mathematics content”—number systems, geometry, algebra—were paired with “levels of behavior”—computation, comprehension, application, analysis. In addition, such factors as attitude toward mathematics, anxiety, motivation, and self-concept were also tested, and information regarding teachers, schools, and communities was gathered (Howson et al., 1981, p. 192). The NLSMA, in short, tried to catch the effects of the “new math” movement with as wide a net as possible.

Howson et al. (1981) were critical of the NLSMA effort. They pointed out that “NLSMA could have been planned and executed with greater care and certainly at less cost.” Moreover, they wrote,
Some of the most important lessons NLSMA taught were that one cannot correct for problems of planning and design by using a large sample and elaborate statistical techniques, that no study is large enough to answer more than a limited number of questions, and that the biggest temptation in a longitudinal study is to spend so much of one’s resources gathering and organising the data that one has no energy left to analyse it. Begle’s second law about mathematics education applies especially to NLSMA:

Mathematics education is much more complicated than you expected even though you expected it to be more complicated than you expected. (pp. 194–195)

As an assessment of the “new math” projects, the strategy of the NLSMA in very general terms seems to have been this: formulate criteria for student achievement; test students who have been exposed to “new math” materials and teaching; assess whether students in fact are achieving according to the aforesaid criteria. This appears sensible enough. The usual objects of assessment are student performance, knowledge, and understanding, and in assessing reform these are also relevant objects—indeed, ultimately, these are the relevant objects. But there is a subtlety here. Although the assessment of reform rests on student assessment, it is not identical with it. A reform movement has its own ends and values and may have its own conception of what student performance, knowledge, and understanding means; it must be assessed within the framework its own values and view of its subject. Thus, it is not enough to cast a wide net to assess a reform, even if it is a very good net; it must be the right net and its contents, as Howson et al. (1981) suggested, must be appropriately analyzed.

The point is important. Usiskin (1999–2000, pp. 5–6) pointed out that the “new math” movement was judged a failure by a public misled by the results of large-scale standardized examinations, namely, those of the SAT and the National Assessment of Educational Progress (NAEP). The results apparently were judged according to a rather naïve logic: The SAT is a well and thoughtfully constructed examination of student achievement; SAT scores were low in the “new math” period; therefore, the “new math” reform was a failure. Of course, the conclusion does not follow: Reasons having nothing to do with the teaching of mathematics may well have accounted for the lower scores. In fact, in the same years, the verbal scores dropped even lower than the mathematical scores. So why should the public attend to such simplistic logic? To say it was misled by the opponents of reform only begs the question, for one must then ask what made the opponents’ arguments persuasive. We tend to think that the answer has to do with the way standardized tests present themselves: They give the impression of being completely objective and, more important, universally applicable. In other words, it does not matter whether or not the SATs were designed to test the progress of the “new math,” it could test any reform. The truth is, without considering the values and goals of the “new math,” one cannot assume that the SAT is right measure for the reform’s assessment.

In contrast with the SAT scores, Usiskin remarked that “By the early 1970s, we were producing more students majoring in mathematics and majoring in science than ever before” (p. 5), and this was indeed one of the goals of the “new math.” On the other hand, for the same reason, one might argue that the fact that the “new math” did not help slower students should not be called a “true sign of failure” (Usiskin, 1999–2000, p. 5) because helping slower students was never an explicit goal of the “new math.” Yet such an argument is not entirely satisfying because if helping slower students were not a goal of the “new math,” then this is a point on which the “new math” ought to have been criticized. This is crucial because it shows that, in assessing a reform, not only does student achievement need to be examined in a way that reflects the movements’ values and goals, but the values and goals themselves must be examined. Clearly, a statistical study is quite inadequate in this regard; it might tell that teachers or students value one thing or another, or work in a certain way, but it
cannot determine whether it is right to value such things or right to work in the way they do.

Another form in which a reform movement can be assessed is what Howson et al. (1981, p. 195) called an “official biography.” The most well-known examples, although they were not necessarily intended to be rigorous assessments (as Wooton (1965, p. vii), at least, made a point of saying), were certainly Wooton’s account of the SMSG, and Thwaites’ (1972) account of the SMP. In principle, however, there is no reason why such “official biographies” could not be rigorous assessments in the way that the biography of an historical figure can also be a work of rigorous scholarship. We all too often look to the exact sciences (especially in their measurement-oriented, 19th-century guise) as our only model of rigor (see Kilpatrick, 1992, pp. 30–31); humanistic fields such as history also have their own brand of rigor. Indeed, it may well be that a written quasi-historical account is the best way to take stock of the reform movement’s accomplishments and reflect on the rightness of the goals it set for itself and the values it assumed. An account such as this can be factual and, at the same time, reflective in a way tomes of statistics can never be. The problem with this kind of “official biography” is that it does not fulfill the requirement of being ongoing assessment, but perhaps it can solve this by being periodic—assessment and reassessment; perhaps, it can take the form of a rigorous journal. Admittedly, though, such possibilities have never actually been executed and need to be more fully explored.

Assessing a reform means looking at the reform as a whole. But what is the whole of reform? Some characterization of this was the purpose of this section of our chapter. Without committing ourselves to a precise definition, which, as we said in the introduction, may not be possible, we end this part with a tentative general statement of what reform and reform movements are about.

Reform movements, whether by design or necessity, take in almost all aspects of mathematics education; they take in content, and thus curriculum development; means, and thus teaching methods, learning environments; societal needs; and, connected with all of these, values. What distinguishes a reform from other attempts at improving education is the necessity of seeing all these aspects as an integrated whole. Thus, we can say that the reform process aims at a systemic improvement of education, that is, it takes as its object not curriculum or teaching style or learning environments individually, but an entire matrix combining content, means, social needs, and values with the populations of students, teachers researchers, parents, and politicians. To make a biological analogy, the reform process is directed toward the ecology of education (see Goodlad, 1987, passim). The ecological viewpoint of the living world takes into account physiology, genetics, chemistry, and so forth insofar as these contribute to making the living world a single complex system; the reform process is similarly directed toward a single complex system.

With the discussion of assessment, we have in effect begun to skirt the edge of the second question of our discussion: What is research into reform? To this we now give our full attention.

RESEARCH INTO REFORM

With the possible exception of “curriculum development,” most examples of mathematics education research—mathematics teaching practices, theories of learning, classroom cultures, and so on—are crucial components of reform but are not directed towards reform per se. Needless to say, to date there is no explicit subfield of research in mathematics education called “reform research.” The main purpose of this section, then, is to give a sketch of what such research ought to be.
Research as a Component and Result of Reform

To begin, let us add a few words to what has already been said about research and reform. To the extent that mathematics education is an applied field, its ultimate end must be the improvement of mathematics education. In this sense, almost all mathematics education research could be connected with the reform of mathematics education. Certainly this is true of efforts directed toward curriculum design and development, learning materials, and learning environments. Moreover, as the crucial place of constructivist theories in the NCTM Standards shows, the importance of basic research to reform is also not to be doubted. So, although basic research cannot determine the values toward which a reform strives, as discussed above, it is plain that reform would be empty without research, and in the more recent reforms, at least, the influence of research is been weighty indeed. The curious fact is that the influence has tended to be mutual, that is, not only has research shaped reform, but reform has also shaped research.

This is clear when one considers that the period in which a genuine community of researchers in mathematics education began to crystallize was precisely the period of major reform in the 1950s and 1960s (see Kilpatrick, 1992, pp. 23–29). And this was true not only of the research community in the United States, but also in Europe. One need only look at the development of the Commission Internationale pour l’Étude et Amélioration de l’Enseignement des Mathématiques (CIEAEM), which, beginning in 1950, simultaneously became a nucleus of reform, as its name promised, and, under the leadership of Gattengo, an active forum for research aiming to bring together “epistemologists, logicians, psychologists, mathematicians, pedagogues…” (Félix, 1998, p. 18).

A telling example of how reform affects research was the NLSMA. Indeed, the NLSMA, for all its faults and despite its questionable value as an instrument of assessment for the “new math,” was truly a consequence of the “new math” reform and did have a tremendous influence on the course of educational research. About this, Kilpatrick (1992, p. 29) said:

[The NLSMA] brought together psychologists and mathematicians to develop instruments for assessing reasoning ability, the ability to apply mathematics in nonroutine contexts, and attitudes toward various facets of mathematics—instruments that were used in both subsidiary and ensuing research studies. It developed and refined new techniques for analyzing the multiple effects of complex treatments on nonrandomly chosen groups. And it trained quite a few researchers in mathematics education. In this connection, it is worth noting that some of the basic themes developed later by the NLSMA were already beginning to take form in 1959, only a year after SMSG itself was founded. On a recommendation from the Advisory Committee, Begle set up an ad hoc committee consisting of three distinguished psychologists who “outline a plan for some needed research [regarding psychological problems relevant to the teaching of mathematics]. SMSG was particularly interested in evaluating the effectiveness of its new mathematics programs in relation to children’s attitudes, motives, anxieties, and skills” (Wooton, 1965, p. 56–57, 95–96). The example of the NLSMA is important because it shows that the growth and development of a research community during the reform years of the 1950s and 1960s was not by chance, but actively encouraged as a natural part of the reform process.

Such encouragement, of course, is equally characteristic of the modern Standards era, as can be seen in this statement by Romberg and Collins (2000):

Propelled by the need to educate all of America’s students to levels of achievement in mathematics and science not thought possible over a decade ago, the challenge of the
reform movement is to create classrooms where all students have the opportunity to understand mathematical and scientific ideas. This assertion is based on the belief that there is a direct and powerful relationship between student understanding and student achievement. In fact, the way to achieve the high expectations that we have for all students rests on their understanding of important mathematical and scientific ideas taught in school classrooms by professional teachers. This challenge needs to be addressed directly by researchers so that real change in the teaching and learning of mathematics and science occurs in the next decade. A sustained research program, conducted collaboratively with school personnel in school classrooms, needs to be carried out. (pp. 82–82)

This statement shows clearly how reform sets an agenda for action, informed by research as well as the social importance of mathematics education, and in doing so simultaneously sets an agenda for research itself, thus illustrating and reinforcing the idea that the reform movement and the research community mutually influence and shape one another. We need to consider now reform research, its characteristics, and its place in the general community of researchers and educators.

“Reform Researchers”

In the brief description just given, we looked at the crystallization of a community of researchers in times of reform. It is one of the laudable aspects of the research community that it is, truly, a community; it is not so differentiated that researchers in the different branches of mathematics education research are unable to find common ground and a common language. Reform researchers must be members of this same community; however, their relationship to it is unique because of the nature of the inquiry to which they are dedicated.

At the end of the first section, we used the ecology analogy to emphasize that reform is concerned with the change of a whole complex system. What have just seen is that the research community is an essential part of that system, defining it and being defined by it. Thus, just as the human beings who study ecological systems are simultaneously part of such systems, researchers of reform are, in the same way, part of the object they study. This also can be said, to some degree, about other branches of research in mathematics education, but there is a difference. In most kinds of educational research, the effect of researchers must be taken into account only because the ideal situation in which researchers have no effect on their subject—in which they are unseen, unheard, and unfelt observers—can never be fully realized. In reform research, by contrast, the researcher is an essential part of the object of research. Moreover, because researchers are no less part of the general research community than any other kinds of researchers, they are as much defining and being defined by reform, the very thing they are studying. These considerations have immediate implications for the question of methodology.

Methodology

Traditional research methodology aims to keep researchers apart from the phenomena being studied, both with respect to the researchers’ physical presence and to their values and particular worldview; the methodology of reform research, by contrast, must conceive the researcher as taking part in the process of reform and possessing definite values and goals. In some sense, the way reform researchers are themselves entangled in their subject is parallel to the position of the sociologist or historian, and, indeed, the study of reform is in some degree the study of a historical process. Interestingly enough, historiography, like educational research, at one time aimed to be “simply a science, no less and no more” (Bury, 1956, p. 223), and, like educational
research, eventually came to accept, for the most part, a view such as this of the historian, E. H. Carr (1967):

Human beings are not only the most complex and variable of natural entities, but they have to be studied by other human beings, not be independent observers of another species.... The sociologist, the economist, or the historian needs to penetrate into forms of human behaviour in which the will is active, to ascertain why the human beings who are the object of his study will to act as they did. This sets up a relation, which is peculiar to history and the social sciences, between the observer and what is observed. The point of view of the historian enters irrevocable into every observation which he makes; history is shot through and through with relativity. (p. 70)

The lesson of this is that reform research must give an account of a process from an insider’s point of view; therefore, it must, like the historian’s account, be reflective and even introspective. The methodology that flows from this alone is something similar to what we described at the end of the section on assessment, namely, the “official biography” or the “rigorous journal.” But reform researchers are not only insiders, they are active insiders; an account is not enough, rather, it must be the sort that brings researchers and the other players in the reform process into the circle of defining and being defined by the reform. The methodology of reform research must therefore answer both the need to be reflective and the need to be active.

There are some existing models for this kind of research. One that comes quickly to mind is action research or participatory research. Carr and Kemmis (1986) described action research as follows:

Action research is simply a form of self-reflective enquiry undertaken by participants in social situations in order to improve the rationality and justice of their own practices, their understanding of these practices, and the situations in which the practices are carried out. (p. 162)

Even from this short description, it is plain that action research contains many of the elements that we have described as desiderata for reform research: It is reflexive, that is, there is no clear division between the researcher and the researched; it considers ends and values; it is directed toward action (see Dick, 1999; Elliott, 1978; Russell, n.d.). Moreover, action research also employs such means as journal writing, which, as we have argued, is in one form or another an appropriate methodological mode for reform research. As Elliott (1978) described, “In explaining ‘what is going on’ action-research tells a ‘story’ about the event by relating it to a context of mutually interdependent contingencies, i.e., events which ‘hang together’ because they depend on each other for their occurrence” (p. 356).

An important aspect of action research is its cyclical or iterative character. It alternates between action and reflection. Moreover, cycles are contained in cycles, “Larger cycles span whole phases of a research program” (Dick, 1999, p. 3). These wider cycles continually broaden and deepen the teachers’ and all other participants’ understanding and can also broaden the range of participation, including “professional” researchers whose own field of action goes beyond the individual classroom. This is important, for it answers two apparent weaknesses of action research as a model for reform research methodology, namely, that the participants may not possess the appropriate theoretical background to reflect deeply enough on their practice and that action research seems to be local in nature, a single classroom, a single school, whereas reform is broad in scope and must see the classroom in a very broad context. The fact that there is cycling at all means that the “theoretical background” which the “professional” researcher supposedly possesses is not a fixed body of knowledge, but knowledge continually being reviewed and revised in the cycling process and that the “professional” researcher and teacher–researcher alike therefore must
view themselves as learners and partners in learning. Moreover, this partnership in learning, which ideally should be ever expanding (see Dick, 1999, p. 4), affords the researcher the broad view appropriate to the phenomenon of reform.

The approaches for reform research methodology given here are consistent with general contemporary trends in mathematics education. For example, in describing “Some Shifts in Emphasis in Educational Research in Mathematics and Science,” Kelly and Lesh (2000, p. 37, table 2.1) noted, among other shifts, less emphasis on “researcher remoteness or stances of ‘objectivity,’” and more emphasis on “researcher engagement, participant-observer roles”; less emphasis on “researcher as expert: the judge of the effectiveness of knowledge transmission using prescribed measures,” and more emphasis on “researcher as coconstructor of knowledge: a learner-listener who values the perspective of the researcher subject, who practices self-reflexivity”; less emphasis on “simple cause-and-effect or correlational models,” and more emphasis on “complexity theory; systems thinking; organic and evolutionary models of learning and system change.” Underlying these trends is the general understanding that research must be characterized by a diversity of methodologies. It is evident, thus, that if general educational research is marked by a diversity of methodologies, the same must be true, a fortiori for reform research the complex object of which demands many views to arrive at some sense of the whole. Thus, different methodologies such as those contained in the historiographic and action research models, presented earlier, must be brought together as complementary parts of a whole reform research effort.

Some Questions for Reform Research

**Method.** What kind of questions should occupy the reform researcher? First is the very question we have just discussed, the question of methodology. But this, in fact, is already built into the action research approach, because in that approach, the continual review and modification of methodology is part and parcel of the cycling characteristic of action research. Indeed, without this continual preoccupation with the question of methodology, the researcher, instead of cycling, is likely just to go around in circles.

**Communication.** A second question has much to do with the function of researchers in the reform process. In the student–teacher–researcher–parent–politician complex, the researcher has a central role both in building of an intellectual foundation for the reform and in communicating the principles of the reform to all those affected by it. We touched on this issue of communication in the section above on the implementation of reform. Both Usiskin (1999–2000) and Battista (1999) made clear to what extent a lack of communication between researchers and the public and between researchers and teachers can threaten the success of reform. Battista, in particular, implied that this lack of communication is more than the absence of communication, but almost active noncommunication—mostly on the part of the public toward the research community: “Too often the educational programs and methods used in schools are formulated—by practitioners, administrators, laypeople, politicians, and professors of education—with a total disregard for scientific research” (Battista, 1999, p. 10). One must ask, however, whether the research community itself has done enough to foster an atmosphere of communication.

Whatever the case, it is clear that the problem of communication is critical and requires understanding to solve. For reform researchers, therefore, this problem of communication must be a central concern, especially in one of its special forms—the problem of theory and practice. Indeed, the degree to which theory is put into practice is a reflection of the mutual comprehension of researchers and teachers. Obviously,
it is in the language of practice that the public, as well as teachers and students, will come to know reform.

There are four main channels through which research communicates to teachers and to the public: (a) journals, (b) teacher-training courses, (c) textbooks and other learning materials, (d) popular media. Teacher training is by far the most direct and in some ways the ideal channel for communication, as we discussed earlier. Textbooks and other learning material, however, are the most common channels for communication and reach the greatest number of teachers and students (competing, perhaps, only with the popular media). Moreover, they are probably the most immediate determinants of practice. The process by which textbooks are produced and adopted also provides a good example of the difficulties of communication, and, hence, ought to be among the foci of reform research.

Ginsburg, Klein, and Starkey (1998) examined this process of textbook production and dissemination in the United States and highlighted the complicated dynamics existing among researchers, government and professional groups, publishers, and teachers. It is difficult to find the first thread in this complex web of relationships, but the role of government is a good place to start. Ginsburg et al. (1998) explained that many states “require publishers to receive official approval before their textbooks may be offered for sale” (p. 432). The approval criteria used by many state governments relies heavily on the NCTM Standards and are informed by the research community. The extent to which ideas from research have influenced such approval can be seen in the California Framework discussed in some detail by Ginsburg et al. (1998):

Drawing on its interpretation of constructivism, the California Framework requires what some in the state have seen as a radical approach to the creation of textbooks. The Framework decrees that, to be adopted by schools in California, curriculum materials must stress meaningful learning, the construction of knowledge, independent thinking, and extended investigations and explorations; and the Framework requires publishers to downplay rote learning, memorization, and the passive absorption of knowledge.

(p. 433)

Editorial staffs of publishers accommodate the state criteria by studying the Standards, by reading NCTM journals such as Teaching Children Mathematics, and attending professional conferences. Moreover, “The editorial staff are often former teachers and other individuals with a sincere interest in helping children learn” Ginsburg et al. (1998), p. 434. In this connection, the role of the editorial staff in producing a textbook goes beyond being a mere forum where books are accepted or rejected; the editorial staff and, for that matter, the marketing staff have great influence not only on the final form of a textbook but also on its content.

With government approbation and encouragement of reform efforts on the one hand and an earnest willingness of publishers to produce acceptable material on the other, the circumstances seem perfect for a smooth working relationship between researchers, teachers, and the public and a real opportunity for constructive communication. Unfortunately, governments must consider voters, and publishers must consider sales. This would remain only a hypothetical issue if there were complete unanimity in support for the general direction and specific details of the reform, but this is never the case. There are those who object to change in principle, who think that what is traditional is best a priori. There are those who simply do not understand the reform at hand, and there those who do understand it but reject the values it promotes. Some of these objections are rational and some are not, but they all can contribute to a heated atmosphere, which the popular media are ever willing to report and which politician and publisher alike cannot ignore. The brunt of these pressures and counter-pressures is largely borne by the publisher, and “The result,” noted Ginsburg et al. (1998, p. 437) “… is a complex series of compromises. Given the goal of
maximizing sales (and profits), publishers naturally wish to have their textbook cake and eat it too. To some extent, publishers try to satisfy all sides in the controversy. So instead of producing textbooks with a clear strategy and clear outlook, publishers tend to produce textbooks rife with mixed messages.

The help such textbooks can provide in informing teachers’ classroom practice is moot, and the damage they may cause in the hands of teachers who have qualms about the merits of reform can be great. One must never forget that textbooks must be used, and the way they are used depends on the readiness and understanding of the user. As Ginsburg et al. (1998) put it,

But consider the extent to which the research-oriented textbook pages make demands on teacher. These pages do not offer a cookbook for teaching; they are not “teacher-proof.” Just the opposite is true: the constructivist approach depends on the intelligence of the teacher, who must construct an understanding of the child’s learning in order to foster it. To use these pages, teachers must think in a flexible manner, responding to the needs and unique constructions of individual children. Textbook pages are merely a resource; they can set direction but cannot guarantee what teachers will do. (pp. 440–441)

So communication between researchers and teachers through the channel of textbooks can break down first at the stage when the textbook is produced and then, partly because of the way the textbook is finally produced, at the stage where the textbook is actually used in the classroom. The problem is compounded by the ability of textbooks to appear to satisfy reform recommendations and of teachers to appear to use them with understanding and in the right spirit—an ability that mirrors that uncanny ability of people to appear as if they are engaged in genuine dialogue when no one is genuinely listening or trying to communicate ideas.

The failure to communicate through the channel of textbooks may be inherent in the nature of textbooks themselves. We have spoken about the obstacles to producing a textbook that reflects clearly and consistently what research has found out about learning and teaching mathematics. Nonetheless, the mere fact that teachers and students are meant to be a final destination for textbooks and that, ideally, a textbook aims to be inclusive and authoritative, means that as a channel of communication, textbooks are essentially one way; researchers, writers, and editorial staff stand on one end, disseminating reform ideas to the teachers and students who stand on the other end. But for theory and practice to meet, communication must be two way and continuous. As Lesh and Lovitts (2000, p. 54) said, “In mathematics and science education, the flow of information between researchers, and practitioners is not the kind of one-way process that is suggested by such terms as information dissemination. Instead, to be effective, the flow of information usually must be cyclic, iterative, and interactive.” On the other hand, at present it is difficult to see what kind of published learning material could bring about this sort of two-way and continuous communication. Yet it is imperative to try and find such material. The theoretical and practical problems with textbooks thus suggest that reform researchers must redefine their own role in communicating ideas about teaching and learning and, at the same time, explore new means of communication; both these ends of course require a deep look at the problem of communication itself.

Change, the Response to Change, and “Pseudo-Change.” The most obvious set of questions with which reform research must be concerned is that set of questions related to change. Reform is all about change; a reform movement is both an agent of change and a response to it. As an agent of change, it must be ever cognizant of students, teachers, researchers, parents, and politicians as a single complex. As a response to change, it must take into account not only the state of mathematical knowledge and of research in mathematics education but also of society and its needs.
Thus, these two aspects of change organize that matrix of factors which, as we said at
the end of the first section, characterize the object of reform.

The basic questions one would think to ask about reform as an agent of change are
questions such as, How can change be effected? Is change truly being effected? When
and how can one judge the results of change? Given the complex picture of reform that
we have been developing until now in which the distinction between researcher and
practitioner, between mover and moved, is blurred, in which there is much interaction
and mutual influence among all those involved in reform, these questions, simple and
obvious though they may seem, must be put somewhat differently. The basic question
in this light ought to be this: How do teachers and researchers learn and change while
at the very same time be facilitators of change?

The question can be asked equally, and more tellingly, in a negative form: What
happens when the cycle of learning and doing is broken? This would be a trivial ques-
tion if the breaking of that cycle meant the unambiguous cessation of the reform effort,
but that is rarely the case. Most often the resulting failure of reform is, rather, a kind of
“pseudo-change” or “pseudo-reform”: materials are produced that seem to conform
to reform recommendations, but in truth they embody the very practices the reform
aims to amend see Ginsburg et al. (1998), pp. 434, 438; teachers use reform material
in class, perhaps even attend inservice training on reform principles, yet often persist
teaching in a way completely against the spirit of the reform (as Robert B. Davis used
to say, “They’ve got the words, but not the music”)—often because of achievement-
oriented administrators who also appear to accept reform ideas. “Pseudo-reform” is
the expression of the failure to change and also a barrier to further change; it is,
thus, exactly the negation of the process of changing and being changed that signifies
healthy reform. Making “pseudo-reform” an object for reflection within the cycle of
learning and doing, accordingly, can at once invigorate and safeguard that cycle.

We might remark in passing that the phenomenon of “pseudo-reform” is proba-
bly also behind the curious fact that writers on one reform or another often refer to
the problem of drills and memorization as if these were advocated by the previous
generation of reformers. Of course, the truth is that hardly anyone who has thought
at all deeply about education ever completely advocated drills and memorization.
This applies not only to modern thinkers, but also to thinkers as far back as Milton,
Montaigne, and even Plato, who, for example, teased his companion Phaedrus for
merely memorizing without truly understanding a tract on love. Even the Jesuits,
who used to say *Repetitio mater studiorum est* [Repetition is the mother of studies], did
not think that memorization was the only mother of studies. The problem seems to be
that it is all too easy (and sometimes all too convenient) to mistake the nonreform prac-
tices, drilling and memorizing, hidden under the guise of reform for the reform itself.
Understanding “pseudo-reform” and learning to recognize it is thus essential not only
for the success of a present reform, but also for the fair evaluation of past reforms.

The next question is, What is the proper pace of reform? This bridges the two
aspects of change and reform. Clearly, it can be taken as a question about imple-
mentation and therefore about reform as an agent of change. For example, one of
the important characteristics of curriculum development projects, which are usually
connected with reform movements, is that they set for themselves a definite period of
time to accomplish their goals (Griffiths & Howson, 1979, pp. 145–146). Prescribing
the amount of time for a project seems to go against the view of reform in which
methods and strategies are continually being redefined, that is, always being emer-
gent; it appears problematic in the way that setting a definite time for an open and
free dialogue is. Yet whether it is for a dialogue or a reform project, a specified time
period is certainly a practical necessity. So the reform researcher must surely ask how
to balance this practical need of defining a schedule for reform with the flexibility
needed for methods and strategies to be truly emergent.
The question of pace also brings us directly to questions related to reform as a response to change. This is particularly true with regard to recent reform movements such as the standards-based reform. These movements have been conditioned by the assumption that the world is rapidly changing and that mathematics education must change with it. Although this may be a slight oversimplification, we can say that there are three, not utterly distinct ways relevant to mathematics education in which the world is understood to be changing: (a) in the state of mathematical knowledge and knowledge in mathematics education; (b) in the demands of society and economy; (c) in the means available for communication, production, and scientific inquiry. These require a corresponding educational response as to content, ends, and means.

The changing state of mathematical and scientific knowledge was obviously behind the “modernization” motive of the new math reform. The changes that this implied for mathematics education had to do largely with the selection of mathematical topics and the mode in which the selected topics should be presented. In any time of reform, however, the way the reform responds to the change in mathematical knowledge is deeply connected with the way mathematical knowledge grows.

The growth of mathematical knowledge is a rather complex process involving both a cumulative body of knowledge—specific mathematical theorems, objects, and techniques—and shifting views as to what is important in mathematics, what should be considered rigorous, and even what is a proper mathematical object—what Elkana has called images of knowledge (see, for example, Corry, 1989; Davis & Hersh, 1981, Elkana, 1981). Moreover, it is often difficult to separate the body from the images of mathematical knowledge. The way reform responds to change here must, in general, reflect both these aspects of mathematical change. Thus, for example, the reduction of classical geometry in the “new math” reform was partly the result of a view of mathematics in which logico-algebraic structure was considered the true heart of the subject; Dieudonné, accordingly, could say, “the whole course [of plane Euclidean geometry] might, I think be tackled in two or three hours—one of them being occupied by the descriptions of the axiom system, one by its useful consequences and possibly a third one by a few mildly interesting exercises” (in Howson et al., 1981, p. 102). A different image of mathematical knowledge has played a part in determining the mathematical content of the standards-based reform. Thus, Kilpatrick (1997, p. 957) pointed out that “a large part of the standards-based reform is built on the view that mathematics itself has become more computational and less formal.” For the reform researcher, then, it is important to maintain a firm awareness that during a period of reform teachers and students may have to accustom themselves not only to new topics but also to new ways of thinking about the general character of mathematics. That said, because the body of mathematical knowledge is cumulative and that new knowledge rarely contradicts the old, school mathematics program tends to have a relatively stable core. Researchers, teachers, and students have some base from which they can build the new ways of thinking entailed by reform.

The nonlinear changes in the images of knowledge share the character of and are partly dependent on equally nonlinear changes in the conditions and demands of society and the economy. This has always been true to some extent, but today the rate of these societal and economic changes is, perhaps, unprecedented. The implications for education are clear. As Hass (1964) said, “change is so rapid in our innovating, industrial society, that today’s education is unsuited for tomorrow’s world and is as outmoded as the Model-T for the world of 20 years from tomorrow the world whose leaders are now in the classrooms of America.” The changing demands of the economy, in particular, is a central preoccupation of the NCTM Standards, and the justification for setting the goal of encouraging of “mathematically literate workers”: 
The economic status quo in which factory employees work the same jobs to produce the same goods in the same manner for decades is a throwback to our industrial-age past. Today, economic survival and growth are dependent on new factories established to produce complex products and services with very short market cycles. It is a literal reality that before the first products are sold, new replacements are being designed for an ever-changing market. Traditional notions of basic mathematical competence have been outstripped by ever-higher expectations of the skills and knowledge of workers; new methods of production demand a technologically competent work force. The U.S. Congressional Office of Technology Assessment (1988) claims that employees must be prepared to understand the complexities and technologies of communication, to ask questions, to assimilate unfamiliar information, and to work cooperatively in teams. Businesses no longer seek workers with strong backs, clever hands, and “shopkeeper” arithmetic skills. (NCTM, 1989, p. 3)

Unlike the changes in mathematical knowledge, it is unclear that in these rapid non-linear changes there is any stable core of content on which a reform curriculum can be built. Obviously, for reform to respond by continually changing the curriculum is neither feasible nor desirable. But what is the right response to such changes, what kind of strategy should curriculum designers adopt to answer these constantly changing economic demands? The overall direction of the recent reforms has been to stress a curriculum that makes the student flexible and adaptable rather than “learned.” Accordingly, the curriculum does not aim to provide specific content but the conditions for learning and doing mathematics. Thus, in the Standards, the NCTM sets as one of the social goals of education “lifelong learning” whereby “Problem solving—which includes the ways in which problems are represented, the meanings of the language of mathematics, and the ways in which one conjectures and reasons—must be central to schooling so that students can explore, create, accommodate to changed conditions, and actively create new knowledge over the course of their lives” (NCTM, 1989, p. 4). In other words, the very general notions stressed by the Standards—problem solving, mathematical reasoning, communication—embody the ideal of an mathematically educated person to be one capable of learning mathematics, that is, one who has a well-based potential for acquiring and using specific mathematical content. This is why the Standards summarizes its outlook with the expression mathematical power. Indeed, “power” and “potential” share the same Latin root, posse, “to be able” or “capable.”

Making modes of thinking, rather than specific objects of thought, the focus of the mathematics program may well be the answer to the problem of providing a relatively stable curriculum able to respond to the unstable ever-changing demands of society and the economy. It is certainly an answer very much in line with a rationalist tradition that, since Descartes at least, gives method precedence over substance—a tradition in which general education in modern democracies is deeply entrenched (see Brann, 1979, pp. 129–149). But that same tradition is fraught with difficulties. Thus, while we may tentatively accept the general approach of current reform program in this regard, what is the right educational response to our rapidly changing society should remain a subject of inquiry for reform research.

The problem of changing means available for communication, production, and scientific inquiry, is, of course, the problem of technology. In some respects, the difficulties for reform arising from the pace of technological change are the same as those just discussed, as are the approaches open to reform to solve those difficulties. A few more words are in order, however. Although there are those who think that the use of graphing calculators and computers can dull mathematical thinking (e.g., Koblitz, 1996), it is fair to say that most see technology as an unavoidable and welcome aspect of the modern world. The Standards certainly puts great stress on the use of technology, and the Principles and Standards makes technology one of its six principles; the
Tomorrow 98 reform in Israel actually made the use of technology its central issue. Roitman (1997) listed four questions that she took to be the relevant questions “about any use of technology in the classroom”:

- What mathematics is reflected in the use of technology?
- What efforts are made to ensure that the mathematics is significant and correct?
- How does the use of technology engage students in realistic and worthwhile mathematical activities?
- How does the use of technology elicit the use or enable deeper understanding of mathematics that it is important to know and be able to do? (p. 7)

The proper consideration of Roitman’s questions really demands an independent research effort and is thus out of the range of reform research. It is clear, however, that the degree to which expectations of a technology-oriented reform such as Tomorrow 98 are fulfilled is proportional to the seriousness with which these questions are asked. Here reform research has a role, for there is a need to develop models whereby these crucial questions are indeed continually being asked and continually informing the cycling process of practice and reflection. Just providing computers or graphing calculators in the schools is obviously not enough. In fact, because computers are physical objects one can point at—and often very attractive ones—merely supplying such technology without teachers and researchers reflecting on their wise use can easily become a variety of “pseudo reform”; the technology becomes visible but ineffectual.

CONCLUDING THOUGHTS: CONTENT, VALUES, AND COOPERATION

What we have said about reform research and technology seems to suggest that reform research is concerned with ensuring only that questions of content are asked, with how they are asked, and with when they are asked, rather than with the questions themselves. This is consistent with some of the other things we have said about reform research, for example, its overriding concern with the existence, manner, and extent of communication. But is reform research really only concerned with the “mechanics” of reform—only that there be communication, but not with what is being communicated, only that there be questions, but not with the substance of the questions? Such a view would be a distortion and definitely not consistent with one other point that we emphasized about reform research, namely, that it be active.

We argue that the active involvement of reform research in reform follows from the very nature of reform in that its object is a complex-interacting system of which researchers are themselves an integral part. The reform researcher is not only studying and monitoring the flow of information within that system from the outside, as it were, but is exchanging ideas with teachers, students, parents, and other researchers, influencing them and being influenced by them, watching change occur and being changed. The reform researcher will, therefore, of necessity be continuously engaged in a dialogue concerned with the content and the values of the reform.

A metaphor that helps clarify the peculiar position reform researchers occupy in this dialogue is that of moderators for discussions among experts. In such discussions, moderators are not expected to have all the expertise of the participants, but they are expected to be able to listen well and ask guiding questions, and this, in turn, demands an intelligent grasp of the content of the discussion. More than this, however, moderators must have an acute awareness of the general picture arising out of the discussion because it is the responsibility of the moderator, not so much that the
discussion reach this or that specific conclusion, but that discussion keeps its general subject and general goals in sight. In short, the moderator must continually coordinate the substance and ends of the discussion.

Where the discussion corresponds to the reform movement the necessity of such a coordination effort is completely evident, for, as should be clear from our considerations of the motives for reform, reform needs to be viewed not as a mere corrective, but as a concrete expression of a total vision of mathematics education. Moreover, this total vision is one arising out of the efforts and thoughts of all those involved in the reform effort and of others too. By the latter, we have in mind, besides those involved in fields of study and action concerned specifically with mathematics education, also philosophers, anthropologists, sociologists, and historians, to name a few.

This is all the more true in light of what was said above about images of knowledge: Our very understanding of mathematics and of the ends of mathematics education is as much determined by culture and society as by logic and objective mental activity. Thus, the point of view of the historian or the anthropologist should be of immense help in defining the values behind the seemingly detached mathematics of the classroom. Mathematics education, in this regard, must never be “provincial.” Reform research, in particular, must envision itself as fostering a sense of openness and of cooperation among those concerned with a wide range of human activity.

In what we have said above, we have really set out only the most basic requirements of reform research and only a very general picture of the reform researcher. For the latter especially, we have had to rely on numerous metaphors: the researcher as ecologist, as complex-system analyst, and as discussion moderator. The need of such metaphors is a sign that in speaking about reform research we are speaking about something genuinely new, something still lacking definition, still lacking precision. We should like to see this new field reach the same level of clarity attained by other areas of educational research. But the only way this can be achieved is if researchers, guided by these general metaphors and released from all other paradigms of educational research, will go out into the field, allow themselves to become immersed in actual reform efforts, and yet maintain enough self-possession to soberly document what they find. We believe this represents a worthy challenge and may, in fact, become emblematic for other future challenges for educational research.

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CHAPTER 16

Democratic Access to Powerful Mathematical Ideas

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Carlos and his family have to move out of his home. His mother lost her job, and the money she made through great effort to pay for their small house is in the hands of the bank. Carlos, a 10th-grade student, is one of the many Colombian youngsters who will finish high school at the beginning of the 21st century. Many of these students seem to be confused about their future. Teachers insist on the importance of schooling and learning, especially in mathematics. Yet how could that help in the real-life circumstances of children like Carlos? On the other side of the world, in Denmark, Nicolai became seriously sick after eating a homemade ice cream. Contracting salmonella via eggs, chicken, or meat products is a relatively common occurrence in Denmark. People blame quality control, but do they, in fact, know what quality control is about? Does the education that Nicolai gains in school help him understand the dangers of his apparently “safe” society?

These are cases of real students in two different countries, and, as we argue, their life experiences are significant to mathematics education. In the task of advancing the field of research on the phenomena connected to the learning and teaching of mathematics, we start with a consideration of the global informational society in a complex social, political, cultural, and economic context. Within this context, both world and local trends intermesh, and new challenges to mathematics education practices and research emerge. Based on the contradictions of this current social order, we propose the *paradox of inclusion* and the *paradox of citizenship* as two central problems that mathematics education must face. With this purpose in mind, we proceed to give meaning to the term *powerful mathematical ideas* in four ways. We then discuss the notion of *democratic access* and question the simple identification of democracy...

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1 Although our names appear in alphabetic order, we want to acknowledge our equal contributions to the writing of this chapter.
with universal access. Finally, we argue that facing the paradoxes of inclusion and citizenship represents a struggle for the provision of "democratic access to powerful mathematical ideas" in mathematics education, both in practice and in research.

PARADOXES OF THE INFORMATIONAL SOCIETY

After the breakdown of the wall between East and West, Fukuyama (1989, 1992) declared "the end of history." This statement resonates with what theories of postindustrial society had been claiming since the 1970s, namely, that the world has reached a state in which the sources of value—and therefore of power—can be described not only in terms of labor and capital, but also and primarily in terms of knowledge and information. This state in the transformation of capitalism has also been called the information society (Bell, 1980). Together with the consideration of value and power, there has been a change in the kind of citizens that this new type of social order requires. People need to be able to deal with knowledge and information in continuous processes of learning. This particular shift is what has been called the "learning society" (Ranson, 1998). In what we refer to as the "informational society" (following Castells, 1999), subsuming both the information society and the learning society, the impact of technology goes beyond industrial production, and, in fact, affects political, economic, social, and cultural structures.

A discussion of the informational society cannot be separated from a consideration of globalization as the process responsible for establishing the "world village." Globalization refers to the fact that events in one part of the world may be caused by, and at the same time influence, events in others parts. Our environment—in political, sociological, economic, or ecological terms—is continuously reconstructed in a process that receives inputs from all corners of the world. Simultaneously, our actions have implications for even the most remote corners of the planet. However, globalization also relates to the apparently shared belief that a given kind of environment is desirable and that there is some kind of universal commitment to the achievement of certain ideals like democracy, market freedom, and individual competitiveness. The myth of the "end of history" can be interpreted as the legitimization of a false universalism (Eagleton, 1996). Together with the discourse of globalization comes a new discourse of colonization. In a similar way that the first European waves of colonization, from the 14th to the 18th centuries, brought new languages, religions, and social orders that trampled down indigenous cultures, the new global colonization also imposes new ways of living, producing, and thinking. D'Ambrosio (1996) saw science, including mathematics, as also playing a role in this cultural invasion; and, of course, mathematics education is not an innocent onlooker of the situation.\(^2\)

Castells (1999) criticized some of the dominant descriptions of the postindustrial society based exclusively on the North American–European context. He emphasizes

\(^2\) As an example of what globalization means, we can analyze the Third International Mathematics and Science Survey (TIMSS) as a representative international study that produced knowledge and information about the state of mathematical and science education in the world. Despite all the discussions about the problems of TIMSS as a legitimate ranking system and means of comparison, one of the conclusions that at several levels seemed to be drawn is the necessity of following the model of high-scoring countries such as Singapore and Japan. Thus, at the International Round Table at the start of the International Congress on Mathematical Education (ICME) 9, a director from the Singapore Ministry of Education explained their conception of mathematics and science education and how they have been able to achieve success. The setting of the whole round table can be interpreted as an attempt to put forward in the international community of mathematics educators a model that is desirable to follow. On the other hand, the systematically scant mention of countries that did very poorly, such as Colombia and South Africa, shows the dispensability of these cases in what is accepted as relevant internationally.
that a theory of the informational society should refer not only to the fact that certain countries and certain regions are becoming closely interrelated, but also to the fact that people, countries, and regions are excluded, apparently for not being of any relevance to the construction of the informational economy. Because access to knowledge is clearly important in the informational society, “the ability to generate new knowledge and to gather strategic information depends on access to the flows of such knowledge and information. . . . It follows that the power of organizations and the future of individuals depend on their positioning vis-à-vis such sources of knowledge and on their capacity to understand and process such knowledge” (p. 60). The access to the flows of knowledge and information constitutes a major division between those in the core of the informational society and those outside it. According to Castells, exclusion is devastating because “the structural logic of the information age bears the seeds of a new, fundamental barbarism” (p. 60). All the outsiders belong to structurally irrelevant areas in the informational society and constitute what Castells called the “Fourth World.”

This observation draws attention to the complex dynamics of globalization. At the same time that we are becoming similar, we are also moving apart. The interplay between the global and the local is a game that connects many parts of the world in a network of flows, and simultaneously excludes regions and people from specific communities and countries in the world. The Fourth World includes not only large regions of Africa, Latin America, and Asia, but certainly also carves out large chunks of Europe, the United States, Japan, and Australia. Many people who either live in poverty or who are isolated from the centers of informational and technological production and exchange in these countries (e.g., political refugees and illegal immigrants in the United States, elderly people in rural areas in Japan, aboriginal communities in Australia, and young drug-dependent and “punk” communities in Germany) are apparently superfluous in this world order.

Nevertheless, the Fourth World has some relevant roles to play for the informational economy. First of all, it supplies spaces for dumping ecological problems and other side effects of industrial production. It also provides a market area to be flooded and a cheap source to supply the material flow of goods needed in the informational economy. The globalization linked with the informational economy seems to continue a provocative exploitation of certain parts of the world as a given. A concern for equity seems not to be part of this type of globalization.

Globalization is also responsible for determining who count as functional people in the free-flowing, informational economy. This social order is characterized by a strong capacity for renewal and flexibility in individuals and social organizations, which manifests itself through an enterprise capacity. Individuals and groups become organized under the principle of continuous learning as a mechanism of adaptation to rapid and constant environmental changes. This idea has implications for dominant current educational conceptions. Learning is conceived as a continuous “learning to learn” to fulfill societal requirements. Notions such as “constructivist teaching and learning,” “active students and teachers,” “rich educational environments,” “technology inclusive experiences,” and, more recently, “accountable and efficient educational services,” together with “satisfied parent and student clientele,” dominate the learning society discourse (Apple, 2000; Masschelein, 2000).

Despite the apparent suitability of some of these “learning to learn” notions, this whole discourse should be carefully questioned. There is a risk of reducing learning to a mechanism of individual survival, which opposes a conception of learning as a human activity whereby unique beings search for meaning in an attempt to initiate events that contribute to securing a sustainable, durable, common world. In other words, the possibilities of education as a questioning of the self, a judgment of the meaning of life, a construction of a common world, and a criticism of the given order of
things, are highly at stake (Masschelein, 2000). Furthermore, Flecha (1999, p. 67) noted that “the knowledge prioritized by the new forms of life is distributed unevenly among individuals, according to social group, gender, ethnic group, and age. At the same time, the knowledge possessed by marginalized groups is dismissed, even if it is richer and more complex than prioritized knowledge. More is therefore given to those who have more and less to those who have less, forming a closed circle of cultural inequality.”

In mathematics education, the lifelong “learning to learn” ideas have been taken as a desirable goal to be reached in the 21st century. Mathematicians in the 1960s took seriously the duty of setting a mathematical education on which could rest “the ever heavier burden of the scientific and technological superstructure” (Organisation for European Economic Co-operation (OEEC, 1961, p. 18)). Nowadays, a large portion of our mathematics education community is apparently committed to the edification of competent citizens of the emerging and rapidly changing informational society. As the National Council of Teachers of Mathematics (NCTM) Standards 2000 (NCTM, 2000, pp. 3–4) state, the capacity to understand and do mathematics is more relevant than ever because it allows one to “have significantly enhanced opportunities and options” for shaping one’s future. This formulation implies that acquiring mathematical competencies is a condition for being able to adapt, and therefore, both survive and help sustain this type of social development. The need and desire for more mathematically able people, as expressed in the discourse of “more mathematics for all,” may contribute to spreading a utilitarian value of mathematics education that in the long run serves as a tool for the survival of the smartest. The contradiction between the social expectations emerging from this kind of discourse and actual practices where mathematics is used as a social filter determining who has access to further success (Smith, 2000; Volmink, 1994; Zevenbergen, 2000b) in fact gets resolved in favor of those who pass the gates of mathematics. Therefore, without anticipating it, mathematics education may support the dangers of the learning society.

We find that the “informational society” is a contested concept. It contains contradictions, and it can develop in different directions. We attempt to summarize this fact by formulating two paradoxes of particular importance for mathematics education. The **paradox of inclusion** refers to the fact that the current globalization model of social organization, which embraces universal access and inclusion as a stated principle, is also conducive to a deep exclusion of certain social sectors. The **paradox of citizenship** alludes to the fact that the learning society, claiming the need of relevant, meaningful education for current social challenges, at the same time reduces learning to a matter of necessity for adapting the individual to social demands. The paradox of citizenship concerns in particular the notion of *Bildung*, which refers to the development of general competencies for citizenship, especially the capacity to act critically in society, and in this way have an impact on it. This paradox refers to the fact that, on the one hand, education seems ready to prepare for active citizenship, but, on the other hand, it seems to ensure adaptation of the individual to the given social order.

Although from our field of research and practice (mathematics education), we cannot solve the paradoxes, we find it necessary to face them. If not, mathematics education could act blindly in the further development of current society. We engage in the task of exploring the significance of these two paradoxes from the particular

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3We could go further in this argument by asking who actually benefits from the expansion of the learning society discourse. Plausible explanations about the forces associated with recent reform trends in several countries can be found in, for instance, Apple (1996, 2000).

4Here we are inspired by Young (1998), who presented the idea of “learning society” as a contested concept.

5For a discussion of the notion of *Bildung*, see Klafki (1986) and Biesta (2000). There is no adequate English translation of the German word *Bildung*, although “liberal education” has been suggested.
perspective of mathematics education by examining the notions of “powerful mathematical ideas” and “democratic access.” We do so by referencing two examples.

TWO EXAMPLES

Terrible Small Numbers

Salmonella poisoning is an everyday danger in Denmark. In one way or another, Nicolai knew that an “innocent” homemade ice cream, prepared with infected eggs, could be enough to make him sick. Students in school hear and can read about salmonella infection. A newspaper article under the headline “We have to live with salmonella” reads as follows:

Experts estimate that a steady number of more than 1,000 Danes will be sick with salmonella each year. The Minister of Food, Henrik Dam Kristensen (Social Democracy) says that we won’t succeed in wiping it out. Danes have to live with the permanent risk of getting sick from salmonella via Danish meat and egg products. . . . This was one of the conclusions from the report given to the Minister by the Danish Zoonosis Center, the advisory institution in these matters. According to the Minister, the paper does not lead to any changes in the strategy against salmonella, but Danes must learn to live with the infection risk. We will still prepare tests and investigations so that we can come as close as we can to zero risk. However, that is not the same as ensuring that salmonella infected eggs, chicken, and pork will not pass the control. Today it is impossible to make people believe that a 100-percent secure control is in place. (Politiken, 2000, our translation)

If risks, as stated by Beck (1992), are an essential constituent of our current world, how could school and especially mathematics teaching and learning provide tools for analyzing those risks in a meaningful way? The project Terrible Small Numbers tries to address this question. Together with their students, the teachers participating in the project collected 500 black film cases to simulate eggs; film cases were selected because they resemble eggs in size, lack of transparency, and the possibility of “opening” then for examination. Inside each egg there was a yellow centicube, except in some of them in which a blue centicube was placed. The blue “yolk” represented a salmonella-infected egg.

During the first sequence of activities, the mix of healthy and salmonella-infected eggs was made in front of the whole class. Everybody knew that out of the 500 eggs, 50 were infected. The students then had to take samples consisting of 10 eggs, and to count the number of infected eggs. Intuitively the students expected to get one blue egg in each sample, but after some experiments they found that in some cases they could get 3 blue eggs—or even more—out of 10. How could that be? Was it because the mix was not done in a proper way? Was it bad luck? The basic question to be addressed by this experiment has to do with the reliability of information provided by samples. How can it be that a sample does not always tell the “truth” about the whole population? And how should we operate in a situation in which we do not know anything about the whole population, except from what a sample might tell? How can we, in this case, evaluate the reliability of numerical information?

In a second sequence of experiments, students were presented with two types of eggs, Spanish and Greek, to buy for retail sale in shops. In both types, there were some infected eggs, but this time the students did not know how many. To make a decision

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6This project is described in Alrø et al. (2000a, 2000b), in Danish. It is a collaboration between two Danish teachers, Henning Bodtkjer and Mikael Skärsjö, and three researchers, Helle Alrø, Morten Blomhøj, and Ole Skovsmose. Terrible Small Numbers has been tested in different classrooms, but here we primarily provide a general overview of its main ideas.
about which type of eggs to buy for retailing, they needed to run a quality-control test. It was impossible to test all eggs because eggs opened in the quality control could not be sold. Furthermore, it was expensive to check eggs for salmonella, so the students’ (the “retailers”) budget was affected by control costs. They had to consider carefully how many Spanish and Greek eggs needed to be sampled to make a decision about which type to buy. The concern for making a responsible decision was confronted with the interest of making a healthy business.

A third sequence of activities dealt with the evaluation of the risks of getting salmonella from food products with eggs in the ingredients. The starting point for the preparation of the products was a mixture of 500 eggs, 5 of which were infected. A preliminary question was to calculate the probability of finding a blue egg; it was not difficult to arrive at \( \frac{5}{500} = 0.01 \). Now, if we want to make an ice cream portion out of six eggs—and of course we would like them all to be healthy—the probability of getting a salmonella-free portion is \( (1 - 0.01)^6 \) and therefore the risk of infection is \( 1 - (1 - 0.01)^6 \). To get to this formula was not simple. The students began by suggesting that if the probability of getting an infected egg is 0.01, then, when picking 6 eggs, the probability must be 0.06. However, by finding the proper formula, the students had an opportunity to contrast mathematical calculations with empirical experimentation.

The project tried to provide ground for a discussion of the difference between ideal mathematical calculations and empirically obtained figures, as well as a debate about the possibility of calculating risks in general. The notion of risk can be summarized in mathematical terms by the equation

\[
R(A) = P(A)C(A).
\]

Here \( A \) represents an event. The risk, \( R(A) \), is the product of the probability that \( A \) happens, \( P(A) \), and the consequences of \( A \) happening, \( C(A) \). In other words, the risk of eating an ice cream dessert equals the probability of being infected times the “cost” of being infected, and naturally the “cost” increases with the size of the dessert because many more people may taste it.

**Macro-Figures Becoming Macro-Dangers**

A country in an unstable economic and political situation is a perfect scenario for witnessing the macro-dangers of macro-figures. Colombia, in the last decade of the 20th century, represented a deeply troubled society, in conflict between democratic consolidation and international globalization demands. In this scenario, where almost premodern, modern, and postmodern living conditions coexist, students struggle to find good reasons for finishing school—if, of course, they have a chance of doing so. Carlos certainly finds it difficult to see the role of so much studying in his future. It is even more difficult now that his family had to leave the house that his mother began paying for some years ago. Recently, the monthly mortgage payments became so high that she had to give up. When she tried to sell the house, she could not recover a single cent of what she had invested, and her best solution was to give it back to the bank as part of the debt payment.

Carlos was not the only student who, between 1998 and 1999, lost his home. Such was the story that many people lived in Colombia. For the first time, people were concerned about what the UPAC (Unidad de Poder Adquisitivo Constante [Unit of Constant Buying Power]), introduced in 1971, could mean in their lives. Certainly a mathematical investigation in the classroom could be of help. In what follows, we imagine the general guidelines of a project, “Macro-figures becoming Macro-dangers”
with 10th or 11th grade students.\textsuperscript{7} The project may allow students to reflect about the use of mathematics as a power resource through economic and social models.

Where do we start? We could ask students to ask their families and friends about the UPAC and its predicaments. We want the project to be of relevance for the students’ actual situation. As one of the essential inquiry sources, we can collect receipts for mortgage payments during the last one or two years. We can ask for help from the social science teachers to get information about the UPAC system and the reasons why back in the 1970s the government adopted it. Is it possible to discover the assumptions of the system? Are they still valid? The UPAC system, which was intended to promote private savings and housing acquisition, was designed under the assumption that, on the one hand, devaluation, inflation, and interest rates could be controlled by the government (Currie, 1984; Perry, 1989), and on the other hand, that the country would have a steady economic growth.

In the case of mortgage payment, the UPAC system operates in the following way. To calculate nominal interest ($n$) on a mortgage, the system considers inflation ($i$), the effective interest rate ($e$), which is estimated at 6\% annually, and a risk factor ($r$). The nominal interest, $n$, is then determined by the formula (Vélez, 1997):

\[ n = (1 + i)(1 + e)(1 + r) - 1. \]

For instance, in normal conditions, if $i = 0.06$, $e = 0.06$, and $r = 0.01$, then $n = 0.13$, which would be a reasonable case. At a time of deep economic crisis the nominal interest gets out of control due to variations in inflation, the effective interest rate and the risk factor, as actually happened in Colombia in the period between 1997 and 2000. For instance, in a crisis situation, if $i = 0.18$, $e = 0.20$, and $r = 0.10$, then $n = 0.55$, generating an aberrant situation. In the specific Colombian case at the end of the 1990s, when people could not afford to cover the payments (e.g., due to unemployment) and were forced to sell their property, they lost all their savings because the value of real estate decreased as part of the crisis itself.

After a first exploration, we could start looking at specific cases—that of Carlos’s, family if he and his family agree—and make groups based on students who have similar cases. The main purpose of the work could be to advise specific families in the process of negotiating new payment systems with the bank. Based on the payment receipts gathered, students can study the connections between the different figures in a given period of time—the total amount of the mortgage, the interest rate, the proportion of the debt that has actually been repaid, the payments for interest, and so forth. We could go as deep as needed into the mathematical exploration of the situation.

Then we could enter into a discussion about the consequences of the model. We could prepare a report for the families, explaining what happened during the time they paid their mortgages and proposing suitable alternatives to deal with bank proposals about the renegotiation of their mortgage and the adoption of the new model proposed by the government.\textsuperscript{8} As one of the aims of the project, we would like to grasp the potential that a mathematics class investigation could have for initiating changes in

\textsuperscript{7}This example builds on discussions with Colombian teachers during the seminar “Cómo desarrollar una educación matemática crítica en el salón de clase” led by Ole Skovsmose in Bogotá, Colombia (October 8–9, 1999), on Paola Valero’s presentation “Desenmascarar las matemáticas: Un reto para los profesores del próximo milenio” in Portimão (Portugal) during ProfMat 99 (November 10–14, 1999), and in particular on follow-up discussions with Jaqueline Cruz and Verónica Tocasuche, secondary school teachers in Colombia, and with Pedro Gómez. These ideas have not been implemented yet.

\textsuperscript{8}From January 2000 the UVR system (Unidad de Valor Real [Unit of Real Value]) replaced the UPAC. The UVR established a simpler index based on the national basic cost of living.
the students’ lives. Is what we all gain during the development of the project enough to act politically around the families in trouble?

Does this experiment illustrate essential aspects of what to consider in an inquiry in the mathematics classroom? Is it important to make this project a reality? Likewise, what can we say about “Terrible small numbers”? If mathematics education should face the paradoxes of inclusion and citizenship of the informational society, such questions become important. To discuss in more detail the possibilities of tackling the paradoxes, however, we must first explore what could be the meaning of “powerful mathematical ideas.” We then discuss the different aspects of providing “democratic access” and then return to the paradoxes.

POWERFUL MATHEMATICAL IDEAS . . .

To say that something is powerful is tantamount to affirming that it can exercise power. If we state that mathematical ideas can exercise power, we should try to clarify the following questions: What do we mean by “power”? What is the source of the power of mathematical ideas? What are the consequences of that power? In what follows, we put forward different possible interpretations.

. . . Logically Speaking

Mathematical ideas can be seen as powerful from a logical point of view. In this sense, power refers to the characteristic of some key ideas that enable us to establish new links among theories and provide new meaning to previously defined concepts. In this sense, one can certainly assert that plenty of powerful mathematical ideas have emerged throughout the history of the discipline.

In particular, we can associate the notion of powerful mathematical ideas with abstraction. A concept may be interpreted as powerful to the extent that it provides new insight into a different set of concepts. The notion of group illustrates the logical power of making abstractions. A group can be defined as a set, \( M \), consisting of certain elements, and an operation, \( * \), which to any pair of elements from \( M \) associates an element from \( M \), and which fulfills certain properties. Exemplars of groups are then recognized all over mathematics, a basic one being the set of integers together with the operation “addition.” A wide range of other mathematical structures, besides group, are recognized, such as ring, vector space, metric space, topological space, all defined solely by their formal properties and not by any qualities of their elements. Such formal structures make it possible to bring an understanding obtained in one area of mathematics to apply in an seemingly completely different area. In this way, abstractions have led to a class of powerful mathematical ideas, logically speaking. The power or strength of those ideas, then, can be defined as an intrinsic and essential characteristic of their position in the hierarchy of mathematics, which allowed them to influence other ideas so as to reaccommodate and redefine them. Once a more abstract mathematical idea provides a new conceptualization for previously existing notions, the building of mathematics is restructured via the legitimacy of the new ruling and organizing principle. In this sense, powerful mathematical ideas, logically speaking, have an intrinsic power exercised within the realm of mathematics.

Such powerful mathematical ideas can be expressed in a logical architecture, as exemplified by the work of the Bourbaki group. A closer look at this strictly modernist edifice reveals also what “powerful,” in a logical sense, could mean for mathematics education. If mathematics education is conceived as having the role of enculturating students into established mathematical knowledge and its ways of working, then it is easy—as the proponents of the New Math Movement in the 1960s thought—to
generate a list of powerful mathematical ideas around which to organize the curriculum. By means of such logically basic ideas, all other ideas could be defined.

Although the particular approach and aims of the modern mathematics education wave have almost disappeared from school curricula, there is still a dominance in practice of the idea that mathematics curricula consist of a list of essential, powerful mathematical ideas and topics to be learned. The amazing similarity and stability in the structure of national mathematics curricula across the world (Kilpatrick, 1996) show the strength of the shared belief in the logical power of mathematical ideas. Independently from the orientation of the approach to school mathematics, such as “Back-to-Basics” in the United States and National Numeracy Strategy in the United Kingdom, which stress the traditional priorities of mathematical topics, or the NCTM Standards, which represent a more progressive curricular proposal, all these views try to grasp the essence of powerful mathematical ideas from this logical point of view. Much mathematics education simply assumes that mathematical ideas are powerful primarily in a logical sense. This justifies that mathematics teaching can concentrate on providing students access to “real” mathematics, either by following the school mathematics tradition or even by a progressive establishment of a scaffolding, which makes it possible for the students to construct mathematics by and for themselves.

From this perspective, what can we make of the projects Macro-Figures Becoming Macro-Dangers and Terrible Small Numbers? Could they lead to powerful mathematical ideas, logically speaking? Certainly we could imagine possible ways of strengthening a mathematical focus. In the case of Terrible Small Numbers, one could have gone deeper into the mathematical significance of expressions such as $1 - (1 - p)^x$ and into other probability notions and the connections among them. This could have brought the students into a whole exploration of probability theory. In the planning of the Macro-Figures Becoming Macro-Dangers project, one could start considering the equations:

$$n = (1 + i)(1 + e)(1 + r) - 1 = (i + e + r) + (ie + ir + er) + ier.$$  

In particular, by making this algebraic reduction, it becomes clear that $n > i + e + r$. Furthermore, the project could provide a nice entrance to algebra, and once more it can be illustrated that abstraction is an essential element of powerful mathematical ideas. The calculations could also open a route directly into the exploration of exponential functions because the project makes it relevant to consider how a function like $f(t) = (1 + n)^t$, with $t$ referring to time, operates.

The logically based interpretation of powerful mathematical ideas legitimates doing mathematics for the sake of the internal characteristics of mathematics. It supports the desirability of allowing students to experiment and play with ideas and ways of working that in themselves appear powerful. Nevertheless, this perspective embraces some risks. It could accentuate the paradox of inclusion because it will justify the provision of an abstract curriculum that, as much research has documented, systematically closes the possibility for the majority of students of participating in a meaningful mathematics education experience (Boaler, 1997). This perspective could also contribute to exacerbating the paradox of citizenship because mathematics education could end up offering knowledge that appears relevant for students to their further career opportunities, but for which the relevance beyond this is limited.

... Psychologically Speaking

We could also associate power with the individual’s experience in learning mathematical ideas. In this sense, power is determined in relation to learning potentialities. From this perspective, what counts as significant ideas is what students can grasp
and make meaning of in the process of developing mathematical thinking. In fact, the
majority of research in mathematics education in the 1980s and 1990s is an important
source for the identification of this kind of powerful ideas.

Influenced by the work of Piaget, and more recently of Vygotsky, on the de-
velopment of human cognition, mathematics educators have formulated different the-
etical frameworks to describe what the learning of mathematics is about. These
theories have also served the purpose of describing basic principles for what should
be achieved through the mathematical schooling experience. Verschaffel and De Corte
(1996) offered an example in the case of arithmetic. First, they stated the leading princi-
pies for arithmetic learning and teaching in school in terms of learning mathematics as
a social and cooperative constructive activity, the role of meaningful contexts, and the
progression toward higher levels of abstraction and formalization (pp. 102–103). Then
they formulated some major aspects that need to be given more attention in connection
with, for example, the acquisition of number concept and number sense (pp. 105–111).
These aspects include counting at the expense of logical operational skills in the early
grades, allowing an awareness of multiple uses of numbers, promoting number sense
and estimation, and going beyond whole numbers. In contrast to a logical interpreta-
tion of powerful mathematical ideas, such items do not emphasize the mathematical
content involved in the learning process, but focus instead on the mental operations
that go together with the acquisition of the mathematical notions. In the case of alge-
bra, Kieran (1992) provided a list of similarly powerful mathematical ideas.

One important notion emphasized in the learning of algebra, and of more com-
plex mathematics, is that of the duality between conceptions of mathematics as pro-
cesses and as objects (Sfard, 1991), which in the French didactique des mathématiques
version is formulated as the dialectic between mathematics as tools and as objects
(Douady, 1987). This discussion, which has certainly influenced the understanding
of mathematics learning and teaching, combines a mathematical analysis about the
nature of mathematical objects with an analysis of learning processes. In this way, the
point of the power of mathematical ideas is connected to the degree to which they can
be integrated into the students’ understanding through processes of interiorization,
condensation, and reification. The whole issue of understanding (Sierpinska, 1994)
is, therefore, the key to defining the potential that mathematical ideas can have once
located in the domain of human learning.

In addition, an emphasis on affective, motivational, and idiosyncratic aspects of
both students’ and teachers’ understanding of mathematics—and of its learning and
teaching—is also considered a central part of the generation of powerful mathematical
ideas, psychologically speaking. The realization that meaningful mathematical
ideas are only acquired—or constructed—if the individual has a favorable mental
disposition to engage in the process of learning generated a complementary set of
ideas such as the importance of students’ and teachers’ attitudes and beliefs toward
mathematics and its teaching and learning. In this sense, some metamathematical
thinking notions, such as competencies in problem solving, metacognition, and sense
making (Schoenfeld, 1992), came to go hand in hand with mathematical ideas. This
combination constitutes powerful clusters in a psychological sense.

For mathematics education all these principles implied the advance in ideas of re-
form along the lines of, for example, the NCTM Standards proposals, which represent
a combination of powerful mathematical ideas in both a logical and psychological

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9 For a discussion of the influence of Piagetian and Vygotskian ideas on mathematics education see

10 One of the signs of the influence of this work is the extent, in quantity and quality, to which Sfard’s
paper has been quoted in research in mathematics education since it appeared in 1991.
sense. To illustrate this combination, we can see how the description of the Standards (NCTM, 2000) plays with the identification and integration of mathematical topics (e.g., number and operations, algebra, data analysis, and probability), mathematically related activities (e.g., problem solving and communication), and competencies in those topics and activities (i.e., understand numbers, ways of representing numbers, relationships among numbers, and number systems, use mathematical models to represent and understand quantitative relationships, monitor and reflect on the process of mathematical problem solving, and communicate their mathematical thinking coherently and clearly to peers, teachers, and others).

Considering our two projects, following the psychological interpretation, we could discuss the role of the contextualization on which the projects are based. The projects bring into the classroom concrete situations that the students can use as a basis for understanding. In this sense, each project provides a frame for the students to become familiar with mathematical notions that intervene in the situation. Its main role is to bring students into mathematics as a facilitator and as a motivational device. In the case of Terrible Small Numbers, the students are familiar with the issue of salmonella. The proximity of the topic to their lives can provide the possibility of making connections between already internalized concepts and new ideas to come. The experimentation with the samples of eggs opens further links to which the ideas of probability and risk can be connected. In Macro-Figures Becoming Macro-Dangers, the extraction of basic data for the mathematical analysis from real sources can be viewed as an especially engaging activity, which can motivate students to learn the mathematical aspects behind the real cases. In particular, the students could observe a new significance of making algebraic reductions. They can reveal connections that are not so easy to identify if only numerical calculations are used.

In most cases, the psychological interpretation of powerful mathematical ideas rests on the assumption that human learning processes are universal, even though strong cultural and social differences may affect meaning construction. It also assumes that those ideas are therefore transferable into diverse situations and that, given this transferability, they constitute useful knowledge. We find this interpretation problematic in light of recent studies that have evidenced and developed a radically different view of knowledge and human cognition. First of all, recent studies have shown (Lerman, 2000) that the individual’s social and cultural situatedness—in particular ethnic, social, or gender groups at a given historical moment—has an impact on her cognitive development. Secondly, it has been suggested that learning is not a mental process but participation in communities of practice (Lave, 1988; Lave & Wenger, 1991). From this perspective, there is no possible knowledge transfer but different types of participation and action in different contextualized situations (Boaler, 1997; Wedege, 1999). A view of mathematical ideas from a broader perspective is necessary.

... Culturally Speaking

If students should experience the relevance and meaningfulness of their learning in relation to their sociocultural experience, it is necessary to consider what counts as powerful from the situated learners’ perspective. We could then try to relate powerful mathematical ideas to the opportunities for students to participate in the practices of a smaller community or of the society at large. These possibilities have to do with the students’ foreground, which refers to the way students interpret and conceptualize—explicitly or implicitly, consciously or unconsciously—their future life conditions given the social, cultural, economic, and political environment in which they live.\(^{11}\)

\(^{11}\)For a discussion of the notion of foreground, see Skovsmose (1994).
also refers to the students’ interpretations and conceptualizations of their possibilities
to engage in meaningful action. Naturally, the foreground is modulated by the back-
ground of the students, that is, their “socially constructed network of relationships
and meanings” (Skovsmose, 1994, p. 179) that belongs to their personal history, but
the foreground provides resources and reasons for the students to get involved—or
not—in their learning as acting persons. In other words, the foreground allows stu-
dents to focus their intentions on the activities connected to learning. We see intentions
as primarily constructed from the a person’s foreground. So mathematical ideas can
become powerful to students in as much as they provide opportunities to envision a
desirable range of future possibilities.

Many studies have tried to identify what “powerful mathematical ideas” could
mean from a cultural perspective. In this context, “cultural perspective” refers to rad-
ical and political interpretations, for instance, as described in Frankenstein (1995).12
She tried to identify issues that specifically concern the political situation of working-
class, urban adults involved in remedial mathematics programs and showed how
questions about issues such as unemployment, military expenditure, taxation, and
economic policy can be dealt with as a central part of mathematics education. Being
able to handle such questions means developing a relevant competence for acting
politically as critical citizens. In this way, powerful mathematical ideas become de-

dined first of all with reference to the situation of the learner in a given sociocultural
situation. This radical perspective is also present in many of the chapters in Powell
and Frankenstein (1997).

Knijnik (1996, 1997) provided another example of culturally and politically power-
ful mathematical ideas in the case of the Brazilian landless movement. From an ethno-
mathematical approach, she, together with teachers from the community, found ways
of bridging the gap between academic mathematics and people’s popular mathemat-
ical knowledge as a way of enhancing possibilities of social change. The emergence
of a “synthesis-knowledge” that rescues and values popular understandings but also
raises awareness about its limitations, is one of the results of relevant pedagogical
work in mathematics for the community.

Mukhopadhyay (1998) also presented an interpretation of mathematics education
as a tool for adopting a critical stance toward current popular culture. She exemplified
her point of view with a mathematical investigation in the classroom about Barbie
dolls. This investigation, starting from making a model of Barbie of “normal” height,
can promote the adoption of a critical attitude toward the stereotypes with which
we are confronted and which have an influence on youth behavior, such as women
wanting to have a body like Barbie but having serious eating disorders in an attempt to
accomplish this goal. Generally speaking, mathematics education becomes powerful
in a cultural sense when it supports people’s empowerment in relation to their life
conditions.

Both Terrible Small Numbers and Macro-Figures Becoming Macro-Dangers illus-
trate what it could mean to consider the political dimension of the students’ culture.
Danish students know about salmonella poisoning, and many Colombian students
may have experienced the consequences of the disturbance in the logic of the UPAC-
system. Therefore, we know that it is possible to relate the content of mathematics edu-
cation to the students’ background. Nevertheless, it might be easy to miss the relation

12A much more narrow interpretation of “cultural” is found in, for instance, Seeger, Voigt, and
Waschescio (1998), in which the culture of the mathematics classroom is interpreted as first of all referring
to interaction and communication in the classroom. Other interpretations of culture are present in the
work of Cobb and colleagues, for whom the mathematics classroom is the micro-community of practice
where sociomathematical and more general social norms are built (Cobb, 2000).
with their foreground. How could our two projects touch students’ learning intentions by touching their foreground? We imagine that for some Danish students experimenting with the meaning of quality control in food products could generate a learning intention related to their capacity for making decisions about types of aliments appropriate for consumption. For Colombian students, especially for those who actually lived through the negative consequences of the break down of the UPAC system, we can imagine at least two significant ways in which their foreground is touched. For some, talking about the issue itself can be so painful that a resistance to get engaged in the topic will dominate. In this case, learning intentions could emerge in opposition to the proposed learning environment. On the other hand, for some students the project could generate learning intentions in relation to their capacity for helping their families make a critical decision about housing and real estate acquisition.

The issue of touching students’ foreground is a delicate point in some of the ethnomathematical approaches. Some studies identify mathematical competencies built into the students’ culture (for instance, competencies related to basket or fabric weaving and ornamentation in some Mozambican communities; Gerdes, 1996, 1997) as a starting point for mathematics education. However, there is no guarantee that, although belonging to the cultural background of a particular group of students, these geometric competencies will be considered relevant, engaging or motivating. Students’ intentions for learning might be related, first of all, to their foreground. Can ethnomathematics be criticized for providing restricted access to mathematical ideas or access to mathematical ideas without sufficient potential for touching the students’ foreground? We need to point in the direction of the potential of mathematical ideas for developing critical citizenship and mathemacy as efficient tools for a critical reading of mathematics and of how mathematics may operate in the social environment.

... Sociologically Speaking

Powerful mathematical ideas can be investigated from a sociological perspective as well. Such ideas can be defined in relation to the extent to which they are used as a resource for action in society.

Mathematics does not exist as independent knowledge in society. Social actors, not only mathematicians, use mathematics as a descriptive and a prescriptive tool. Mathematics, including its applied forms such as engineering mathematics and mathematical economics, is part of the available resources for technological action, involving planning and decision making. We use the term technological action (Skovsmose, 1994) in the broadest possible sense, including making decisions about, for example, how to manage the economy of the family, establishing a new security system for electronic communications, investigating traffic regulations, organizing insurance policies, instituting quality control of mechanical constructions, providing a booking system for airlines, testing of algorithms and computer programs, and many other activities that today are present in most working places. In various ways, mathematics constitutes a resource for such actions.

To express this rapidly growing multitude of “actions through mathematics,” we highlight three characteristics of the way in which society operates with and through mathematics in technological enterprises. First, using of mathematics, it is possible to establish a space of (technological) alternatives to a situation. Mathematics also

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13 Teachers, working from a culturally empowering perspective, may face cases of students resisting the teachers’ “empowering” game because the students can envision traditional teaching as a valuable contribution to their foreground.

14 For a critique of ethnomathematics, see Vithal and Skovsmose (1997).
provides a limitation on this space of alternatives, however. In this sense, mathematics serves as a source of technological imagination, which is limited by many blind spots. Second, mathematics allows us to investigate particular details of a not-yet-realized plan. However, hypothetical reasoning about details of imagined constructions, supported by mathematics, also lays a trap because mathematics imposes a limitation of the perspectives from which hypothetical situations are investigated. In particular, risks can emerge from the gaps in hypothetical reasoning, which might overlook whole sets of consequences of certain technological implementations. Finally, as a resource for technological action and decision making, mathematics becomes an inseparable part of our present reality and of other aspects of society. We come to live in an environment created and supported by means of mathematics. In particular, the development of the informational society is closely linked to the spread of mathematical based technologies.\footnote{For a discussion of mathematics in action and the notion of the formatting power of mathematics, see Skovsmose (1999) and Skovsmose and Yasukava (2000).}

Talking about powerful mathematical ideas, sociologically speaking, we do not simply have in mind a long list of mathematical modeling examples or applications of mathematics that have been presented in many textbooks serving the purpose of motivating students and illustrating that mathematics can be an useful subject.\footnote{De Lange (1996) presented a discussion of applied mathematics in education. We find that his view of applied, realistic mathematics is in many respects different from what we see as sociologically relevant and powerful.} We want to draw attention to the fact that mathematics operates as an integrated part of many technological actions and that such actions, as any other, may have unpredictable positive or negative consequences. The quality of the consequences of a technological action is not guaranteed by the quality of the mathematical base behind it. The three aforementioned characteristics of how we operate with and through mathematics may help one to grasp the complexity of mathematics in action and draw attention to the basic uncertainty associated with any mathematical idea put into operation in any technological context.\footnote{For a discussion of uncertainty about mathematics, see Skovsmose (1998, 2000a).}

Therefore, a critique of mathematics in action is necessary.

We can illustrate some of the aspects of powerful mathematical ideas, sociologically speaking, by considering Macro-Figures Becoming Macro-Dangers. This project builds on an actual situation in which a mathematical model has been the basis of an economic policy that has great social impact. A primary task for the students could be to reveal the connections between the blind spots of the hypothetical reasoning of the model and the emergence of certain social uncertainties. A particular issue is to consider the relationship between the growth on the one hand of the functions $f_i(t) = (1 + i)^t$, $f_e(t) = (1 + e)^t$ and $f_r(t) = (1 + r)^t$, and, on the other hand, the growth of the function $f_n(t) = (1 + n)^t$, where $n$, $e$, $i$, and $r$ are connected by the formula $n = (1 + i)(1 + e)(1 + r) - 1$. By studying these functions, we witness some elements of the logic of the UPAC system. We experience the drama of “actions through mathematics.” The system of mortgage payment is determined by this logic. In particular, it becomes relevant to clarify to what extent the growth of $f_n(t)$ gets out of control (economically speaking), even though the growth of $f_i(t)$, $f_e(t)$ and $f_r(t)$ seems “reasonable.” Thus, the project can illustrate how mathematical calculations used for social decision making can provoke new risk structures for certain groups of people.

Terrible Small Numbers also shows the relationship between risks and mathematics. In this project, students were brought into a situation in which economic and epistemic interests were confronted. This contradiction is exemplary in many technological design processes. How much additional investigation is needed to make an educated decision? How large a sample of eggs must be investigated to decide...
whether to put the eggs on the market? In many cases, mathematical modeling makes it tempting—and possible—to jump into conclusions about what to do; such conclusions may bring new risk structures to our future.

Thus far in our analysis, we have tried to show different interpretations of “powerful mathematical ideas.” Each one can be related to central notions such as the level of abstraction in the mathematical architecture, the meaningfulness of acquired mathematical notions, the way learners can experience an empowerment as citizens, and the critical concern about how mathematics operates as a resource for action in a technological environment. Each one of these interpretations can suggest a response to the paradoxes of inclusion and citizenship. Thus, if the logical and the psychological interpretation dominate, the paradoxes seem to disappear. What could be more important in the mathematics classroom than bringing students to master the highest level of abstraction in a meaningful way? Considering the sociological and the cultural interpretation of “powerful”, both paradoxes reappear in a strong version. Mathematics education cannot ignore them. The discussion of providing democratic access to powerful mathematical ideas becomes more complex when we also consider the notion of democracy, however. We discuss this issue in the following section.

DEMOCRATIC ACCESS

All students, everywhere in the world, have the right to education. We can go further and say that all students in the world should have the chance to learn mathematics. Democratic access, in this sense, refers to the actual possibility of providing “mathematics for all.” However, the idea of “democratic access,” understood as the right to participate in mathematics education, is more complex. Here we discuss in more detail what the expression can represent.

“Democratic access” designates the possibility of entering a kind of mathematics education that contributes to the consolidation of democratic social relations. As we have argued elsewhere (Skovsmose & Valero, 2000), a critical view on the connection between mathematics education and democracy situates the notion of democracy in the sphere of everyday social interactions and redefines it as purposeful, open political action undertaken by a group of people. This action is collective, has the purpose of transforming the living conditions of those involved, allows people to engage in a deliberative communication process for problem solving, and promotes collection—that is, the collective reflection or thinking process by which people “bend back” on each other’s thoughts and actions in a conscious way (Valero, 1999).

Democratic access in mathematics education, in the sense we have indicated, is played out in many different arenas in which the practices of mathematics education take place. We comment on three such arenas that we believe are fundamental: the classroom, the school organization, and the local and global society.

In the Classroom

The mathematics classroom is a microsociety in which democratic relationships between students and teacher and among students must be present if education is to provide any form of democratic access. Democratic relationships that allow collaboration, transformation, deliberation, and collection are central in opening possibilities for a critique of mathematical contents in the class and of their significance in social actions based on them.

Communication in the classroom can follow many patterns, but to establish a spirit of democracy, dialogue and critique are indispensable components. Thus, a mathematics classroom governed by bureaucratic absolutism or by a communication that does not incorporate possibilities for politicizing the mathematical education
experience does not represent democracy. Alrø and Skovsmose (1996) provided an example of a communicative model with a democratic concern in mathematics education. The inquiry cooperation model refers to a variety of communicative acts supporting an inquiry process. Such a process cannot simply be guided by the teacher; rather the students act in the process of investigation in cooperation with the teacher. The elements of the inquiry cooperation model are: getting in contact with, discovering, identifying, thinking aloud, challenging, reformulating, negotiating, and evaluating.

The nature of some of these acts can be clarified with reference to the project Terrible Small Numbers. When the students carry out the experiment related to the quality control of eggs, the process is not reduced to a set of exercises organized in a certain sequence. The openness allows students to “own” the learning process and to experience what it could mean to be responsible for making decisions. When the students work on their own and the teacher wants to intervene, students should not feel threatened in their ownership of the process of investigation. The teacher has to get in contact with students and then she can challenge them: How could it be that in some of the samples there are more that one egg with salmonella? The students can try to identify sources for explanation: Maybe the teacher did not mix the eggs sufficiently? Discoveries can be made: Could it be that samples do not always “tell the truth” about the whole population? During the process of negotiation in which different possible explanations are considered, thinking aloud is possible. Thinking aloud is a way of providing public access to a line of thought, and it can be open for negotiations and reformulations. Any result of such a process can be evaluated.

In the section Sociologically Speaking . . . , we outlined three aspects of actions through mathematics: Mathematics helps to open possibilities by providing the basis for a technological imagination; mathematics supports investigation of particular aspects of not-yet-realized constructions, and, when realized, mathematics operates as an integrated part of the technological device. If such aspects of mathematics in action should be addressed critically in the mathematics classroom, then mathematical content needs to be contextualized, not only in terms of the provision of a “task context” but mainly in terms of a “situation context” (Wedege, 1999). In Terrible Small Numbers, it would not be possible to introduce a discussion about reliability and make students experience what it would mean to make decisions if the figures were not strongly related to actual situations happening in the social, political, economic, or cultural environment in which learning takes place. In general, we find that contextualization is a precondition for problematizing “trust in numbers.” Such problematizing is essential for establishing a critical citizenship.

The contextualization is not simply a motivational device—although it might be motivating. It is a condition for establishing a discussion of how mathematics can operate as a source of power in a sociological sense because it invites a critical examination of how mathematics in fact is put into operation. A rich contextualization could help an inquiry cooperation model enter the classroom, and in this way influence the structure and content of the discussion.

We are aware that Terrible Small Numbers took place in a school situation and that this particular context provides a frame for interpreting the activities. Students worked with eggs that were not real, and their calculations had no actual consequences. Although they calculated the risk of producing an ice cream dessert out of six eggs, there was no real ice cream production in the classroom. Still, there is an important difference from a traditional exercise context in which a problem can refer to prices, goods, and amounts to be bought, but in which these prices, goods, and quantities operate in a completely different way from real prices, goods, and quantities. The traditional school mathematics exercise is accompanied by a set of metaphysical assumptions, notably that the description provided by the text is exactly true and it cannot be challenged. If a problem makes us buy apples and their price is set at $3.10 per kilo, it will
not make sense if one student knows that the same apples can be bought around the corner for $2.30. If we are asked to buy 3 kg of apples, it does not make sense to question whether we can expect the scale to show exactly 3 kg—although we all know that apples are big units and it is difficult to have a weight of exactly 3 kg. The information provided by the text of the exercise is what we need for solving the problem, and the problem has one—and only one—correct answer (Mukhopadhyay, 1998; Skovsmose, 2000b).

An essential task of the contextualization is to crack the metaphysical assumptions of the exercise paradigm. This metaphysics was challenged by Terrible Small Numbers, and this is essential for a critique to make sense. Opening the classroom for in-depth reflections is a condition for mathematics education to be part of a democratic endeavor.

In the School Organization

Although many opportunities for establishing democratic access to powerful mathematical ideas are present in the classroom, this is not the only or the main arena for establishing such an access. In fact, recent research has acknowledged the importance and necessity of understanding classroom practices in connection with the whole context of the school organization and, more generally, the educational institution (Krainer, 1999; Perry, Valero, Castro, Gómez, & Agudelo, 1998; Stein & Brown, 1997; Valero, 2000). Teachers and students in the classroom are not isolated from the way mathematics teachers and school leaders work to shape mathematics education through curriculum planning and teachers’ professional development. So, when we discuss democratic access, we must also consider how mathematics education practices in the school as a whole operate and create opportunities for, or obstacles to, this endeavor.

In the context of the school organization, we draw attention to the importance of who organizes the curriculum and how it gets organized. Let us assume that we, as well-intended policymakers, want to provide a curriculum that ensures students get democratic access to powerful mathematical ideas—indeed, independently of what interpretation of “powerful” we have in mind—and that we, as a result, offer a detailed plan including topics and ways of working in the classroom. Let us assume, furthermore, that this curriculum is put into operation. The detailed planning itself will obstruct the realization of our democratic intentions because the top–down model closes possibilities for the people involved in the actual curriculum development to own the process.

This conflict points to the basic complementarity in curriculum thinking. The very process of planning, carefully and in detail, access to any kind of ideas obstructs the possibility of making this access democratic. The latter presupposes that teachers, students, and school leaders are acting subjects in identifying, planning, and implementing the curriculum. (Naturally, other groups such as parents, could be considered as well.) This view implies that certain curricular decisions need to be taken in the community of participants in the school mathematics practices. The way a curriculum is organized, then, depends on the relations inside a network of school mathematics practices (Valero, 2000), where teachers as individuals, mathematics teachers as a group, the students, and school leaders can share their expectations about the meaningfulness of a mathematics educational experience. Who participates in formulating a curriculum and how such a formulation is implemented are

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18Vithal (2000a) elaborated on the notion of complementarity in mathematics education as the possibility of “bringing together irreconcilably conflicting but necessary positions or theories” (p. 307) to provide a better and fuller understandings of what we study in mathematics education.
constantly implicated in an unsolvable and necessary tension between specificity and freedom.

The planning of Macro-Figures Becoming Macro-Dangers is a microcurriculum design process in which basic components are identified and developed in close connection to the classroom. Practicing teachers, as a response to their own and their students’ experiences, identified the idea of the project. It was not a suggestion from a textbook or an external authority, but it emerged from a situation that needed understanding because it was affecting the life of the school community. The potential for collaboration among teachers and students in the development of the project, as well as for transforming their understandings, and eventually their situation concerning mortgages, are key elements in the project. Were it predetermined that the projects should serve as an introduction to, say, algebraic calculations, then the significance of the projects could easily be lost. Then it would be impossible for the students, or for the teachers, to maintain ownership of the project. The experimental character of the curriculum design process exemplified by macro-Figures Becoming Macro-Dangers highlights the importance of creating a “laboratory for curriculum development” (Vithal, 2000a). This notion refers to cooperation between different participants in the network of school mathematics practices to build an open curriculum planning process that acknowledges democratic concerns.

**In the Local and Global Society**

There is a contradiction between establishing mathematics education in terms of democratic access to powerful ideas and, at the same time, letting that education serve differentiation functions in society by, for instance, ranking students in a way that significantly influences their future career possibilities. The emphasis on high-stakes assessment or classroom assessment in most countries can clearly contribute to the paradoxes of inclusion and citizenship (Morgan, 2000). Furthermore, mathematics education—at least in certain forms—generates different situations of exclusion according to gender, race, language, class, and abledness (e.g., Grevholm & Hanna, 1995; Keitel, 1998; Khuzwayo, 1998; Rogers & Kaiser, 1995; Secada, Fennema, & Adajian, 1996; Zevenbergen, 2000a). If we want to end this exclusion, then we should allow entry into mathematical learning to all.

The difficulty of establishing mathematics education as a democratic resource can be illustrated clearly by the following dilemma. Ethnomathematics (D’Ambrosio, 1996; Powell & Frankenstein, 1997) has represented a challenge to Eurocentrism first by demonstrating that all cultures, not least indigenous ones, demonstrate a deep mathematical insight and second, by showing that building on this knowledge makes it possible to reconstruct a mathematics education that does not recapitulate the priorities of colonisation. This has led to the formulation of ethnomathematical curricula for disadvantaged populations such as, for example, the landless people in Brazil (Knijnik, 1997) or Mozambican peasants (Gerdes, 1997). Could it be, however, that offering ethnomathematical education to certain disadvantaged groups prevents them from being active members of the informational society and therefore dooms them to a life in the Fourth World?

In a similar way, critical mathematics education (Skovsmose, 1994; Vithal, 2000a) has been proposed as an educational philosophy to address the risk of a mathematics education that contributes to the creation of citizens uncritical toward the devastating effects of mathematics in society. Nevertheless, in a research and development project intending to open possibilities for critical mathematics education with immigrant students in Catalonia (Gorgorió & Planas, 2000), the researchers perceived certain interpretations of that type of education as a “soft” program that could be suitable for these particular kinds of students. This view contrasts with the position—nonexplicit, but nonetheless easy to elicit in actions and proposals—of the educational authorities
defending the need for “hard-core” mathematics education programs for those students expected to succeed within the educational system, in particular, the local students. We see that this interpretation could lead to a situation in which so-called critical mathematics programs serve as a second-rate curriculum for immigrants and political refugees because, after all, it may not provide them with the “hard” mathematical knowledge needed for climbing the ladder of social prestige in that community. Here we directly face the paradox of inclusion. In this way, an attempt at inclusion could result in growing exclusion, and a concern for citizenship could come to establish citizenship among the excluded.

In the creation of the Fourth World in the present informational society, as described by Castells (1999), the barbarism of the paradox of inclusion is associated with mathematics education. Mathematics education could help to secure access to the informational society as well as to establish and legitimize exclusion from it. For a teacher in an underresourced educational system, it is difficult to provide new possibilities in life for the students beyond what is already well known to them as their background. Thus, a fundamental discussion about mathematics education in many developing countries concerns the relocation of resources as a way of distributing possibilities in radically new ways. If this does not happen, then, for instance, historically Black schools in South Africa are doomed to belong to the Fourth World. The situation in South Africa is exemplary for the problem of how an unequal distribution of resources obstructs democratic ideals.

One particular aspect concerning mathematics education and the informational society is the use of technology in teaching and learning. We must consider how mathematics becomes installed in more and more technological devices (Wedeg, 2000) and how it operates “behind the screen,” making it possible to use mathematized tools without presupposing a deep understanding of their underlying mathematical structure, maybe even without being aware of the fact that a complexity of mathematics is in operation. The implication of this is that the necessary competencies to operate with these technologies has split people into two categories: those who can operate on the surface of the technology and those who can construct and reconstruct it. Both competencies are essential, and therefore it is important to discuss how mathematics education operates with regard to this split. Furthermore, we must discuss how mathematics education accomplishes its global function in a world where access to computers is still reserved for a select few. Given the nature of the information society, mathematics education occupies a sensitive position in which possibilities in the information age are distributed among students, regions, and nations. In mathematics education, the barbarism of this distribution is particularly visible.

Globalization also concerns the particular content of what is learned. Macro-Figures Becoming Macro-Dangers addresses issues that represent general aspects of how risks and uncertainties are distributed. The project tries to illustrate how large-scale economic figures and decision making can have particular effects and contribute to the creation of macro-dangers. This transformation from figures to dangers is a basic feature of globalization, where large-scale decision making distributes risk and uncertainties in formidable ways. In this sense, Macro-Figures Becoming Macro-Dangers has a particularly exemplary value.

**FACING THE PARADOXES**

Let us recapitulate the paradoxes of the informational society as we originally described them. The paradox of inclusion refers to the fact that the current globalization, which proclaims universal access and inclusion as a stated principle, can also be associated with processes of exclusion. As part of the development of a universal
framework for global connections, strong processes of exclusion and isolation are simultaneously taking place. Among other things, this brings about a “Fourth World,” many new citizens of which are currently to be found in mathematics classrooms. The paradox of citizenship refers to the celebration of a learning society that emphasizes the need for relevant and meaningful education for the further development of social, political, and cultural structures, although in reality that education may have only a functional relevance for the system.

Does mathematics education in fact face the paradoxes? Up to now, we have referred mainly to mathematics education as a field of practice, but now we concentrate on mathematics education as a research field. In Fig. 16.1, we present the space for investigating democratic access to powerful mathematical ideas.

Reviewing research literature in mathematics education, there are unfortunately different ways in which the field ignores the paradoxes of inclusion and citizenship. One way of doing so is by concentrating on particular interpretations of powerful mathematical ideas, mainly the logical and the psychological ones in which the emphasis is placed on the development of mathematical thinking independent of any context in which it takes place. The second way is by ignoring that mathematics education is part of a democratic endeavor, or simply (and rhetorically) assuming that mathematics education, due to the very nature of mathematical thinking, constitutes a profound democratic enterprise. In this view, powerful mathematical ideas have an intrinsic democratic value (Skovsmose & Valero, 2000). A third and more moderate way is by considering only selected aspects of what democracy can involve. In particular, much discussion has focused on democracy in the classroom but ignored the other arenas in which meaningful participation in political action through different kinds of powerful mathematical ideas is built.

Gómez (2000) carried out a classification of the papers published in 1997 in three main international journals in mathematics education, the Journal for Research in Mathematics Education (JRME), Educational Studies in Mathematics (ESM), and Recherches en Didactique des Mathématiques (RDM), and in the proceedings of the 21st meeting of the International Group for the Psychology of Mathematics Education (PME). Although he acknowledged that his sample is not representative of all international research published in the area, he considered it an indicator of the type of research carried out. He concluded that

“mathematics education research production is centered mainly on cognitive problems and phenomena; that it has other minor areas of interest; and that it shows very little production on those themes related to the practices that influence somehow the teaching and learning of mathematics from the institutional or national point of view.” (p. 95)
Translated to our space of investigation, Gómez’s results indicate that there is a high concentration of work in the lower, left-hand portion of Fig. 16.1. To check these results, we classified the papers published in JRME, RDM, ESM, *For the Learning of Mathematics* (FLM), *Suma*, and the *International Journal of Mathematics Education in Science and Technology* (IJMEST) published between January 1999 and October 2000, in the different portions of our area of investigation. The results are indicated in Fig. 16.1.

This distribution shows that the majority of papers are concerned with interpretations of powerful mathematical ideas in a logical and psychological sense and in the arena of classroom interactions. There is, however, also a considerable number of papers highlighting cultural interpretations. However, there are areas, such as the society and school arenas for democratic access and the sociological interpretation of powerful mathematical ideas, that either have not been explored to any great extent or that are a low priority for publication in research journals. Naturally, we are not claiming that each and every research project in mathematics education should address the full range of issues mentioned. However, it is highly problematic that dominant research trends in mathematics education operate within a limited scope of the space of investigating democratic access to powerful mathematical ideas. Such a paradigmatic limitation effectively obstructs the possibilities for mathematics education to face the paradoxes of the informational society.

What could it mean, then, for mathematics education research to face the paradoxes of inclusion and citizenship? We attempt to offer possible answers by raising clusters of questions that point in the direction of under- or nonresearched issues.

1. Democracy, understood as a collective, political action for the purpose of transformation, is lived through everyday experience, including the mathematics classroom. How do particular forms of mathematics education, including interaction and communication in the classroom that they generate, acknowledge democratic values? Which are the forms of interaction in the classroom that open possibilities for politicization and critique of both the mathematical content and the interaction itself? How does mathematics education acknowledge that the microsociety of the classroom is related to the local and global society where students live? Are forms of learning in school related to forms of learning in workplaces and organizations and in everyday situations?

2. The contextualization of school mathematics is an important gateway into cultural and sociological interpretations of powerful mathematical ideas. Do we deal with a contextualization that primarily observes the metaphysics of the exercise paradigm, or must we deal with deeper, real-life references? Does the contextualization of school mathematics touch on both the students’ background and foreground in significant ways? Do we try to illuminate issues in which the content of mathematics education prepares the students to operate as critical citizens in a context where mathematics and mathematically based decision making are in operation?

3. The dynamics of school mathematics practices, understood as the complex interaction among teachers, school leaders, and students in the school organization, needs exploration. A particular issue concerns who participates in the curricular decisions and where they take place. Does curriculum planning and implementation open possibilities to bring into the classroom different interpretations of what powerful mathematical ideas mean? What do teachers as a group and school leaders

19 At this point, we could also enter into the discussion of what it means to research a situation that “does not exist” because there is also a connection between deficiencies in the area of investigation and deficiencies in actual practices in school mathematics. Because this discussion is broad and it is not our intention to tackle it here, we merely mention the work of Skovsmose and Borba (2000) and Vithal (2000a, 2000b), who have tried to approach this question.
value as an appropriate mathematics education, given their students’ backgrounds and foregrounds? In particular, it is important to consider how local curricula can operate in society. Could a particular curricular design and implementation constitute “second-rate” mathematics education, which dooms students to exclusion or to uncritical acceptance of society? How does the process of exclusion of certain social groups—defined in terms of gender, race, class, and ability—operate in the school organization as a whole?

4. It is relevant to consider how information and communication technologies (ICTs) open and reorganize new learning possibilities (Balacheff & Kaput, 1996; Borba, 1997, 1999). What do ICTs mean in boosting culturally and sociologically powerful mathematical ideas? Part of this understanding has to do with identifying the state of actual global distribution of ICT learning possibilities. Obviously, we have to work with an unequal distribution of ICT facilities around the world. What does this mean for the role of mathematics education in underresourced classrooms and schools? In particular, what does this imply for the formation of the “Fourth World”? Does the reorganization of learning possibilities also include a reorganization of inclusion, as well as exclusion, from the informational society? Is research in mathematics education too often set up in such a way that certain social and economic resources are taken for granted, although they can be taken for granted only in certain (privileged) parts of the world?

5. As we have indicated previously, mathematics operates as a resource of power in a variety of actions and decision making in all areas of life. Does mathematics education open possibilities for students to see this resource in operation? How can “actions through mathematics” be illustrated in mathematics education? How far do we go in making mathematics education a critical activity, addressing both the wonders and the horrors of actions through mathematics? What does it mean to offer a mathematics education that tries to illustrate such contrasting aspects of powerful mathematical ideas?

6. Finally, through mathematics education in all its arenas—the classroom, the school, and society—we contribute to the construction of public images of mathematics and mathematics education. How does this process of building social images and ideologies of mathematics and mathematics education happen in the different practices of mathematics education? Which are the characteristics of the discourse that we construct so that it can actually attribute so much power and democratic relevance to our subject? What are we doing in the classroom, in schools, in society, to strengthen mathematics as a powerful form of knowledge? What are the broadest social consequences of our practices? Could one be the reproduction of a world in which the paradoxes of equality and citizenship can easily survive?

If, as mathematics educators in research and practice, we are concerned about the lives and experiences of students like Nicolai and Carlos, we should consider even more seriously the importance of broadening our interpretations of what democratic access to powerful mathematical ideas means. Furthermore, we should keep in mind the necessity of tackling the inclusion and citizenship paradoxes in our endeavor during the coming century.

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Since the earliest days of education research, scholars have attempted to develop means by which their work can be made more relevant and influential to teachers. Dissemination of research findings to the practitioner community therefore has historically been a prime goal for educational scholarship. In fact, the National Science Foundation, the largest single source of monetary support for mathematics education research in the United States, has made dissemination of research to a wider audience than the community of scholars a primary criterion for funding. In addition to support for research, the National Science Foundation, the Office of Educational Research and Improvement (of the U.S. Department of Education), private foundations, and corporate sponsors contribute significantly more funds for research-based staff development and systemic reform (i.e., dissemination) than for research itself.

We realize that the audience for this chapter is international and therefore has little interest in the sources of funding for U.S. projects. However, the allocation of federal funds dictates, to a large extent, the relative worth to which education-related projects are held in the United States. This emphasis on dissemination exemplified in the current U.S. political and economic scene is echoed in other industrialized nations across the globe. In part, this attention on dissemination stems from a panglobal economic
and political need for improving the capability of workers to mine data, predict patterns, forecast trends, as well as merely understand the workings of a digital environment. Mathematics is one key to success in these endeavors, and research in mathematics education is a key source of inspiration for instituting change. Despite this political emphasis on dissemination, however, the general practice of “staff development” as it relates to deep and widespread infusion of research-inspired practices into the mathematics classroom is primitive at best. Short workshops, the most common form of “professional development,” are typically delivered as self-contained presentations and do not anchor themselves in the ongoing concerns of teachers; for this reason, they do not contribute to any significant change in teachers’ practices (Gamoran, Secada, & Marrett, 2000). It is easy to see why such short-term solutions are attractive despite their relative impotence. Economically, the cost of bringing in a scholar from a university, government, or private company for a day or two is inexpensive. A 2-day workshop can be enveloped into the school’s allotted staff development days (where teachers are under contract), and therefore the school will not have to pay extra to bring in substitute teachers or to pay stipends for teachers who attend the sessions.

Moreover, culturally, “dissemination”-as-workshop fits with a socially agreed on convention for what counts as knowledge and what counts as teaching. Knowledge is a “thing.” Teachers package and present knowledge to students. Students receive the knowledge and apply it to the diverse situations to which it is applicable. The workshop format merely extends this flow of goods to include another tier: Education researchers as knowledge producers and distributors. Given that this genre is largely ineffective (i.e., knowledge does not “trickle down” from the elite to the masses), an alternative conceptualization of the relationship between research and practice, indeed between researchers and practitioners, must be forthcoming if education research is to have any practical value beyond that of the ivory tower.

In addition, there has been debate on the adequacy of methods by which researchers (here meaning the “ivory tower” type) come to know, that is, construct, the knowledge base on which reform documents are occasioned (Schoenfeld, 1999). Experimental studies in particular have been decried as being too esoteric or divorced from the reality of classrooms and ethnographic or qualitative work has been projected as being more relevant to practice because of its “thick description.” A recent study by Kennedy (1999), however, found that research genre has little to do with the perceived utility, relevance, or interest level of research for teachers. Instead, she found that teachers seem to be capable of reading research from a variety of genres and, in fact, create meaningful analogies for informing their own practice regardless of whether the work was nomothetic, idiographic, narrative, or expository.

So what is it that makes some research more applicable than others to teachers? It appears that the substantive question the study addressed makes all the difference. The most influential studies, according to the teachers in Kennedy’s article, were those that addressed the relationship between teaching and learning. In other words, studies that engage teachers in answering the question, “What do the tasks, tools, verbalizations, and structures I design occasion in the thoughts and practices of my students?” This is what teachers are all about.

CLASSROOM RESEARCH

Those of us who are interested in such questions can take heart from these findings. It follows that the better we address the relationship between what teachers do and what children learn, the better able we will be to inform practice. It also follows that the closer the fidelity of our work to the conditions of work of teachers and the characteristics of students teachers might expect to meet, the better our arguments
about the conditions under which such relationships apply can be made. To ensure such fidelity, classroom researchers are increasingly developing relationships with teachers that amount to that of collaborative partnerships focused on the improvement of teaching and learning on the one hand, and systematic study of teaching and learning on the other (e.g., Kelly & Lesh, 2000). In this chapter, we define “classroom researchers” as those who do research in classrooms as opposed to on classrooms. This distinction is important, we think, to bound the kind of work we do as a discourse around the problems arising in teaching and learning with the particular lens of problematizing the mathematical and scientific practices as they are enacted by teachers and students. Usually, such work is situated in a larger effort such as curriculum development, design of technologies, the study of equity and diversity, or, broadly put, systemic reform.

In particular, the systemic research in mathematics reform, which studies systems as varied as self-contained classrooms (e.g., Jacobson & Lehrer, 2000) to large national-level reform projects (e.g., Webb & Romberg, 1994), has broken the age-old tradition of “researchers” studying “teachers” impacting “students.” Instead, a new model of teacher as professional is emerging. Under such a model, teachers are seen as knowledge producers, curriculum designers, and policy analysts with a unique configuration of knowledge, skills, and practices that have merit in the larger order of knowledge production, curriculum design, and policy analysis. Action research, design experiments, and teacher research (e.g., Ball, 2000; Cochran-Smith & Lytle, 1999) are becoming an accepted form of critical inquiry into instructional practices, and collaboration with teachers in the collection, analysis, and interpretation of classroom data is becoming an acceptable and desirable norm (Doerr & Tinto, 2000).

Even a cursory analysis of the characteristics of these genres of scholarship reveals a redefinition of the role of an academic researcher on the one hand and a classroom teacher on the other. To illustrate this contrast in roles, we cite two relatively recent innovations in research methodology: teacher-as-researcher (Henson, 1996) and the classroom teaching experiment (e.g., Cobb, 2000; Steffe & Thompson, 2000). Classroom research, as we present it, is a sizeable superset containing each of these established genre and as such includes teachers with this philosophy in mind.

In the teacher-as-researcher literature, the practice of teaching is seen as a process of inquiry. That is, the properties of a sociocultural setting, as they are impacted by introduced notations, forms of argument, and emergent meanings, afford interesting questions for reflection, experimentation, and engineering. Moreover, the questions of interest in this approach to inquiry are those of the teacher, stemming from her or his more or less immediate needs for solving problems related to her or his practice. The methods of analysis are designed to fit the situation at hand and are modified as the situation dictates. Lastly, the dissemination of produced knowledge takes the accepted forms of the teaching profession—primarily narrative in style, discursive in form, and informal in tone (Cochran-Smith & Lytle, 1999).

As an example, in a recent project in the Phoenix, Arizona, urban community, we engaged a large number of teachers in the study of children’s mathematical thinking as a basis for the reform of their own practice based on the model of cognitively guided instruction (Jaslow, Middleton, & Hertzog, 2000). A subset of these teachers elected to participate in a 15-week course that used the model of teacher-as-researcher as a mechanism for supporting their unique needs and questions of interest but also for disseminating their understandings across a larger audience. The first author of this chapter was the instructor of the course and served as a technical advisor for questions of method. The makeup of the set of teachers was diverse, ranging from five teachers in a single school to a mentor teacher and her “mentee” to a graduate assistant teaching mathematics methods for elementary teachers. Over the course of the semester, initially tentative questions of interest became shared dilemmas that the class worked
over to formalize and operationalize. Because the questions of interest originated in the practices of the participants, each had an immediate applicability to the reform of those practices—that is, as information about the qualities of children’s thinking became available that suggested alternate courses of action might be fruitful, the teachers were able to make those changes, document their impact, and disseminate their analyses to their colleagues in the class on a weekly basis. Moreover, because the work was collaborative, the models of practice that individual teachers developed were situated in either a series of studies across age levels or were applied across diverse settings (e.g., language groups), such that the group work had a coherence that was transportable across classroom boundaries. As the course progressed, the questions of interest of one group became questions for discussion for all groups. Thus, as time went on, an initially ad hoc agglomeration of individual teachers or (small groups) became a community with emergent properties that defined an epistemology of scholarship and an identity rooted in both teaching and research (although still primarily in teaching). Many of those teachers became leaders for reform in their own local communities, continuing the Cognitively Guided Instruction (CGI) work even after the initial grant had expired (Lehrer, in preparation).

With regard to the classroom teaching experiment genre, Cobb (2000) suggested that researchers who collaborate with teachers in classroom research serve as leaders in a local pedagogical community defined as the research and development team. This kind of leadership gives the researcher considerably less control over the innovations under study than either laboratory studies or studies in which the researcher is the teacher. However, the conditions under which the study may progress maintain a closer fidelity to the conditions under which the innovation is expected to be implemented. Because the members of the research and development team come together with varied backgrounds and expertise (by the way, we maintain that teachers’ “wisdom of practice” is as legitimate and important to the quality of classroom research as that of researchers), the development of this community is likely to take a different trajectory than the researcher originally envisioned. Like any social system, a number of emergent properties will arise from the give-and-take that force the team to adopt each other’s practices. For example, in a recent study by the first author of this chapter (Middleton, Poynor, Wolfe, Toluk, & Bote, 1999) the entire focus of the classroom inquiry evolved from an initial examination of bilingual strategies used in teacher questioning to a detailed, phrase-by-phrase analysis of the structure of teachers’ questions to ascertain why the level of discourse in the classroom was mathematically deeper than other classrooms with similar circumstances. This shift occurred when the original question ceased to be helpful for the classroom teachers, and their own action research spurred them to raise questions about how to stimulate students to reveal their thinking and how to use those revelations as the basis for class discussions.

**EMERGENT COMMUNITIES**

This chapter treats the development of teacher partnerships as the study of emergent communities. The analytic lens we use hinges on the study of relationships among teachers, researchers, and others involved in serious educational inquiry. We focus this lens on two levels of emergence: that of the relationships among teachers and researchers on a research team, attempting to both understand and reform mathematics and science instruction under the conditions that make the research fundamentally situated in practice, and that of researchers studying relationships among teachers and others involved in such systemic reform. That is, teacher partnerships are both a mechanism and an object for serious inquiry in mathematics and science education.
The former level produces findings that inform teaching, learning, and development of curricular sequences (e.g., Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997). The latter produces findings that impact policy, staff development, and support mechanisms. Both levels are necessary; each informs the other. The purpose of the research in each level is quite different, however, and the relationships between the researcher and teachers is dramatically different.

What do we mean by an emergent community? In short, emergent systems are self-organizing systems that become successively more adaptive in response to increasing systematic complexity—the more complex the system becomes, the more adaptive it becomes (Union of International Associations, 1995). An emergent community, then, engages itself in the kind of activity Wenger (1998) ascribed to communities of practice, namely the definition and pursuit of a common enterprise; mutuality of engagement in shared activity; the accumulation of a history of shared experiences and interacting trajectories that shape identity; the development of interpersonal relationships; the production of local regimes of competence; the management of boundaries; and the opening of peripheries that afford different degrees of participation. Emergence, in fact, is one of the key features distinguishing a situated sociocultural perspective on learning in general, as opposed to a primarily individual or exclusively cultural perspective (e.g., Lave & Wenger, 1991; Wenger, 1998).

A good analogy for understanding how the emergent properties of a social system arise and then are constrained by the activity in which participants engage is that of improvisational jazz (see also Sawyer, 1999). When a combo engages in the performance of a composition, each musician is allowed considerable freedom to “take it,” meaning that each can alter the rhythm, tempo, melody, and key of their own contribution to the performance. Because the improvisation is not fixed but emerges from the mutual reaction of each musician to the other, each performance, each coalescence of the particular community of musicians, has a unique aesthetic quality. Although no two performances of the “same” composition are alike, the structure of the piece is not entirely free, however, and so it could also be said that no two performances are entirely different either. The conduct of the performance is structured and channelized by the composition, facilitating coherence in the improvisational behaviors of the group (e.g., Wenger, 1998). This analogy of course stems from the emergent perspective on education research (Cobb, 2000). This particular lens asserts that activity is at once both communal and individual whereby group norms, like a jazz composition, channelize the practices of individuals but in which the qualities of that practice are determined by the configurations of individual improvisation.

It must be said that having a composition or norms of behavior through which the group’s activity is constrained does not ensure coherence or even quality. Individuals in the combo (or classroom or research team) may attempt to upstage each other, may misinterpret each other’s actions, or may have too little experience in improvisation to behave competently under the constraints of that particular genre. The composition merely sets the rules by which the group’s activities can be judged by a wider community of musicians and provides a common purpose by which the behavior of the community can be coordinated.

In much the same way, to be coherent the activity of a research team, must be united by a common purpose. The “teachers” and the “researchers” (this distinction often becomes blurred) must view the innovation to be studied as a common value and should “improvise” in a way that fulfills the common purpose (Cobb, 2000). Moreover, the two cultures to which a classroom research team must simultaneously appeal, teaching and education research, have different rules for establishing expertise, different practices, and different language. To maintain the epistemic fidelity of the work, the “composition” or group norms that guide the research team must be ontologically embedded in both of these cultures. This is made possible by the fact that
people have the capacity to be several things at once: The work of Lampert (1998) and Bowers, Doerr, Masingila, and McClain (1999, 2000) clearly attest to the duality of being a teacher and a researcher at the highest levels of both professions. Such duality of being forms the initial ontological basis for the emergence of community. As communities begin to form, the coherence that emerges is expressed in the aesthetic relations between the epistemic proclivities of participants and their ontological grounding in the community they bring forth (Maturana & Varela, 1980; Varela, Thompson, & Rosch, 1991; Sawada, 1991).

In establishing relationships between researchers and teachers on an investigative team, we acknowledge that bad things can happen as well as good—any idiot can screw up a good thing, so to speak—nevertheless we blithely suggest that the benefits potentially outweigh the costs and provide a model of emergent systems that is clearly proponent toward the enfolding of teachers into the research community. Keeping this in mind, the properties of research teams as emergent systems include the following (per Sawyer, 1999).

**Gestalt Novelty**

By *gestalt novelty*, we mean that the system displays a new form as opposed to mere combinatoric novelty. In classroom research, the convergence of perspectives across practitioner/scholar lines can fundamentally redefine what a “teacher” is or what a “researcher” is. Certainly the design research literature shows a fundamental shift in what the nature of research is and in what kinds of activity are legitimate research endeavors. As the discussion of the emergent perspective maintains, individuals have freedom within boundaries established by group norms. The probability that an individual will do a particular thing at a particular time, therefore, is conditional on the actions of his or her peers—that is, the affordances and constraints that they, the collective, place upon the moment (Sawyer, 1999). Rearrange the characteristics of the collective—that is, the distributed expertise—and the resulting activity will be markedly different.

In emergent domains like classroom research, one necessary outcome of the activity is the creation of a novel product. One of the results of the process of research, then, is an object that takes on an existence determined by, but independent of, any individual creator. In our own research, these products could be called designs, theories, or models (Middleton, Lesh, & Heger, in press). Such products are intended to influence the future creative acts of the broader membership of the discipline via viewing, analysis, and personal identification (Sawyer, 1999). As such, their publication and distribution constitute social acts (e.g., Mead, 1910) by which the research team intends to steer the thinking of the discipline in some new direction. Historically, these products have been targeted toward the ivory tower (with curriculum development and teacher-oriented products relegated to lower status in the community). The degree to which products influence instruction must then depend largely on the willingness of the research team to translate its work into the language and perspective of the teaching community or the initial situatedness (ontological grounding) of the research team within the teaching community such that the novel products created are by definition the kinds of things teachers produce, read, and accord legitimacy.

**Concrescence**

We use *concrescence* to mean the integration of previously disconnected entities into a metastable entity. The research team, as described by Cobb et al. (1997) epitomizes the notion of concrescence. Other examples are embodied in the Lehrer and Schauble projects and in our own work (Middleton et al., 1999). Although the team as already
established has been described in numerous sources (e.g., Henson, 1996), the processes of its concrescence has not. A first stab at making sense of the attraction of ostensibly different and disconnected entities into a community would most likely involve investigating the ways in which information is traded (i.e., the ways in which boundary objects such as inscriptions, labels, and gestures serve as objects of focus by which communities can organize their interconnections) and brokering (i.e., the ways in which people introduce elements of one practice into another). In much the same way that classroom communities use inscriptions to coordinate their communal action, by placing an object that has embodied meaning for one community into the interstices of two communities, agents with feet in both teaching and research can serve as brokers of meaning across their respective boundaries. Sustained mutual engagement in such brokering, then, should build temporally stable relationships among community members (Wenger, 1998).

**Systemic Extension**

By *systemic extension*, we mean the case in which the organization of elements is maintained in subsystems despite incorporation of a new level of organization. Teachers fundamentally retain the purposes and practices of teachers; researchers still study teaching and learning. Yet where the two overlap in the form of new emerging communities, teaching is modified according to empirical evidence, whereas research becomes concerned with developing and supporting new teaching practices. This new level of organization constitutes the development of a new community with novel discourse patterns involving the practices of both, thus structure-determining the emergence of novel, but not completely strange, patterns (Gordon Calvert, in press).

**Normative Innovation**

By *normative innovation*, we mean the appearance of an additional level of organization requiring the institution of new norms. Of course, this redefinition of who a teacher/researcher is and what he or she does, generates new tasks, tools, norms, and language to facilitate the new relationship that has been established, some of which originate in teaching, some in research, and some of which are generated on the spot to meet the immediate—that is, pragmatic—demands of design.

**Subsystem Specialization**

By *substrate specialization*, we mean the modification of previous subsystems in terms of articulation or differentiation of structure and normative innovation, contributing to increased complexity, efficiency, and elegance of structure and behavior. As the system becomes more complex, subsystems that may previously have been limited in scope may in fact become more general but, compared with the growing system, represent an increasingly narrow focus. We see this specialization in the body of research in the content areas: although the general system and norms for classroom practice may be common, the norms and practices related to mathematics, science, language arts, and other subject areas become more and more focused on the particularities of the fields and how children come to understand important ideas in each of these fields. In turn, this research specialization, when confronted with the demands of teaching, may engender a concomitant specialization in teacher expertise. Unfortunately, the norms of practice within the teacher/researcher community may have evolved without attention to the larger sociopolitical norms that govern the hiring, rewards, and retention practices of school systems, and therefore such specialization may not be valued in the community at large and may in fact exist in contradistinction to these values.
Negentropy

By *negentropy*, we mean a general increase in variety and organization through an increase in decisional degrees of freedom due to communication and transformation of information from one subsystem to another. Overall, we are beginning to see cases that illustrate the development of local expertise in the education system—some represented by individual teachers, some by pockets of innovation within larger systems—whereby systems differentiate teacher leaders who have specialized knowledge, and these teachers drive innovation across the system in a number of fields (e.g., mathematics, science, language arts, and so on) simultaneously. In other words, although it is virtually impossible for any teacher to develop the kind of expertise in teaching and learning that we envision for all content areas, it may be possible, even desirable, for a teacher to develop deep pedagogical expertise in one or two disciplines. The broader system, then, has the initial conditions for developing networks of local experts who collectively embody deep pedagogical content knowledge in all of the disciplines to which children’s educational experience might ascribe.

THE ROLE OF THE RESEARCHER IN AN ONGOING PARTNERSHIP WITH TEACHERS

In the fall of 1994, the first author of this chapter was just starting his career as an assistant professor. He had moved to a new region of the United States, where he did not yet have the same level of contact and cooperation with local school districts he had enjoyed as a graduate student and postdoctoral scholar. He was anxious to make contacts so that his research agenda could get up and running with little lag time.

At the same time, a local school district, which we will call the Jefferson Elementary School District, had just hired a new superintendent and curriculum coordinator.\(^1\) The new administrators came into a district with historically high marks on state and national mathematics examinations. Patrons were largely middle- to upper-class and could afford to send their children to exclusive private secondary schools. The demographics of the district were beginning to change, however. To illustrate this trend, the poverty level of students entering the district in 1994 was under 20%. Between 1994 and 2001, the poverty rate had grown to approximately half the student population in several of the schools. Despite this trend, parents in the district still saw their peers in terms of neighboring high-income communities.

Not wanting to rest on the laurels of past success, the new administrators initiated a mathematics reform policy, citing the lack of a district curriculum guide and changing characteristics in the student body as evidence of the need for change. Beginning that semester, a small group of innovative teachers in each of the district’s schools was identified to serve on the Math Cadre, a committee charged with designing a new curriculum guide based on the latest research on teaching and learning mathematics. It was during these activities that the lifestrings of the researcher and the teachers in the district became hopelessly entwined.

Anecdotally, this particular district did all of the right things in developing a reform agenda. They spent the first year of work reading up on the body of mathematics

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\(^1\)For readers unfamiliar with the typical administrative structure of school districts in the United States, the superintendent is the chief executive officer of the district and is responsible for policy, financial oversight, and public relations. Curriculum coordinators, often associate superintendents, are the middle management that serve as a vital conduit by which the policies of the district administration can be translated into action on the part of teachers, but also where the needs of teachers can be communicated more directly to administration. In short, the superintendent sets policy, and the curriculum coordinator gets things done.
education research, from primary sources and quality reviews (e.g., Grouws, 1992). Being versed in neither the language nor the practices of the research community, the teachers requested some help in interpreting some of the more esoteric writing and unfamiliar content. The curriculum supervisor, who was a language arts specialist, hired the first author as a “content expert” to conduct workshops on children’s mathematical thinking and its implications for reform in light of the (then) recent publication of the National Council of Teachers of Mathematics (NCTM) Standards (1989, 1991). The workshop series seemed like an ideal venue for finding teachers with promising practices and for trying out ideas related to documenting reform as it developed organically in the system. Little did the “expert” realize that these first introductions would blossom into a full-time partnership.

One of the characteristics that make partnerships of this type approachable is that researchers and teachers have complementary resources with which they can bargain in an exchange of goods. Historically (Middleton & Webb, 1994), this type of university–school partnership has functioned primarily for this exchange of goods. The researcher packages “knowledge” and distributes it to teachers (the consumers) in exchange for a small stipend — access to classrooms or assessment data of some kind. This exchange allows the researcher to package more “knowledge,” and so the cycle continues. In the beginning of our partnership, this kind of relationship had some benefits for both the teachers in the district and the researcher. In particular, it enabled the teachers to design a district curriculum guide that was based on the existing research on students’ mathematical thinking, choose a defensible set of curriculum materials based on that guide, and begin to reform their own practice. For the researcher, the benefits were primarily relegated to access to classrooms where he could observe and critique teachers’ practice.

The problem was, the Jefferson teachers kept wanting more information, and the researcher kept becoming more and more puzzled about what constitutes a reform-oriented trajectory of teacher change. Increasingly, the cases the researcher cited were generated in the very classrooms of the teachers who wanted to change their practice. It became difficult to continue an authority role when the people to whom the generated knowledge was to be dissiminated knew as much or more than the so-called authority. Moreover, the justification the teachers proferred for making changes in the classrooms were often the very theoretical constructs the researcher was trying to articulate. Clearly, a new kind of relationship had to be established.

A couple of years into the partnership, two graduate students and the researcher decided to do a case study of one of the teacher leaders (Koellner, Bote, & Middleton, 1998). We spent about 15 weeks in her classroom, documenting her assessment practices and the kind of feedback she gave to her students. We had access to her years of journal writing, to her personal narrative, and to almost daily observation of her classroom. So much of the narrative in the case came from the mouth of the teacher that we could not, ethically, write about her without including her as a coinvestigator on the project. In doing so, an amazing thing happened. The teacher took ownership of the practices we reported: Both promising and problematic. As her case (in written form) made its way through the larger project, an outside observer viewed the teacher’s classroom and commented on an article about “Mrs. Watson” she might want to read. The teacher replied, “I am Mrs. Watson.” “Yeah, we all are!” the researcher replied. To which Mrs. Watson said, “No! I mean Mrs. Watson is me! That’s my classroom! I made those mistakes 2 years ago, and I am still making them!”

Henson (1996) listed three basic levels of teacher involvement in research on practice, ranging from “teacher as helper” through “teacher as junior partner,” all the way to “teacher as researcher.” The relationship with Mrs. Watson at this time could be described as “teacher as junior partner,” moving toward “teacher as researcher.” We had not yet come to the common purpose (Cobb, 2000) around which the legitimate
activities of both our research team and the teacher’s practical and intellectual needs could exist at more or less the same level of importance.

When the relationship between the researcher and the system becomes closer, the nature of their activity takes on a different form—one more akin to engineering than to science—“with targeted goals, explicit deliverables, prototyping, iterative design, and direct and widespread field testing” (Confrey, 2000, p. 90), in which the questions of interest of the “researcher” and the “teachers” are one and the same. The goods exchanged are therefore empirical, analytic models of thinking and practice, and the role of the actors in the system reflect their own local expertise. Harking back to our jazz ensemble, each player on the research team has a unique focus and set of skills that he or she brings to bear on studying the reforms. We would add that research on systemic reform must include those actors in the system who have an impact on, and insight into, the workings of the system. Included in this set of actors, of course, are the teachers on whose shoulders the onus for reform ultimately falls. Cobb (2000, p. 331), in fact, suggested that “the highest priority should be given to establishing with the collaborating teachers relationships that are based on mutual respect and trust.” Without such trust, the model of distributed expertise that we propose would have no subsystem specialization and therefore could exist only under a differential power structure depicted as “teacher as helper” or “teacher as junior partner” (e.g., Henson, 1996). At these levels, the teacher typically does not see the results or the results tend to be summative and offer little immediate feedback for the improvement of his or her practice, and so the question remains, “What is the real purpose of this work?”

Over the years, as the mutual needs of the researcher and the district became more enmeshed, the roles each played in the relationship changed dramatically. The model of research evolved to embody classroom teaching experiments as the primary design activity. Therein, the researchers provided psychological and mathematical structure for the development of curricular sequences. In particular, the researchers had the time to take recommendations by the teacher(s) and revise the tasks, tools, and sequences to directly reflect the ad hoc hypotheses of the research team for moving children from one level of mathematical understanding to another. For their part, the teachers provided explicit knowledge of their children, interpreting their verbalizations and inscriptions, which were often idiomatic to the particular child or class, and provided alternative tasks that served to bridge between levels of understanding when the research team over- or underestimated the length of the gap (e.g., de Silva, 2001). This current relationship begins to approach the “teacher as researcher” model as described by Henson (1996), in which the teacher develops, either singly or in concert with others, the focus of the research and the methods and tools of analysis and is integral to the creation of the explicit products of research. This illustrates the reflexive nature of systemic extension, whereby teachers’ roles as teaching agents are retained despite the ever-increasing complexity of the relationship, and subsystem specialization whereby teachers develop deeper and proportionally narrower expertise in particular disciplinary subject matter.

As a case in point, the particular set of studies we are describing (de Silva, 2001; Middleton, de Silva, Toluk, & Mitchell, 2001; Toluk, 1999; Toluk & Middleton, 2000, 2001) began with individual teaching experiments examining the development of children’s understanding of quotient (i.e., fractions as division, division as fractions). From these initial cases, we developed a set of instructional sequences designed to engage students in confronting the extension of whole number division to include fractional quotients and simultaneously to see situations that can be expressed as fractions as cases of partitive division. The classroom teacher helped construct, revise, critique, alter, and enact the sequences to embody the underlying design under the conditions of her own classroom; she interpreted childrens’ inscriptions, verbalizations, and actions; and finally, she reflected on the ad hoc alterations of the sequences that
she ad libbed during instruction. Moreover, her understanding of the development of rational number understanding deepened significantly from the typical part–whole focus to one that incorporated quotients and measures. Without her integral role, enacted so beautifully, our research understanding of how a critical concept in rational number and in subsequent algebraic applications can be fostered in a classroom culture would remain almost nonexistent. Without our role, perhaps more cumber-somely handled, the teacher would have had no opportunity to refine her pedagogical content knowledge and concomitant practices to engage students in a fundamental, but generally neglected, mathematical topic.

The shift in focus to research as design and to the unit of analysis as students’ thinking constituted the most fundamental shift in worldview as embodied by this relationship. Now when teachers are asked how they make instructional decisions, they juxtapose two elements of design: (a) a hypothetical learning trajectory (e.g., Cobb et al., 1997) that situates the classroom activity within the field of mathematics and its development and (b) the evidence, both from the extant body of research and teachers’ prior experience, of the kinds of knowledge that children bring to the particular teaching and learning environment (de Silva, 2001). Any decision based on these elements can then be called a design-based decision because it takes into account the parameters of the design and the unique environmental conditions under which the design is destined to play out, and it makes adjustments of fit to both design and environment. The shift in research worldview manifested itself in the world of practice as a general model for understanding and enacting the teaching and learning mathematics.

The trend in this particular district of teachers engaging in substantive scholarly work and being enfolded into the research community (i.e., normative innovation leading to negentropy, as we described in a previous section) has evidenced itself through the involvement of a sizeable number of teachers in the district in research and development relationships with other classroom researchers interested in design including Lehrer (2001) and Cobb and McClain (1998).

Segue

This first case illustrates the emergent properties of a community of teacher/researchers in an easily definable unit: a single elementary school district, serving a single geographic region in a single city. Our next case broadens the scope to describe a community of teachers/researchers with more amorphous boundaries: university professors of mathematics and science, university professors of mathematics and science education, mathematics and science instructors at 2-year postbaccalaureate institutions (i.e., community colleges), and mathematics and science teachers in public schools spread across a wide geographic region (Wyckoff, 1994).

As stated in the first paragraph of this chapter, in the United States, the federal government spends more money on staff development and systemic reform than on research. The largest funding programs currently proposed require the involvement of personnel from each of these diverse communities, in the form of a legal partnership (which, of course, doesn’t necessarily constitute a real partnership) for the reform of teaching and learning at all levels. Implicit in these requirements is the assumption that there is a kind of fractal similarity in what occurs at the highest levels (by which is meant the university mathematics and science departments) all the way down to the lowest levels (by which is meant public school classrooms; U.S. Department of Education, 1999). In fact, this assumption proves to be true prior to establishment of any cross-institutional relationship. Practices are indeed similar. Teachers typically engage in didactic instruction at all levels of education. This model of instruction runs counter to constructivist research on children’s thinking, however
(Bransford, Brown, & Cocking, 1999); sociocultural research on classroom practice and situated learning (Greeno & Goldman, 1998; Lave & Wenger, 1991), and the conception of teacher-as-researcher. How, then, can such a diverse group of teachers and researchers redesign the substance of their work to incorporate this research base, in a way that is still self-similar but the form of which is fundamentally different than currently embodied (i.e., improved)? This question guided the evaluation of the Arizona Collaborative for Excellence in the Preparation of Teachers (ACEPT).

**THE ROLE OF THE EVALUATION IN INSTITUTIONAL PARTNERSHIPS FOR SYSTEMIC REFORM**

ACEPT is one of 22 “Collaboratives for Excellence in Teacher Preparation” (CETP), each funded for 5-year periods to collaborate with local educational institutions to prepare mathematics and science teachers (Wyckoff, 1994). ACEPT began operation in the summer of 1995 and as with all such projects, provisions for its evaluation were required.

It is clear that any system as complex as standards-based, cross-institutional mathematics and science teaching reform cannot be adequately researched by chatting with leaders, carrying out interviews, conducting focus group discussions, administering surveys, visiting a few classrooms, and the like. Our perspective on the matter is that a phenomenon as dynamic and complex as teaching and learning at any level cannot be revealed by an intelligence that is not deeply grounded in the phenomenon itself. This groundedness often is what is so exciting about classroom research: We would like to be learning as much about collaborative reform as the people engaged in it. We hope that new and perhaps cutting-edge insights into reform will emerge out of the relationship. In terms of evaluation criteria for systemic projects, what seems important is the creation of an “evaluation space” that not only allows this to happen but that would make it inevitable. Such an evaluation space is a form of “design space” (Wenger, 1998) within which the ongoing authentic activity of innovators can be actualized, documented, and fed back into the system for continual improvement (for recent examples of innovation in evaluation, see Chelminsky, 1997; Chen, 1990; Fetterman, 1996; House & Howe, 1998; Patton, 1994; Pawson & Tilley, 1997; Ryan & DeStefano, 2000, to name only a few). Needless to say, this approach to evaluation did not come early or easily in the project. In fact, the project was more than half complete when the inadequacy of the more traditional evaluation performed by an outside firm was raised as a serious impediment to its improvement.

For 4 years, ACEPT had been attempting to reform mathematics and science content and methods courses to make them more inquiry-based, delving deeper into the understanding of fundamental math and science concepts. These reforms were occurring at least in pockets in the university departments as well as surrounding community colleges, and a few workshops were presented at local schools. By the midpoint of the 4th year, ACEPT staff were able to proudly document large numbers of faculty and college students who had been affected by the reforms simply showing up to class. At this point in time, the National Science Foundation sent a visiting committee to evaluate the status of the project. ACEPT faculty dazzled committee members by presenting findings related to the increase in the numbers of college students enrolled in the reformed courses and shared anecdotes of student enthusiasm for the engaging pedagogy. Yet the visiting committee stumped the ACEPT faculty with one simple question: “What good has it done?” The bottom line for the visiting inspectors was whether there was evidence that justified the federal government’s investment as manifested by student learning. The project was forced to examine the fractal similarity in its reforms at all levels.
To answer this question in the context of a multifaceted reform effort, we suggest that the system be thought of in the following manner. It begins with a set of principles around which the collaborative partnership is organized. For ACEPT, these principles featured “inquiry-oriented standards-based” learning as the chief outcome goal. Congruence (coherence) across levels of reform begins to become systemic if it could also be said that the teaching practices that support “inquiry-oriented standards-based” learning are themselves, inquiry-oriented and standards-based. Furthermore, if the programmatic reform (i.e., the institutional structure of teacher support, faculty development and so on) that supports the inquiry-oriented standards-based teaching is itself inquiry-oriented and standards-based, then the congruence becomes even more systemic. Continuing in this fashion, if the evaluative activity that assesses the program reforms is itself inquiry-oriented, then the congruence becomes even more systemic. At a fifth level, if the meta-evaluation activity that validates the evaluation is itself inquiry-oriented, then everything about the project (i.e., that the project can affect directly) can be said to be systemically in accord with inquiry. These five levels of systemic congruence can be depicted with some nested ellipses (see Fig. 17.1). The many levels of self-similarity function as a design in which novel patterns necessarily emerge in structure-determined fashion.

In the middle of the project, despite collaboration developing across academic departments at the collegiate level, ACEPT had done little to build sustaining partnerships with school districts. Without this type of active collaboration, ACEPT was not able to track beginning math and science teachers into the field and assess the influence of their collegiate experiences. So although professors felt good about altered syllabi, the introduction of new technologies, and active learning, to answer the question “What good has it done?” we were forced to shrug our shoulders. Our embodiment of systemic spatial congruence (SSC) had a major discontinuity at the third and fourth levels.

Because it was assumed that the first principles of ACEPT were inquiry-orientated and standards-based, the planned evaluation could be said to have been in accord with the first principles of the “object” being evaluated (the evaluand). Like a fractal, this evaluation says in its overall structure what each cluster or component of ACEPT might say in the specificity of its own reformed activity. As such, it (i.e., its systemic extension) is systemically in accord with itself. A significant bonus of such congruence is the increased likelihood that the processes of the evaluation will not get in the way of the subtle complexities inherent in the reform. In being in accord with the first principles of the reform, the chances that the evaluation plan will uncover (discover, recover)

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2 The reader may have noticed that the phrase *inquiry-oriented standards-based* was not used to describe inquiry into the evaluation of program reform. Instead, only the phrase *inquiry-oriented* was used. This is because there are currently no accepted standards for the evaluation of preservice teacher preparation programs. This is one reason why the evaluation of large systemic programs is so problematic. This is not to deny the substantial contributions of evaluators such as Guba and Lincoln (1989), whose book on “fourth generation evaluation” provides beginning standards for systemic spatial congruence evaluation.
the “essence” of the reform is optimized if not maximized. Additionally, the systemic nature of the evaluation ought to support the detection of generative systemic effects (emergent communities and their properties) inherent in but not easily distinguishable within the messy ambience of authentic reform activity.  

This approach to evaluation research can be read as saying that a teacher who considers her own teaching an important object of inquiry would be better able to support inquiry-oriented learning than a teacher who places her students in the role of inquirers but does not view her own activity as inquiry-oriented. For example, by engaging in practices such as action research, inquiry-oriented teachers would likely view professional development as an inquiry-oriented activity. Such teachers would be researchers at the same time as they were teachers and would learn about their teaching each day their students learned through inquiry. Their own teaching (as well as the learning of their students) would be construed as research-in-action and would involve the process of facilitating communities of inquirers. The premise and promise of systemic congruence maintains that learning, teaching, program reforms, evaluation of reforms, and evaluation of evaluation would create a scaffoldlike space instantiating at every level the inquiry processes that ground the evaluand itself. In this sense, inquiry-orientation is pervasively present—hence, the term systemic spatial congruence for the evaluation design. In the same sense, the evaluation space might be said to be systemically inquiry-oriented.

**BROKERING ACROSS COMMUNITIES: EVALUATION AS FACILITATION**

To address the major discontinuity mentioned earlier, a group made up of recent public school teachers and representatives from each of the sciences and from mathematics and education convened. Although the responsibility of this group was to “conduct” the evaluation, they became known as the Evaluation Facilitation Group (EFG).

The term facilitation was used as a deliberate break from the traditional model of “outside” evaluation, instead turning the responsibility for evaluation back squarely on those making the innovations. In accord with the SSC evaluation model, the task for this team was to facilitate the evaluation. Such a process is different from that of conducting the evaluation. The difference between conducting and facilitating is precisely as salient as the difference between providing workshops for teachers to implement research and working with teachers as a design team to learn while doing. The intent of facilitation was to assist cluster members (a cluster was a unit within ACEPT that had a common disciplinary focus, such as the math cluster, the physics cluster, the biology cluster, the student teaching cluster and that was composed of education researchers, content experts, and teachers) in actualizing SSC in the clusters’ own settings. Operationally, this entailed exploring possible solution paths in SSC with the innovators themselves, paths that might sensitively and succinctly, and often subtly, open up evaluative strategies appropriate to the complex reforms occurring in their cluster.

Operationally speaking, one or more members of the EFG worked with innovators (who were teachers, whether at the public school level or at the university) within each cluster to help them create a comfortable space for considering and generating evaluation approaches that would provide helpful evidence for them to assess and subsequently improve their own teaching practice. Over time, assessment tools,

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3The literature surrounding systemic reform often uses the term alignment for what is referred to here as congruence. We choose not to use alignment because it is used for linear systems as well as nonlinear ones. We suggest that the emergent properties of systemic reform projects in general are nonlinear in their generativity, although they are often managed as if they weren’t.
observational protocols, training sessions, and ostensible products became focused particularly on ideas or solution paths that were suggested by cluster members.

In general, members of the EFG assisted cluster members to accomplish the following:

- ascertain the current focus and extent of evaluation practices and provide critique;
- use the SSC model to frame the clusters' thoughts about evaluation possibilities congruent with their disciplinary focuses with their unique practices and forms of depiction;
- generate potential evaluation strategies that arose naturally as responses to cluster priorities regarding the concurrent goals of teaching, researching, publishing, documenting achievement, and so on;
- generate specific questions or hypotheses about cluster reforms that had a high probability of revealing some aspect of excellence in the cluster and generate specific evaluation pathways that might explain the conditions of, and practices that lead toward, this revealed excellence;
- design common protocols that could contribute to an aggregate or cross-cluster portrait of the ACEPT project as a whole;
- select, develop, modify, and validate special instruments or techniques appropriate to answering the questions of interest of the innovators themselves;
- integrate data collection process with ongoing teaching and learning activities; and
- showcase excellence across institutional lines.

The members of the EFG, through their activity, therefore served as brokers (e.g., Wenger, 1998) between the education research community, the National Science Foundation contingent wanting evidence of success, and the innovators who cared little for the NSF but cared a great deal about their students' quality learning. The central model of SSC provided a coherent focus for individuals from diverse disciplines to coordinate their activities with EFG representatives providing “translation services” and technical expertise in methods of social science data collection and interpretation.

How this brokerage played out in the schools is exemplified in the following case of another Southwestern school district partnering with the ACEPT grant.

Ross is a district accustomed to forming partnerships with other entities. The district has had a history of good relationships with local industries such as Motorola and Intel that provide project funding and unique professional development opportunities. Collaborations with state universities have provided master's degree programs for teachers within the district. With each of Ross's partnerships there exists one consistent mechanism: All these partnerships thrive on the central efforts of some key person(s) to initiate and maintain the alliance (i.e., a district-level broker). In this case, the district's science resource specialist took it upon herself to coordinate district activity with the ACEPT project. ACEPT worked with the resource specialist and her staff to draft a tailored letter for teachers defining the research as a collaborative effort. The letter highlighted that resource specialists during the school year would visit the new teachers (an preexisting feature of Ross’s beginning teacher evaluation) and the EFG member would simply be an additional visitor. The letters were then hand delivered to prospective teachers by resource staff. The letters also indicated a date by which the teachers needed to respond; once the date passed, the district resource specialist made sure that those who did not respond were contacted again. Nearly all new mathematics and science teachers opted to have classroom visits during the school year.

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4 As defined by the norms and practices of each disciplinary cluster.
After clearing the hurdle of accessing teachers, ACEPT and Ross began to work together in a manner that can be described as collegial vibrance. The plan called for teams of evaluators (one Ross and one EFG representative) to visit the beginning teachers. Both ACEPT and Ross had their own observation protocols for making classroom visits. After members of each staff reviewed the other’s instrument, the Ross staff became intrigued with ACEPT’s Reformed Teacher Observation Protocol (RTOP) instrument Piburn, Sawada, Turley, Falconer, Benford, Bloom, and Judson (2000). The EFG had recently created the RTOP to capture what the university mathematics and science faculty considered attributes of a “reformed” classroom. As such, it embodied an aesthetic that could be assessed through observation, given what was a taken as shared understanding of what the items in the protocol were in fact referring to. In contrast, Ross used a general observation instrument that favored a good deal of scripted teacher and student talk. In developing an interest in the tool, both staffs had viewed a series of videotaped classroom lessons. The videotapes then spurred vigorous dialogue concerning the viewed teacher’s pedagogy and classroom environment. At this time, the RTOP was still a draft instrument and Ross’s input was highly respected as the team, meaning EFG and Ross teachers, arrived at consensus on how to interpret particular RTOP items. During training, the RTOP was not viewed as exclusive university property; it was a pliable document requiring concurrence of interpretation as the relationship underwent the process of systemic extension.

As mentioned in the introductory paragraphs of this chapter, in a typical research and development partnership, one party trades services in exchange for receiving a return commodity; the terms of exchange are usually well defined. The difficulty with a barter-based partnership of this type is that all parameters are typically defined at the initial bargaining session. The Ross resource specialist did not view such explicit partnerships as worth the efforts of the district. Rather, she pushed for the ongoing mutuality of engagement in evaluation. As a result, the experiences and outcomes that emerged were not (nor could they be) anticipated at the outset of the association. During the following academic year, the ACEPT and Ross staffs met regularly to collectively reflect on teacher practices (with video evidence as boundary object) and determine pointed suggestions for the design (yes, this appears at all levels of SSC as well as classroom research) of staff development that would occasion reform in the design of instructional sequences. This type of professional sharing became the cornerstone of interaction. Partners from both institutions thoroughly enjoyed the benefit of discussing with another evaluator how best to help a new teacher. Before this, Ross staff members worked in isolation, unable to fully relate what was observed in a particular classroom to the reform literature. For the part of EFG, their musings had developed at the university level divorced from the perspectives of teachers immersed in their practice. For ACEPT, the discussions became a valuable source of ideas and case studies to enrich the depth of research for the purposes of both institutions.

The high comfort level between the two staffs was illustrated when members of the Ross staff requested that the EFG members observe a teacher who was not part of the research. The teacher had been experiencing classroom management problems, and the Ross staff wanted fresh input to guide this teacher. Additionally, camaraderie was built through informal lunches, sharing of information related to professional organizations, and networking with math and science resources.

It is difficult to determine what the final testament to the Ross partnership because it is an entity that continues to flourish. Yet a recent significant change that took place in the nexus of the two communities is the development of a professional development school that will operate entirely in the Ross school district. In the future, professors and accredited personnel of Ross will jointly instruct preservice education courses, supervise student teaching experiences, and support students as they begin their
teaching careers. Successful relocation of the postbaccalaureate program from the Arizona State University campus to Ross is not viewed as a transplant from one locale to another. Rather, all involved anticipate a transformation of the fundamental nature of teacher education, placement, and induction to occur. In a definite sense, a new community integrating preservice and inservice development has emerged in the Ross school district.

In much the same manner, the EFG worked with college and university professors to reform their own practice in undergraduate science and mathematics courses. One goal of congruence is that the kinds of practices we engender in each level of the educational enterprise (e.g., from kindergarten to graduate study) are in accord with practices in all the other levels and in accord with disciplinary approaches to inquiry (in science) and forms of argument (in mathematics). The EFG members served as brokers between and among science and mathematics faculty to develop the RTOP protocol as a common aesthetic of reformed teaching. This activity, the building of an evaluation tool, served to orient the attention of the disciplinary experts to the core features of practice that defined the reforms they wished to make.

A FEW CAVEATS

Two facets of the larger case of ACEPT should serve to warn researchers of potential pitfalls in their work. First, with the increasing mobility of teachers at all levels across schools and districts (and universities) and the preponderance of administration abdicating their positions in favor of other positions (Middleton & Webb, 1994), the core personnel of a research and development partnership may change suddenly and without warning, taking with it its collective knowledge. Such was the case of a community of mathematics instructors at the undergraduate level at ASU. Because only one instructor remained from the initial group of four after 3 years of coherent work reforming their curriculum and practice, the collective distrust of the EFG evaluation in general kept adequate research data from being collected, in effect severing the feedback loop into the system. Second, this distrust of evaluation procedures, or misinterpretation of the purposes and practices of research, led to unwitting or even deliberate sabotage of the data. Both of these conditions can be debilitating to a systemic research and development project, returning the relationship to where it started in its infancy.

SUMMARY

The preceding cases generate a number of considerations when attempting to build quality relationships with teachers for the purpose of research on systemic reform. Taken together, they provide a first cut as to the necessary and sufficient conditions for the coalescence and emergence of mutually informative research and development partnerships for systemic reform. Finally, at the end of this section, indeed the end of the chapter, we present the thesis on which all of these arguments depend.

Partnerships Emerge. They Are Not Built

As the perspective implies, when viewed as an emergent system, it is unclear whether productive partnerships can be “built” at all. Instead, our cases illustrate the gradual concrescence of roles and personnel in a complex social system that emerges as a community with dramatically different parameters (i.e., gestalt novelty) than existed or were even possible in the first introduction of partners. The question for design becomes one of what configurations of institutions, roles, tasks, tools, and central
purpose have a reasonable probability of organizing themselves into a meta-structure with emerging stability. In particular, the common purpose for a community is quite different from a collective vision. A collective vision may indeed be important, but the teachers and researchers in our cases seemed to coalesce around more pragmatic endeavors. In particular, the creation of products such as curriculum designs, research reports, analytic tools, and video cases seemed to bring small groups of people together. With the support of district- and university-level brokers (a subset of the teachers and researchers themselves), these groups could explore and negotiate the focus of the research and creative activity to exist at the nexus of their two communities. This in turn constitutes the development of local knowledge (e.g., Lytle & Cochran-Smith, 1994), which serves to guide the decision making of the research-team-as-reformers. It is these local understandings, then, that form the basis for determining issues of transportability between and across local conditions.

The Research Partnership Must Be Valued by the Teacher(s) and by the Researcher(s)

As trite as it may sound, mutual respect and trust must be emerging properties of the relationship between those who wish to effect teaching and learning, and those who wish to study it. Fear of evaluation, distrust of expertise, and lack of mutual contribution to the project can undermine or even sabotage a working partnership. Indeed, these properties may be the most important characteristics emerging in the collaboration. Fear of evaluation was prevalent in our initial interactions with teachers. Because university researchers are seen as authoritative sources to judge good teaching, and perhaps because the popular press is full of "studies" that criticize the practices of teachers, teachers justifiably feel anxiety at the prospect of having their classes observed.

Distrust is by no means confined to teachers. Our experience has taught us that scientists and mathematicians distrust the content expertise of education researchers, and for their part education researchers suspect that content experts often don’t have an enlightened understanding of pedagogy. Neither group feels that teachers as a whole are versed in their respective fields, and teachers quite frankly resent it. Inherent in these perspectives is an underlying feeling that members of the other disciplines cannot be fully trusted. These feelings make it imperative that researchers who wish to serve as brokers across communities attend seriously to the prime motivations for people in each area to engage in classroom research. Scientists and mathematicians engage as a service to their field and to improve their own instruction. As such, their questions are often centered around determining the effectiveness of changes they make to courses and programs. Teachers engage for similar reasons: to improve their own practice and for professional empowerment. Their questions often center around assessment as well, for example, “How do students think when they are engaged in particular tasks or content?” Education researchers engage because research is the core activity of their profession. Their questions often are more theoretical and esoteric, such as, “How is information traded across individuals within small groups and from small groups to the collective?” The key, we think, to bridging these communities, is to ensure the authenticity of engagement for all members of the partnership. Table 17.1 highlights the key questions researchers might ask themselves as they attempt to engage members of the teaching and professorial communities in collaborative research, keeping in mind that each community will have slightly different needs, roles, and ways of interpreting the relationship.

To maintain a mutuality of engagement in the task, it becomes imperative that all partners understand the nature and intent of the research methods. In fact, the ethics of involving those we study in the interpretation of their own behaviors is becoming
TABLE 17.1

Authentic Participation in Reform as Embodied in Questions for Evaluation

<table>
<thead>
<tr>
<th>Authentic Participation</th>
<th>Key Question for Researchers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relevant participation</td>
<td>Who participates? What role does each member of the research team play?</td>
</tr>
<tr>
<td>Addressing local conditions and processes</td>
<td>What conditions and processes will the project address?</td>
</tr>
<tr>
<td>Coherence between means and ends of participation</td>
<td>What is the common purpose of the research?</td>
</tr>
<tr>
<td>Focus on broader structural inequities</td>
<td>How does the project legitimize and enhance the work of participants?</td>
</tr>
</tbody>
</table>

*Note. Adapted from Anderson (1998).*

a central issue in classroom research. Tobin (2000), for example, discussed the “moral practice of research” and included the full disclosure of research goals to all members of the research team. To this we add that the co-construction of the research question and design eliminates the need for disclosure and simultaneously legitimizes the involvement of members from whatever community is under study.

**Research as Design Orients the Relationship Around Common Practices**

The perspective of research as design emerged as a promising central focus of activity that simultaneously legitimized the knowledge and practices of classroom teachers and the knowledge and practices of professional researchers. The reasons for this appear simple: Design is a basic activity of both teachers and classroom researchers; designs, the results of such activity, are transportable across situations with varying parameters and therefore are applicable to further research and teaching, and design is fundamentally teamwork that is, relationship-based, requiring the coordination of distributed expertise on a common problem. As Wenger (1998, p. 235) pointed out, “Design creates fields of identification and negotiability that orient the practices and identities of those involved to various forms of participation and non-participation.” We do not want to paint the picture as quite this simple, but coming to understand the first principles of design research in classrooms will require unpacking design as simultaneously activity, product, and social structure. In our case, because design was the organizing principle and students’ thinking was the outcome we wished to impact, the inscriptions and verbalizations which arose from, and yet also determined, students’ thinking became boundary objects through which the design activity of the teacher and the researchers could be coordinated.

In terms of large-scale systemic projects such as ACEPT, design occurs on multiple levels. The systemic spatial model is one metaphor that a group of researchers and teachers (professors and educators in the public schools) used to understand the alignment of goals for learning, classroom reforms, program articulation, and formative and summative evaluation. Although that model was designed for the explicit need for evaluation of a particular project, the notion of a kind of fractal structure by which the practices of innovators in the trenches is philosophically congruent with, and behaviorally constituent of, each successive level of complexity in the system is echoed in other approaches to partnerships. Lesh and Kelly (2000), for example, used a similar approach when they described multiliter teaching experiments as shared research designs with mathematical models being the focus of design activity at the
student level, the design activity of students being the focus of the design activity at the teacher level, and the design activity of teachers being the focus of the design activity at the research level. These shared designs support the selection of productive ideas or ways of thinking from nonproductive ones, the propagation of productive ideas among members of the research team and across all levels of the project, and the preservation of productive ideas by means of creating jointly authored products.

Partnerships Need Brokers to Bridge Activity Between Heretofore Separate Communities

Brokerage, as we are describing it (e.g., Wenger, 1998), is a form of mediation between novices in one community, and knowledgeable practitioners in another. As such, it mediates an entrée by interested parties into a new configuration of practices and norms. Brokerage is substantially different from the practices of the Master in a community of practice (e.g., Lave & Wenger, 1991) in that brokerage is intercommunal, attempting to leverage aspects of both communities that, in the process of concrescence, can form the basis of a third community. In mathematics education research, the researcher often serves as that broker, having in a previous life been a teacher of mathematics. Others in the affected communities, however, serve as brokers, especially as the complexity of the emerging system grows. In the ACEPT project, the EFG served as brokers, each with the disciplinary knowledge to work specifically with content experts and education experts on a common evaluation. In the case of the Jefferson and Ross school districts, the curriculum coordinator or supervisor, in a position of middle management, served as brokers between institutions with different norms of operation. We maintain that without such people willing to take the middle, play the bridge, and actively broker knowledge and practices across communities, productive partnerships cannot be occasioned.

The Commitment to Mutual Support Must Be Maintained for a Sustained Period of Time

In all of these discussions, the element of time seems to be somewhat problematic. Although the concrescence of our relationship with the Jefferson community has taken 7 years to become highly productive mutually, it is unclear whether such a relationship necessarily takes quite that long to emerge. ACEPT became productive after 4 years, whereas the underlying partnership with Ross was begun nearly 30 years ago when a young mathematics education professor taught the current curriculum coordinator in her methods course to become a mathematics teacher. Nevertheless, given that the level of sophistication of the partners in terms of knowledge of reform, widespread interest, and generally quality practices was high to begin with, our experience suggests that partnerships for research and reform, in the best of circumstances, emerge on a time scale amounting to years instead of months or weeks. The researcher must take the time to know the culture under study, including the culture of the classroom, the teachers as a collective, and the school. The researcher must, in effect, become part of the community. We have used the term ontological grounding to refer to this kind of authentic knowing. For scholars who place stock in such things, the 8-week, in-and-out observational study, unless situated within an ongoing agenda for reform, coupled with the commitment to maintaining and growing the teacher–researcher relationship, may not lead to any immediate benefits to those who most need them, nor the kind of programmatic and sustained inquiry that provides theoretical insight into what is truly possible in mathematics teaching and learning (for a more extensive example, see Lehrer & Schauble). It also seems to take considerable time for the teacher to become enculturated into the research team (i.e., how to do research), and it takes
even longer for them to learn how to look systematically and critically at their own teaching (Feldman & Minstrell, 2000). This would suggest that the study of teacher change be an integral part of the relationship building stage in the development of a classroom research project and that the burgeoning understanding of the teacher as a researcher be included in the discussion of the case. Emerging communities do not emerge overnight. Patience is key.

CONCLUSIONS

Based largely on our own experience, and supported by research on both large-scale and classroom-level systemic reform efforts, we have reduced our thesis on change-producing research efforts in science and mathematics to only three words: Relationships build reform. More accurately, reform emerges from relationships. No matter from which discipline your partners hail, no matter what financial or human resources are available, no matter what idiosyncratic barriers your project might face, it is the establishment of a structure of distributed competence, mutual respect, common activities (including deliverables), and personal commitment that puts the process of reform in the hands of reformers and allows for the identification of transportable elements that can be brokered across partners, sites, and conditions. After all, the local success of a particular project over the short time span of a grant doesn’t matter much, does it? Money doesn’t last very long. Personnel come and go. Inertia is the tendency of all trajectories of reform. Relationships have the potential to transcend money, time, and even individual reformers if the culture of research and development is fundamentally changed.5

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REFERENCES


5The team of authors involved in writing this chapter is, in itself, an emergent community of mathematics and science teachers, brought together under the charge of making sense of the kind of partnerships in which we are attempting to engage. Two of us (Middleton and Sawada) are university mathematics education professors. Our self-image is that of academics: People who get paid to think. Two of us are public school science teachers (Judson and Turley), on hiatus from our profession to assist the university to coordinate student teaching and mentoring experiences for young teachers in local schools. One of us (Bloom) is a former high school mathematics teacher, working in the Arizona State University mathematics department to reform undergraduate content courses. For the past 6 years, we have been part of a larger effort to develop a community of mathematics and science teachers from kindergarten through graduate school intent on changing their practice to reflect high quality science and mathematics content, reformed-oriented pedagogy, and appropriate use of technology.


SECTION III

Advances in Research Methodologies
Four hundred years post-*Hamlet*, researchers in mathematics education might well invert Polonius’ famous comment: In contemporary research, *though this be method, yet there is madness in’t* seems fairly close to the mark. In this essay, I unravel some of the reasons for the madness and the method in contemporary research, suggest criteria regarding the appropriateness and adequacy of investigatory methods and their theoretical underpinnings, and identify some productive pathways for the development of beginning researchers’ skills and understandings.

Let us begin with an indication of the magnitude of the task. In terms of scale, it is worth noting that the *Handbook of Qualitative Research in Education* (LeCompte, Millroy, & Preissle, 1992) and the *Handbook of Research Design in Mathematics and Science Education* (Kelly & Lesh, 2000) are 881 and 993 pages long, respectively. Neither of these handbooks claims to cover the territory; moreover, there is relatively little overlap between them. The vast majority of the former is devoted to ethnographic research, which is but one of many approaches to understanding what happens in educational settings. The core of the latter is devoted to elaborating half a dozen “research designs that are intended to radically increase the relevance of research to practice” such as teaching experiments and computer-modeling studies. Neither discusses the kinds of quantitative or experimental methods that dominated educational research just a few decades ago (see, e.g., Campbell & Stanley, 1966) and which remain important to understand. Given the broad spectrum of contemporary work, any attempt in a single chapter to deal with educational research methods must of necessity be selective rather
than comprehensive. One must adopt the strategy of identifying and elucidating major themes.

Second, some historical perspective is in order. Mathematics education is solidly grounded in psychology and philosophy among other fields and can thus claim to have a long and distinguished lineage. The discipline of research in mathematics education is itself quite young, however. The first meeting of the International Congress on Mathematics Education (ICME) was held in 1968. Volume 1 of *Educational Studies in Mathematics* appeared in May 1968. The *Zentralblatt für Didaktik der Mathematik* was first published in June 1969, the *Journal for Research in Mathematics Education* in January 1970. In addition, although growth in the field has been substantial, that growth has been anything but evolutionary. As a consequence first of the “cognitive revolution” in the 1970s and 1980s and then of an expanded emphasis on sociocultural issues and methods in the 1990s the field has, within its short life span, completely reconceptualized the nature of the phenomena considered to be central, and it has developed new methods to explore them. The dust has not settled as this chapter is being written. What we take to be foundational assumptions, how we investigate various empirical phenomena, and how we provide warrants for the claims we make are all issues that stand in need of clarification and elaboration. To the degree that space allows, those issues will be addressed in this chapter.

Sections II and III of this chapter provide the broad context for the discussions that follow. Section II provides a brief summary of trends in mathematics education over the 20th century, describing the philosophical underpinnings and research methods of some major approaches to the study of mathematical thinking, teaching, and learning. A main point is that mathematics education research is a young discipline, having coalesced in the last third of the century. This serves, in large measure, to explain the diversity of perspectives and methods; some degree of chaos is hardly surprising during the early stages of a discipline’s formation. Section III summarizes the current state of affairs, with an eye toward the future. An argument is made that the “pure versus applied” characterization of much research may be a misdirection—that educational research has progressed to the point where it can address many basic issues in the context of meaningful applications.

The core of this chapter, sections IV, V, and VI, is devoted to the elaboration of a framework that addresses the purposes and conduct of research. It addresses the role of underlying assumptions in the conduct of research, the implications of (implicit or explicit) choices of theoretical frameworks and methods for the quality of the research findings, and the nature of the warrants one can make regarding research findings.

As a rough heuristic guide for the discussion of methods, the following framework is used. Research contributions will be conceptualized along three dimensions: their trustworthiness (how much faith can one put in any particular research claim?), their generality (are claims being made about a specific context, a well-defined range of contexts, or are they supposedly universal?), and their importance. Section IV provides an overarching description of the research process: the choice of conceptual framework, the focal choice of data and their representation, their analysis, and the interpretation of the analyses. This description focuses largely on places where essential decisions are made and on possible problems regarding trustworthiness and generality when such decisions are made. Section V offers a set of criteria by which educational theories, models, and findings can be judged. Section VI elaborates on the framework discussed above, with an emphasis on the generality dimension of the framework. It offers a series of examples illustrating the kinds of claims that can be made, ranging from those that make no claim of generality (“this is what happened here”), to those that claim to be universal (“the mind works in the following ways”). For each class of examples discussed, issues of trustworthiness and importance are addressed.
Section VII addresses issues related to the preparation of educational researchers. The educational enterprise is complex and deeply interconnected, and simple approaches to simple problems are not likely to provide much purchase on the major issues faced by the field, but beginning researchers have to start somewhere. Is there a reasonable way to bootstrap into the necessary complexity? Are there general chains of inquiry, and pathways into educational research, that seem promising or productive? Brief concluding remarks are made in section VIII.

II. A BRIEF HISTORY: PERSPECTIVES AND METHODS

Throughout much of the 20th century, a range of perspectives and their associated research methods competed for primacy in mathematics education. Some of those perspectives were associationism/behaviorism, Gestaltism, constructivism, and, later, cognitive science and sociocultural theory.

Associationism and behaviorism were grounded in the common assumption that learning is largely a matter of habit formation, the consistent association of particular stimuli in an organism’s environment with particular events or responses. The generic example of behaviorism is that of Pavlov’s dogs, which salivated at the sound of their handlers’ approach, the association between certain noises and their upcoming meals being so strong that it induced a physiological response. Pavlov showed experimentally that the response could be reinforced so strongly that the dogs salivated in response to stimulus noises even when no food was present. Presumably, human learning was similar. E. L. Thorndike’s 1922 volume The Psychology of Arithmetic established the foundation for pedagogical research and practices grounded in associationism. Thorndike’s learning theory was based on the concept of mental bonds, associations between sets of stimuli and the responses to them (for example, “3 × 5” and “15”). Like muscles, bonds became stronger if exercised and tended to decay if not exercised. Thorndike proposed that, in instruction, bonds that “go together” should be taught together. This theoretical rationale provided the basis for extended repetitions (otherwise known as “drill and practice”) as the vehicle for learning.

In broad-brush terms, associationist/behaviorist perspectives held sway at the beginning of the 20th century, at least in the United States. Evidence thereof may be found in two yearbooks, the first yearbook of the (U.S.) National Council of Teachers of Mathematics (NCTM), published in 1926, and the 1930 yearbook of the (U.S.) National Society for the Study of Education (NSSE).

A theory based on the development of bonds and associations lends itself nicely to empirical research. From the associationist perspective, a fundamental goal is to develop sequences of instruction that allow students to master mathematical procedures efficiently, with a minimum of errors. Thus, relevant research questions pertain to the nature of drill—how much and of what type. Such work was relatively new, heralding the beginnings of a “scientific” approach to mathematics instruction. It is interesting to note, for example, that the editors of NCTM’s 1926 Yearbook, A General Survey of Progress in the Last Twenty-Five Years, introduced a research chapter (Clapp, 1926/1995) with the following statement:

Detailed investigations and controlled experiments are distinctly the product of the last quarter century. The Yearbook would not be truly representative of the newest developments without a sampling of the newer types of materials that are developing to guide our practice.” (NCTM, 1926/1995, p. 166)

The 1926 NCTM yearbook contains two chapters that focus on research. The first (Schorling, 1926/1995) invokes Thorndike and provides an extensive summary of
“The psychology of drill in mathematics” (pp. 94–99), including a list of 20 “principles which have been of practical help to [the author] in the organization and administration of drill materials.” The second, mentioned above (Clapp, 1926/1995), represented the state of the art in the study of student learning of arithmetic. An empirical question, for example, was to determine which arithmetic sums students find more difficult. Clapp reported:

In the study of the number combinations a total of 10,945 pupils were tested. The number of answers to combinations was 3,862,332... [The sums] were read to pupils at the approximate rate of one combination every two seconds. The rate was determined by experimentation and the time was made short enough to prevent a pupil’s counting or getting the answer in any other round-about way. ... The purpose of the study was to determine which combinations had been reduced to the automatic level. The results may be said to indicate the relative learning difficulty of the combinations.” (1926/1995, pp. 167–168)

The 1930 NSSE Yearbook (Whipple, 1930) was devoted to the study of mathematics education. Its underpinnings were avowedly behaviorist/associationist:

Theoretically, the main psychological basis is a behavioristic one, viewing skills and habits as fabrics of connections. This is in contrast, on the one hand, to the older structural psychology which still has to make direct contributions to classroom procedure, and on the other hand, to the more recent Gestalt psychology, which, though promising, is not yet ready to function as a basis of elementary education. (Knight, 1930, p. 5)

Thus, in the 1930 NSSE yearbook one saw research studies examining the role of drill in the learning of multiplication (Norem & Knight, 1930) and fractions (Brueckner & Kelley, 1930) and on the effectiveness of mixed drill in comparison to isolated drill (Repp, 1930). Errors were studied in fine-grained detail, similarly to the work Clapp reported. In their study of multiplication, for example, Noren and Knight (1930) analyzed the patterns found in 5,365 errors made by students practicing their multiplication tables.

It is interesting to note from the perspective of these yearbook authors and editors, Gestalt psychology, while “promising,” was not ready for prime time with regard to mathematics instruction. In many ways, the Gestaltists’ stance could be seen as antithetical to that of the associationists:

With the development of “field theories” of learning, of which the Gestalt theory is most familiar to school teachers, the center of interest shifted from what was often, and perhaps unjustly, called an “atomistic” concept of learning to one which emphasized understanding of the number system and number relations and which stressed problem solving more than drill on number facts and processes. (Buswell, 1951, p. 146)

Indeed, insight and structure were central concerns of the Gestaltists. An archetypal Gestaltist story is Poincaré’s (1913; see also Hadamard, 1945) description of his discovery of the structure of Fuchsian functions. Poincaré described having struggled with the problem for some time, then deliberately putting it out of mind and taking a day trip. He reported that as he boarded a bus for an excursion, he had an inspiration regarding the solution, which he verified upon his return.

Poincaré’s story is typical, both in substance and methods. With regard to substance, the outline of the story is the basic tale of Gestalt discovery: One works as hard as possible on a problem, then deliberately putting it out of mind and taking a day trip. He reported that as he boarded a bus for an excursion, he had an inspiration regarding the solution, which he verified upon his return.
is pure gold, without damaging the crown itself. With regard to method, what Poincaré offered is a retrospective report.

Perhaps the best known advocate of the Gestaltist perspective with regard to schooling is Max Wertheimer. Wertheimer’s (1945/1959) classic book, *Productive Thinking*, is a manifesto against “blind drill” and its consequences. Wertheimer described the responses he obtained from students when he asked them to work problems such as

\[
\frac{357 + 357 + 357}{3} = ?
\]

He reported that some “bright subjects” saw through such problems, observing that the division “undoes” the addition, yielding the original number. Wertheimer found, however, that these students were the exception rather than the rule. Many students who had earned high marks in school were blind to the structure of the problem and insisted on working through it mechanically. He continued:

These experiences reminded me of a number of more serious experiences in schools, which had worried me. I now looked more thoroughly into customary methods, the ways of teaching arithmetic, the textbooks, the specific psychology books on which their methods were based. One reason for the difficulty became clearer and clearer: the emphasis on mechanical drill, on ‘instantaneous response,’ on developing blind, piecemeal habits. Repetition is useful, but continuous use of mechanical repetition also has harmful effects. It is dangerous because it easily induces habits of sheer mechanized action, blindness, tendencies to perform slavishly instead of thinking, instead of facing a problem freely. (Wertheimer, 1945/1959, pp. 130–131)

The focus of the Gestaltists’ work, whether in discussions of schooling or in discussions of professionals’ mathematical and scientific thinking (e.g., in Poincaré’s story and Wertheimer’s interviews of Einstein regarding the development of the theory of relativity), was on meaning, insight, and structure. Their methods were “introspectionist,” depending on individuals’ reports of their own thinking processes. These methods, alas, proved unreliable. As Peters (1965) summarized subsequent research, “a wealth of experimental material [demonstrated] the detailed effects of attitudes and interests on what is perceived and remembered. Perception and remembering are now regarded as processes of selecting, grouping, and reconstructing. The old picture of the mind as receiving, combining, and reproducing has finally been abandoned” (p. 694). And, one might add, methods that depended on individuals’ reports of their own mental processes could hardly be depended on.

Following World War II, the “scientific” approach to research in education returned with a vengeance. Given the context, this was natural. It was science that had brought an end to the war, and it was science that promised a brighter future. (The motto of one major corporation, for example, was “progress is our most important product.” There is no doubt that the progress referred to was scientific.) After a decade of worldwide turmoil, what could be more psychologically desirable—or prestigious—than the prospect of a rational, orderly way of conducting one’s business?

The wish to adopt the trappings of science played out in various ways. Among them were the ascendance of the radical behaviorists and the dominance of “experimental” methods in education—and more broadly in the social sciences, so named for the reasons discussed in the previous paragraph. First, we consider the radical behaviorists. As noted above, the “mentalistic” approaches of groups such as the Gestaltists, depending on introspection and retrospective reports, were shown to be unreliable. The radical behaviorists such as B. F. Skinner (see, e.g., Skinner, 1958) took this objection to reports of mental processes one step further. They declared that the very notions of “mind” and “mental structures” were artifactual and theoretically superfluous; all that counted was (observable and thus quantifiable) behavior.
The radical behaviorists, following in the tradition of their classical behaviorist antecedents, took much of their inspiration and methods from studies of animals. Rats and pigeons might not be able to provide retrospective verbal reports, but they could learn, and their behaviors could be observed and tallied. One could teach a pigeon to move through a complicated sequence of steps, one step at a time, by providing rewards for the first step until it became habitual, then the second step after the first, and so on. Out of such work came applications to human learning. Resnick (1983) described Skinner’s approach as follows:

[Skinner] and his associates showed that “errorless learning” was possible through shaping of behavior by small successive approximation. This led naturally to an interest in a technology of teaching by organizing practice into carefully arranged sequences through which the individual gradually acquires the elements of new and complex performance without making wrong responses en route. This was translated for school use into “programmed instruction”—a form of instruction characterized by very small steps, heavy prompting, and careful sequencing so that children could be led step by step toward ability to perform the specified behavioral objectives. (Resnick, 1983, pp. 7–8)

N.B. The holy grail of “errorless learning” persisted long after the behaviorists’ day in the sunshine had supposedly passed. Many well-known computer-based tutoring systems marched students through various procedures one step at a time, refusing to accept as correct inputs that, even if ultimately sensible, were “errors” in the sense that they were not the “most logical” input anticipated by the program.

Behaviorism, both in its earlier and then in its radical form, was one manifestation of scientism in the research culture. As indicated above, scientism was widespread, permeating all of the social sciences during the third quarter of the 20th century. It played out in the wholesale and oftentimes inappropriate adoption of statistical and “experimental” methods through much of the third quarter of the century. Many educational experiments were modeled on the “treatment A versus treatment B” model used in agricultural or medical research.

Under the right conditions, comparison studies can provide tremendously useful information. If, for example, two fields of some crop are treated almost identically and there is a significant difference in yield between them, that difference could presumably be attributed to the difference in treatment (which might be the amount of watering, the choice of fertilizer, etc.). Drug tests operate similarly, with “experimental” and “control” groups being given different treatments. Statistical analyses indicate whether the treatment drug has significantly different effects than the control (typically a placebo).

Unfortunately, the “right conditions” rarely held in the educational work described above. Although it may be possible to control for all but a few variables in agricultural research, the same is not the case for most educational comparisons. If different teachers taught “experimental” and “control” classes, the “teacher variable” might be the most significant factor in the experience. Or, the same teacher might teach the two treatments at different times of the day, and the fact that one group met early in the morning and the other right after lunch (or the teacher’s enthusiasm for one treatment) would make a difference. Or, the “treatment” itself might be ill defined. One could say much more, but there is no need to flog a dead horse; by and large, the field has abandoned such methods. This in itself is unfortunate; the use of quantitative methods may need to be revisited (with care). We return to the issue of experimental methods in section IV.

Slowly, and in various ways, U.S. mathematics education researchers made their way out of the paradigmatic and methodological straightjackets of the 1960s and 1970s. In many ways, work had simply run itself into the ground, and the field came
to recognize that fact. For example Kilpatrick discussed the methodological state of the art in the mid-1970s as follows:

No one is suggesting that researchers abandon the designs and techniques that have served so well in empirical research. But a broader conception of research is needed. . . .

Some years ago a group of researchers gave a battery of psychological tests each summer to mathematically talented senior high school students. . . . The scores on the tests were intercorrelated, and some correlation coefficients were significant, some not. Several research reports were published. . . . As Krutetskii (1976) notes, the process of solution did not appeal to interest the researchers—yet what rich material could have been obtained from these gifted students if one were to study their thinking processes in dealing with mathematical problems. Why were the students simply given a battery of tests to take instead of being asked to solve mathematical problems? It’s a good question. (Kilpatrick, 1978, p. 18)

In a hugely ironic twist, the study of mind was largely resuscitated by artificial intelligence (AI), the study of “machines that think.” Pioneering efforts in AI included computer programs such as Newell and Simon’s (1972) “General Problem Solver,” or GPS. GPS played a reasonable game of chess. It solved “cryptarithmetic” problems. And it solved problems in symbolic logic. Specifically, GPS derived 51 of the first 53 results in Russell and Whitehead’s (1960) famous mathematical treatise Principia Mathematica, and GPS’s proof of one result was shorter than the proof provided by Russell and Whitehead.

To write problem-solving programs, Newell and Simon asked people to solve large numbers of problems, working on them “out loud” so that Newell and Simon could record and later analyze what was done as their subjects worked on the problems. They transcribed the recordings, and pored over the transcripts, looking for productive patterns of behavior, that is, for strategies that mimicked the successful “moves” made by their subjects. Those strategies, once observed and abstracted, were then written up as computer programs.

The irony comes from the fact that AI provided the means for hoisting the behaviorists by their own dogmatic petard. AI programs “worked”; their record of problem solving was clear. More important, all of their workings were out in the open; every decision was overtly specified. By virtue of being inspectable and specifiable, work in AI met all of the behaviorists’ criteria for being scientific. At the same time, the success of the AI enterprise depended entirely on the investigation of human thought processes. Hence, the study of “mind” was legitimized. Studies of human thinking and research methods that involved reports of problem solving were once again scientifically “acceptable.”

Given the climate at the time, the legitimization of such methods was by no means easy. There was a great deal of controversy over the use of problem-solving protocols (records of out-loud problem solutions), and it took some years before the dust settled (see, e.g., Ericson & Simon, 1980; Nisbett & Wilson, 1977). At the same time, a wide variety of methods and perspectives became known internationally, sowing the seeds for the profusion of views and techniques that would flower in the latter part of the century. Piaget had, of course, been developing a massive

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1These are problems stated in forms such as

\[
\begin{align*}
S & \quad E & \quad N & \quad D \\
+ & \quad M & \quad O & \quad R & \quad E & \quad \Rightarrow & \quad M & \quad O & \quad N & \quad E \quad Y \\
D & \quad O & \quad N & \quad A & \quad L & \quad D \\
+ & \quad G & \quad E & \quad R & \quad A & \quad L & \quad D & \Rightarrow & \quad R & \quad O & \quad B & \quad E & \quad R & \quad T.
\end{align*}
\]

A solution to a problem consists of replacing each letter in the given form with a unique digit from 0 to 9 so that when all the replacements are made, the arithmetic sum that results is correct.
corpus of work on children’s intellectual development, both philosophical (e.g., Piaget, 1970) and with regard to various mathematical concepts such as number, time, and space (e.g., Piaget, 1956, 1969a, 1969b). Piaget’s work brought the “clinical interview” to prominence as a research method. The work of Krutetskii (1976) and colleagues (see Kilpatrick & Wirszburg, 1975) popularized the idea of “teaching experiments,” detailed studies of principled attempts at instruction and their consequences in terms of students’ abilities to engage with mathematics. Freudenthal’s (1973, 1983) work lay the foundation for the study of “realistic mathematics,” a central tenet of which is that mathematics instructional sequences should be grounded in contexts and experiences that support the development of meaningful mathematical abstractions.

Broadly speaking, the 1970s and the 1980s were a time of explosive growth. The “cognitive revolution” (see, e.g., Gardner, 1987) brought with it a significant epistemological shift—and with it, new classes of phenomena for investigation and new methods for exploring them. For much of the century, the focus of research in mathematical thinking and learning had been on knowledge, a body of facts and procedures to be mastered. As theoretical frameworks evolved, such knowledge was seen to be only one (albeit very important) aspect of mathematical thinking. Theoretical frameworks (see, e.g., Schoenfeld, 1985) indicated that central aspects of mathematical performance included the knowledge base, problem-solving strategies, aspects of metacognition, and beliefs. They invoked the notion of “culture” in that students were seen to engage in (often counterproductive) practices derived from their experiences in school, and which were quite different from the practices of the mathematical community. Each of these aspects of cognition was explored with a wide variety of emerging methods: observational and experimental studies, teaching experiments, clinical interviews, the analysis of “out-loud” protocols, computer modeling, and more.

There were, however, few “ground rules” for conducting such research, either in terms of investigatory norms or in terms of quality standards. Research in mathematics education had moved from a period of “normal science” to one in which the ground rules were unknown. Fundamental questions, not well addressed, became the following: How does one define new phenomena of interest? How does one look for them, document them? How does one make sense of things such as the impact of metacognitive decision making on problem-solving performance, the relationship between culture and cognition, or what might be an appropriate focus for investigation in the blooming complexity of a teaching experiment? The field began to address such issues (see, e.g., Schoenfeld, 1992).

The flowering of theoretical perspectives and methods continued through the end of the 20th century. Specifically, sociocultural perspectives had long roots. Vygotsky, for example, in both Mind in Society (1978) and Thought and Language (1962), had advanced a perspective that, perhaps in too-simple terms, could be seen as complementary to Piaget’s. Vygotsky and his theoretical allies argued that learning is a function of social interaction:

Human learning presupposes a specific social nature and a process by which children grow into the intellectual life of those around them. (Vygotsky, 1978, p. 88)

Every function in the child’s social development occurs twice: first, on the social level, and later on the individual level; first, between people (interpsychological), and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relations between human individuals. (Vygotsky, 1978, p. 57)

2Vygotsky died in 1934, so the roots of this work extend quite deeply. The 1962 and 1978 dates of publication of Thought and Language and Mind in Society represent the appearance of his work in English translation.
From the 1970s onward, multiple lines of research explored aspects of cognition and culture. For example, a series of studies conducted in Brazil (see Carraher, 1991, for a review) explored the relationships between mathematical understandings in school and in “real world” contexts such as candy selling. The main theoretical perspective adopted by the French for studies of mathematical didactics presumed the existence of a “didactical contract” that is inherently social in nature (see, e.g., Brousseau, 1997). German work, significantly shaped by Bauersfeld (e.g., 1980, 1993), took as a given that there are multiple realities and social agendas playing out in instruction, and that one must attend to “language games” (à la Wittgenstein) in the mathematics classroom. By the time of ICME VII in Quebec (1992), a multiplicity of competing theoretical perspectives had blossomed. The Proceedings of the VII International Congress on Mathematics Education’s Working Group on Theories of Learning (Steffe, Nesher, Cobb, Goldin, & Greer, 1996), for example, contains three large sections: “Sociological and anthropological perspectives on mathematics learning,” “Cognitive science theories and their contributions to the learning of mathematics,” and “The contributions of constructivism to the learning of mathematics,” as well as a fourth small section that includes explorations of metaphor as the possible basis for a theory of learning of mathematics (see also English, 1997; Sfard, 1994).

At the end of the century, one saw a proliferation of perspectives, of theories, and of methods. On the one hand, this was undoubtedly healthy: The field had escaped from the paradigmatic and theoretical straightjackets of the earlier part of the century, and it was virtually bursting with energy and excitement. This is entirely appropriate for a young field that is clearly not in a time of “normal science” (Kuhn, 1970). On the other hand, having let a thousand flowers bloom, it is time for researchers in mathematics education to prune their garden. We must begin to ask and address the difficult questions about theory and methods that will help us move forward.

### III. THE CURRENT STATE OF AFFAIRS

This section focuses on current needs. It begins with some assertions about desiderata for research, and moves on to a discussion of current challenges. It takes as background the current and somewhat chaotic state of research: that there are multiple and competing theoretical perspectives, and a host of methods that are tailored to specific problems but of limited general utility.

The first two assertions regarding desiderata for high-quality research live in dialectic tension:

1. One must guard against the dangers of compartmentalization. It is all too easy to focus narrowly, ignoring or dismissing work or perspectives not obviously related to one’s own. This can be costly, given the systemic and deeply connected nature of educational phenomena. Educators need a sense of the “big picture” and of how things fit together.

2. One must guard against the dangers of being superficial. Superficial knowledge (of information or methods) is likely to yield trivial research. Generally speaking, high-quality research comes when one has a deep and focused understanding of the area being examined and extended experience mulling over the issues under question. Needless to say, it is difficult to strive simultaneously toward both depth and breadth. Yet they are both necessary.

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3The discussion of issues 1 and 2 in this section is a brief reprise of an argument made in Schoenfeld (1999b); see that chapter for more extensive detail. The discussion of issue 3 is taken, with slight modifications, from Schoenfeld (1999a).
The third assertion has to do with the current state of theory and methods:

3. Educational research has reached the point where it is possible to conduct meaningful research in contexts that “matter” and not simply in the laboratory. Indeed, the traditional model of doing basic research and then applying it in context needs to be reconsidered.

The fourth and fifth assertions deal with the conduct of research and serve as announcements of themes to be elaborated at length in this chapter:

4. In conducting research, one must have a sense of where one stands and where one thinks he or she is heading. On stance: One has biases and a theoretical perspective, whether one thinks so or not. These affect what one “sees.” On direction: Methods are not used in the abstract or pulled off the shelf. Any research method is, in effect, a lens or filter through which phenomena are viewed (and possibly clarified, or distorted, or obscured). Thus, the ways that questions are framed should shape the ways that methods are selected and employed.

5. A fundamental issue both for individuals conducting their own studies and for the field as a whole is the need to develop a deep understanding of what it means to make and justify claims about educational phenomena. What is a defensible claim? What is the scope of that claim? What kinds of evidence can be taken as a legitimate warrant for that claim?

The discussion that follows provides some brief examples in support of the first two claims. The third assertion is elaborated at greater length, for it characterizes the evolving contexts within which it is now possible to do basic research and to “make a difference.” The fourth and fifth assertions point to the main substance of this chapter. A few points will be made briefly, as an orientation to what follows.

**Issue 1: The Dangers of Compartmentalization**

It is easy to find examples of the costs of educational myopia. One can point fingers at those administrators who fail to value subject matter understanding, such as those who want to know how (not if) they can “retrain” surplus social studies teachers, in the break between school years, to teach mathematics the following year. Or, one can point to an inverse problem, mathematicians who believe that knowing the subject matter is all that is required by way of teacher preparation. Indeed, major educational movements have been ill conceived and squandered large amounts of money, precisely because of their tunnel vision. Consider, for example, various attempts to provide students with preparation, in school, for productive work lives. This is known as the “school-to-work” problem.

Most proposed school-to-work programs are oriented toward skills: They may proceed by examining the workplace, identifying productive skills, and teaching them directly. Or, they may take a more “contextual” approach, with suggestions that students should engage in apprenticeships, that curricula should be designed to reflect workplace demands, or both. Such approaches are doomed to fail. There is, of course, a pragmatic reason. The skills set is a moving target in that skills learned today will be obsolete tomorrow, and new skills will be needed. More important, there is a deep theoretical reason. The past quarter century of research in mathematics education has shown that skills are but one component of mathematical performance. Problem-solving strategies, metacognition, beliefs, and domain-specific practices are also aspects of mathematical behavior. These are essential components of a theory of mathematical behavior. And it’s not just mathematics; a good case can be made that they are relevant in any domain.

Now, if you want people to be good at X, then you ought to have a theory of what it means to be good at X. A starting place for the dimensions of such a theory
should be established theories of competence from other areas. For example, school-to-work attempts that proceed in ignorance of theoretical frameworks and advances in mathematics education—ignoring, for example, crucial aspects of performance such as metacognition and beliefs—proceed at their own peril. Similarly, mathematics educators must ask, What frameworks or insights from other fields (for example, anthropology or studies of organizational behavior) does mathematics education ignore at its own peril? There is, thus, the need for great breadth.

### Issue 2: The Dangers of Superficiality

Breadth may be essential, as argued immediately above, but there is a significant breadth-versus-depth tradeoff. Expertise comes with focus (which takes time and energy), and the danger of a lack of focus is either dilettantism or superficiality. A case in point is discussed by Heath (1999), a linguist who pointed to the problem of researchers in education using methods from other fields without really understanding them. For example, Heath discussed “discourse analysis,” which, in educational research, often seems to mean “making sense of what people say by whatever ad hoc methods seem appropriate.” Heath noted that there are more than a half dozen different schools of discourse analysis, each with its own traditions, history, and methods. Moreover, each type makes particular kinds of contributions, takes a fair amount of time to master, and should not be used cavalierly by amateurs. Educational researchers who call their ad hoc attempts to make sense of dialogue “discourse analysis” are abusing the term.

Do we need such methods? The answer is a clear yes—more so as time goes on. But those who employ such methods need to be well enough steeped in them to use them with wisdom, and skill. This implies focus and depth.

### Issue 3: The Relationship Between Research and Practice

In *Pasteur’s Quadrant: Basic Science and Technical Innovation*, Donald Stokes (1997) discusses tensions between theory and applications in science and technology. Stokes argued that in both our folk and scientific cultures, basic and applied research are viewed as being in tension. For most, applications would seem the raison d’être of science. Stoke pointed out, however, that in elite circles, “pure” science has been considered far superior to its applications. A quote from C. P. Snow’s essay on “the two cultures” describes how scientists at Cambridge felt about their work: “We prided ourselves that the science that we were doing could not, in any conceivable circumstances, have any practical use. The more firmly one could make the claim, the more superior one felt.” (Snow, 1965, p. 32) (Recall G. H. Hardy’s famous (1967) quote, “I have never done anything ‘useful’…. The case for my life [is] …that I have added something to knowledge.”)

This perspective was reified by Vannevar Bush, who was asked by President Franklin Delano Roosevelt to map out a plan for post World War II scientific research and development. Bush’s report, *Science, the Endless Frontier* (1990), ultimately provided the philosophical underpinnings of the U.S. National Science Foundation (NSF).

Echoing Snow, Bush wrote that “basic research is performed without thought of practical ends” and that its defining characteristic is “its contribution to ‘general’ knowledge and an understanding of nature and its laws. (p. 18)” He went on to say that if one tries to mix basic and applied work, that “applied research invariably drives out pure.” Federal funding should support basic work, he argued; out of that basic work would come a broad range of applications. The tension between basic and applied work is represented in Fig. 18.1. A hypothesized progression from basic research to use-in-practice is represented in Fig. 18.2.
One can point to researchers whose work fits cleanly at various points in the spectrum illustrated in Fig. 18.1, paradigmatic examples being Niels Bohr and Thomas Edison. Bohr’s work on the structure of the atom was conducted without thought of applications; in contrast, Edison disdained theory while pouring his energy into “electrifying” the United States. You can imagine Bohr situated on the far left of Fig. 18.1, and Edison on the far right.

But what about Louis Pasteur? Pasteur’s work in elaborating biological mechanisms at the microbiological level, working out the “germ theory of disease,” is as basic as you can get. But Pasteur did not engage in this activity solely for reasons of abstract intellectual interest. He was motivated by problems of spoilage in beer, wine, and milk, and the hope of preventing or curing diseases such as anthrax, cholera, rabies, and tuberculosis.

At what point on the spectrum in Fig. 18.1 should one place Pasteur? Do we split him in half, with 50% at each end of the spectrum? Or should one “average” his contributions, placing him in the middle? Neither does him justice.

Stokes resolves this dilemma by disentangling these two aspects of Pasteur’s work, considering basic knowledge and utility as separate dimensions of research. He offers the scheme seen in Fig. 18.3.

Pasteur has a home in this scheme, and moreover, considerations of use and the quest for fundamental understanding are seen as living in potential synergy. Note that this conceptualization destroys the linearity of the hypothetical scheme in Fig 18.2. There are times when “basic” work can be done (indeed, may need to be done) in applied contexts. Fundamental research does not necessarily take place before, or in contexts apart from, those of practical use.

This perspective, elaborated by Stokes in the case of science, applies equally well to educational research. The idea, simply stated, is that a significant proportion of educational research can now be carried out in “real” contexts. The careful study of

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4 A caveat: the claim is that a significant proportion of educational research can (and when possible, should) be carried out in “real” contexts. However, at different points in the development of a field, it may be difficult for any one corpus of work to contribute simultaneously to both theory and practice. Sometimes the state of theory is such that it may best be nurtured, temporarily, aside from significant considerations of use (consider the origins of cognitive science, which was nurtured in laboratory studies). Sometimes the need to solve practical problems seems so urgent that theoretical considerations may be given secondary status (consider the post-Sputnik period, during which engineering efforts such as “putting a man on the moon” took priority). Figure 18.3 should be taken as a heuristic guide, with the upper right-hand quadrant representing a desirable site for work, when possible.
“design experiments” or other educational interventions can reveal important basic information about mathematical thinking, teaching, and learning.

This statement, however, raises profound questions for the conduct of research. Issues of how to make sense of “real-world phenomena” and how to justify the claims one makes are thorny indeed. Mathematics education in particular, and educational research in general, have yet to grapple adequately with methods and standards for making and judging such claims.

**Issue 4: On Knowing Where You Are and Where You Are Going**

One point made repeatedly in this chapter is that, whether or not researchers believe that they have theoretical perspectives and biases, they do. (Researchers who think otherwise are like the proverbial fish who are unaware of the medium in which they swim.) This observation is critically important, for one’s framing assumptions shape what one will attend to in research. Needless to say, they also affect the scope and robustness of one’s findings. Sections IV and V of this chapter address these issues at some length.

The second point that needs to be made here is more subtle and is easily misinterpreted: Research methods are best chosen when one has some idea of what it is one is looking for. A research method is a lens through which some set of phenomena is viewed. A lens may bring some phenomena sharply into focus, but it may also blur others at the same time and perhaps even create artifactual or illusory images. Moreover, to continue the metaphor, different lenses are appropriate for different purposes—the same individual may use one set of glasses for close-up work, one for regular distance, and complex devices such as telescopes for very long-distance work. So it is with methods: The phenomena we wish to “see” should affect our choice of method, and the choice of method will, in turn, affect what we are capable of seeing. And, of course, the kinds of claims one will be able to make (convincingly) will depend very much on the methods that have been employed.

Thus, researchers should be aware of the following questions and of the answers they propose to them:
• What theoretical perspective undergirds the work?
• What questions are being asked? What kinds of claims does one expect to make?
• What methods are appropriate to address these questions?
• What kinds of warrants do these methods provide in substantiation of the (potential) claims to be made?

These questions are essentially independent of the nature of one’s work; that is, they apply equally well to naturalistic research intended to provide “rich, thick descriptions,” to experimental methods employing statistical analyses, or to the construction of “models” representing various phenomena. If the researcher does not have good answers to them, there is a good chance the research will be seriously flawed.

It is essential to stress that not all decisions about methods must be made beforehand; the claims above are not intended to be either reductivist or positivist. Research is a dialectic process in which researchers come to grips with phenomena by living with them and understandings evolve over time. One can point to numerous studies in which important phenomena emerged midstream. Indeed, longstanding notions such as “grounded theory” and methodological tools such as the “constant comparative method” (see Glaser & Strauss, 1967) serve as codifications of the fact that sense making is an inductive process. The same is true of work that includes significant quantitative components: in “design experiments” (see, e.g., Brown, 1992) and various teaching interventions (see, e.g., Ball & Lampert, 1999; Schauble & Glaser, 1996), a great deal of data is gathered, and then sifting and winnowing process takes place. What is essential to understand, however, is that the sifting and winnowing are done with the purpose of answering specific (perhaps emergent) questions. If the question isn’t clear by the end of the process, the answer isn’t likely to be either.

Issue 5: What Is Believable, and Why?

This, of course, is the key question the field faces with regard to methods. It is, alas, all too infrequently addressed. One could hardly hope to answer the question in a chapter of this nature, but one can hope to bring it to the forefront and clarify aspects of it. Most of the balance of this chapter (sections IV, V, and VI) is devoted, directly or indirectly, to that enterprise.

IV. A VIEW OF THE RESEARCH PROCESS AND ITS IMPLICATIONS

Section II of this chapter noted that, especially in the decades following World War II, there was extensive use of experimental methods in education—and afterwards, the recognition that such methods had not produced much of lasting value. Partly as a result of those problems, such methods (modeled on those in the physical and biological sciences) have fallen out of favor. It is worth reconsidering the issue of experimental methods, to better understand why they contributed so little in the long run. The reasons for doing so are not merely historical, although postmortems often reveal interesting and useful information. The fact is that many of the problems that plagued experimental studies also have the potential to weaken or negate the value of studies that employ nonexperimental research methods. Those who do not learn from the mistakes of the past are doomed to repeat them.

This section of the chapter begins with a description of a conceptual framework within which one can examine the use of experimental methods. The framework is
FIG. 18.4. A simplified version of the modeling process.

employed to highlight potential difficulties with such methods—places where the work can be undermined if researchers are not appropriately careful. The framework is then expanded and modified so that it applies to nonexperimental methods as well. This will set the stage for later discussions (section VI) regarding the trustworthiness and robustness of educational research findings.

The use of statistical–experimental methods is a form of modeling. A simple diagram (see Fig. 18.4 above) and discussion, taken from Schoenfeld, 1994, highlight some of the issues involved in the use of such models.

It should be noted that statistical tests are conducted under the assumption that the “real-world” situations being considered conform to the conditions of specific statistical models; if they do not, the conclusions drawn are invalid. When the experimental conditions do match those of the statistical model, it is then assumed that the results of statistical analyses conducted provide valid interpretations of the real-world situations. This is represented by the dashed arrow at the bottom of Fig. 18.4.

The essential point to keep in mind when applying statistical models is that they, like any other models, are representations of particular situations—and the usefulness of the model will depend on the fidelity of the representation. The effective use of statistical or other modeling techniques to shed light on a real-world situation depends on the accuracy of all three mappings illustrated in Fig. 18.4: (a) the abstraction of aspects of the situation into the model, (b) the mathematical analysis within that model, and (c) the mapping of interpretations back into the situation. Even if the manipulations performed within the formal system (e.g., calculations of statistical significance) are correct, there is no guarantee that the interpretation of the results obtained in the formal system will accurately reflect aspects of the real-world system from which the model was abstracted.

There are numerous places where these mappings can break down. For example, statistical significance means nothing if the conditions under which the experimentations are done do not conform to the assumptions of the model underlying the development of the statistics; and it means little if the constructs being examined are ill-defined…. Although researchers adopted the language of “treatments” and “variables,” the objects they so named often failed to have the requisite properties: oftentimes, for example, an instructional “treatment” was not a univalent entity but was very different in the hands of two different experimenters or teachers. Similarly, if an instructional experiment used
different teachers for the treatment and control groups, then teacher variation (rather
than the instructional treatments) might account for observed differences; if the same
teacher taught both groups, there still might be a difference in enthusiasm, or in stu-
dent selection. In short, many factors other than the ones in the statistical model
—the variables of record—could and often did account for important aspects of the situation
being modeled. (Schoenfeld, 1994, pp. 700–701)

With some expansion, the scheme given in Fig. 18.4 can be modified into a scheme
that applies to all observational and experimental work, whether that work is qualita-
tive or quantitative in nature. See Fig. 18.5.\(^5\)

The first main change from Fig. 18.4 to Fig. 18.5 is the explicit recognition (seen
along the vertical dimension) that “reality” is never abstracted directly. There is, of
course, the fact that humans do not perceive reality directly; we interpret our sensory
images of the world through conscious or unconscious filtering mechanisms. More to
the point, however, is the fact that any act of codifying (our perceptions of) the real
world represents an act of selection and thus of theoretical commitment. Whatever
perspective the analyst adopts, some things are highlighted, and some are down-
played or ignored. This set of choices, the set of entities and relationships selected for
analysis, will be called the analyst’s conceptual model. It indicates what “counts,” from
the analyst’s perspective.

The second main change is the expansion from experimental methods to general
analytic methods. When one employs classical statistical–experimental methods, one

\(^5\)What follows is complex, but perhaps not complex enough. In Fig. 18.5, each of the boxes is static,
and each of the arrows is unidirectional. In reality, of course, the process of data interpretation is dynamic:
conceptual models and representational systems evolve as one comes to a better understanding of the
relevant phenomena, and the process is dialectic rather than linear. Readers who wish to wallow in the
complexities of the research process, among other things, may wish to explore Latour (1988, 1999).
typically performs standard statistical manipulations (t or z tests, factor analyses, etc.) under the assumption that the data gathered conform to the conditions of some formal statistical model. In “treatment A versus treatment B” comparison tests, for example, one gathers relevant data such as the scores of the two treatment groups on some outcome measure. These are data points in a formal statistical model (the upper left-hand corner of Fig. 18.4). The data are analyzed in accord with the conditions of the model (arrow 2 in Fig. 18.4). If the results are deemed statistically significant, the researcher typically draws the inference that the (significant) difference in performance can be attributed to the difference in the two treatments.

That situation can be abstracted as follows (see the top section of Fig. 18.5). Virtually any record of occurrences can be considered “data.” (The reliability and utility of such data are an issue, of course.) Such records may, for example, be field notes of anthropological observations—audiotapes or videotapes of a classroom or of one or more people engaged in problem solving, interview transcripts, or just about any permanent, record of events. This record (which already represents a “filtering” of events through the researcher’s conceptual lens) is then represented in some way for purposes of analysis. Videotapes may be coded for gestures or for nature and kind of interactions between people. Field notes and interview transcripts may be annotated and categorized. Then, the analyst pores over the data. There are myriad ways to do so, of course. Coded data may be analyzed statistically, as in the “process–product” paradigm, in which the independent variables were typically counts of specific classroom behaviors by teachers and dependent variables were measures of student performance on various tests (see, e.g., Brophy & Good, 1986; Evertson, Emmer, & Brophy, 1980). Statistical analyses may be integrated with observational data, as in The AAUW Report: How Schools Shortchange Girls (American Association of University Women, 1992) or Baler (1997). Patterns of behavior may be “captured” in a model, as in Schoenfeld (1998). Or patterns of observations and other data may be woven together in a narrative, as in Eisenhart, Borko, Underhill, Brown, Jones, and Agard (1993). No matter what the form, the point is that inferences are drawn, within a conceptual framework, on the basis of data captured in a representational system. Following such analyses, the researchers map their findings back to (their interpretation of) the phenomena they are investigating.

Seen from this perspective, all instances of interpretation and analysis, quantitative and qualitative alike, are seen to be similar in some fundamental ways. The interpretive pathway begins with the (conscious or unconscious) imposition of an interpretive framework. It continues with the selection and representation of data considered relevant to the question at hand and the interpretation of those data within the conceptual and representational framework. The interpretation is then mapped back to the “real world,” as an explanation and interpretation of the phenomena at hand.

Having established this general framework, I now point to some of the difficulties involved in traversing the pathways indicated in Fig. 18.5. Those difficulties point to potential pitfalls in quantitative and qualitative studies alike.

**Along the First Arrow: Focal Choices Reflect (Perhaps Implicit) Theoretical Commitments**

The first and perhaps most fundamental point that must be recognized in the conduct of educational research is that what researchers see in complex real-world settings is not objective reality but a complex function of their beliefs and understandings. In any setting, infinitely many things might catch one’s interest. Where one’s attention settles is shaped by what one believes is important and what one is prepared to see. Take, for example, the question of “what counts” in mathematical understanding. For the behaviorist/associationist, the central issue in mathematics learning is the efficacy
with which one masters standard procedures; understanding was defined to mean “performing the procedures well.” From this point of view, Clapp’s 1926 study of students performing a total of 3,862,332 arithmetic sums, described in the previous section, makes perfect sense. Indeed, from the radical behaviorist’s point of view, any invocation of mental processes above and beyond the strength of “bonds” developed by repeated practice was nonsense. In contrast, the Gestaltist Wertheimer looked for signs of structural understanding. Where traditionalists saw arithmetic competence, Wertheimer (1945/1959) saw “blind, piecemeal habits... tendencies to perform slavishly instead of thinking (p. 22).” Or, consider an event such as an hour’s mathematics lesson. From the “process-product” perspective (see, e.g., Brophy & Good, 1986), what counts are teacher behaviors and student performance on various outcome measures; other aspects of classroom interactions might be ignored. From the situative perspective, a central issue may be the joint construction of mathematical meaning in the classroom, through discourse (Greeno & the Middle-School Mathematics Through Applications Project Group, 1998). From a cultural perspective, one might focus on the typicality of certain instructional practices within and across nations (Stigler & Hiebert, 1999). From a microsociological perspective (see, e.g., Bauersfeld, 1995), one might focus on the nature of the classroom culture and the role of language in the classroom as a “medium between person and world.” Researchers viewing the same phenomena from within these different traditions might attend to, and “see,” very different things. The analytic frames they then construct, the conceptual models in Fig. 18.5, will then differ widely.

Along the Second Arrow: Do the Data, as Represented, Reflect the Constructs of Importance in the Conceptual Model?

Perhaps the most accessible cases in point regarding this issue come from the statistical and experimental paradigms. As noted in section II, “treatment A versus treatment B” comparisons are meaningless if the treatments are ill defined (e.g., the case of “advance organizers”), or if variables other than the ostensible “independent variables” in an experiment aren’t the only ones that affect performance on the outcome variables.

The question of data representation is central in all paradigms, however. To start with an obvious point, field notes are clearly selective; they represent the observers’ focus and biases. Less obvious but equally important, “objective” records such as videotapes also represent observers’ focus and biases. If there is one camera, where is it focused? On an individual or on a group? On the written work produced by individuals or groups or on their faces as they talk? (With more than one camera, one can get more “coverage,” but issues of focus remain.) Given a videotape record, what gets coded, and at what grain size, when the tape is analyzed? Compare, for example, the fine level of detail in the transcript coding scheme set forth by Lucas, Branca, Goldberg, Kantowsky, Kellogg, and Smith (1980) with the rather coarse-grained coding scheme found in Schoenfeld (1985). Both schemes were aimed at understanding “problem solving,” yet by their very nature, they supported very different kinds of analyses. Or compare the two transcripts of “Leona’s puppy story” by Sarah Michaels (pp. 241–244) and James Gee (pp. 244–246) in their discussion of discourse analysis (Gee, Michaels, & O’Connor, 1992). Michaels presented the story in narrative form, with a range of markers to indicate changes in pitch and intonation, timing, and more. False starts and repairs are included, in an attempt to capture a large part of the “spoken record” in written notation. In contrast, Gee stripped such markers from the text. He presented a cleaned-up version in “stanzas,” as a narrative poem, an “ideal realization” of the text. Here, too (and this is the point of the authors’ examples),
two different transcripts of the same oral record support two very different types of analyses. The form of representation makes a difference.

In short, the constructs in the conceptual model may or may not be well defined, and the ways data are represented may or may not correspond in straightforward ways to those constructs. To pick an example from the social realm, “turn taking” is easy to code, but coding “mathematical authority” and “social authority” (see, e.g., Cobb, 1995) is a much more delicate issue. The choices of what to code, and the accuracy, consistency, and grain size of the coding, will have a critical impact on the quality of the analysis.

Along the Third Arrow: What’s Meaningful Within the Representational Scheme? What Can Be Said About the Quality of the Inferences Drawn?

The third arrow asks the deceptively simple question, “What conclusions can be drawn within the given conceptual system, using the data as represented?”

Putting technical language aside for the moment, there are a series of common-sense questions that are natural to ask when someone proposes to make some judgments from a body of data. Those include the following:

- Are there enough data on which to base a solid judgment?
- Is the means of analysis consistent (i.e., will anyone trained in the analytical methods draw the same conclusions from the same data)?
- Does the data-gathering mechanism tap into stable phenomena (i.e., will someone be likely to produce similar data when assessed at different times, and will their interpretation be consistent)?

In terms of classical statistical methods, these questions are related to technical issues of sampling, reliability, and validity. There is, of course, a huge body of statistical and psychometric theory and technique that addresses those issues. Unfortunately, however, the theoretical underpinnings and the conditions of application for those theories and techniques mesh very poorly with evolving epistemological understandings regarding theories of competence in subject matter domains. In days gone by, tests such as the U.S. National Assessment of Educational Progress (NAEP) simply used “content by difficulty matrices,” in which test items reflected mastery of particular topics at various levels of difficulty. Currently, the situation is much more complex. Theories of mathematical understanding include aspects of competence such as the ability to employ problem-solving strategies, to employ self-regulatory skills effectively, and more. “Performance assessment” items may cross topic areas; a problem may be accessible to a solution via algebraic or geometric means, for example, or be solvable numerically or symbolically. Under such circumstances, standard psychometric techniques are woefully inadequate to provide knowledge profiles of students. New methods need to be developed (see, e.g., Glaser & Linn, 1997; Greeno, Pearson, & Schoenfeld, 1997).

In terms of the broad spectrum of research methods available to (mathematics) educators today, the questions highlighted above are both fundamental and extremely difficult. Section V of this chapter is devoted to addressing such issues.

Along the Fourth Arrow: Are Results Derived in the Representational System Meaningful in the Conceptual Model?

Arrow 4 is the mirror image of arrow 2, completing the analytic loop within the conceptual model. The pathway from arrow 2 through arrow 4 represents the gathering,
analysis, and interpretation of data, given the assumptions of the conceptual model. The main point here is that, no matter how fine the analysis within the representational system may be, the overall analysis is no better than the mapping to and from the conceptual model.

As one case in point, consider econometric analyses of school district expenditures vis-à-vis the effects of class size. Ofttimes precise data regarding actual class size are unavailable. In early studies, researchers used proxies for these input data (e.g., the ratio of “instructional staff” to students in a district, or some fraction thereof). But such ratios had the potential to be tremendously misleading because some districts’ figures included nonteaching administrators, and some did not. As a result, the input variables had no consistent meaning. The output variables were often standardized tests of basic skills or other “achievement tests.” Typically, these tests were only marginally related to the actual curricula being taught, and thus were dubious measures of the effectiveness of instruction. In short, both input and output variables in many such studies were of questionable value. No matter how perfectly executed the statistical analyses on such data might be, the results are close to meaningless.

Within the standard statistical paradigms, there are also well-known examples of “sampling error,” one example of which was a mid-20th-century telephone poll of voters in a U.S. presidential election. What the pollsters failed to realize was that telephones were not universal and that by restricting their sample to people who owned telephones, they had seriously biased the sample (and, indeed, made the wrong prediction). Much more recently, there is the fact that a large number of medical studies were conducted using only male patients. The samples were randomly drawn, and the statistics were properly done; the findings applied perfectly well to the male half of the population. The problem is that the findings were also assumed—in many cases incorrectly—to apply to the female half of the population as well.

Issues may be more subtle with regard to qualitative data, but they are there all the same. In the example just given, the poll was accurate for the population sampled, but not for the population at large. Similarly, some Piagetian findings, which were once thought to be universal, were later seen to be typical of middle-class Swiss children but not of children who had radically different backgrounds. Sampling error is every bit as dangerous a flaw in qualitative as in quantitative research.

The same is the case for issues of construct validity. In Piagetian clinical interviews, for example, children’s performance on certain (wonderfully clever) interview tasks was taken as evidence of the presence or absence of certain cognitive structures. Further studies revealed that although performance on certain tasks might be robust, the robustness was in part a function of the task design; other tasks aimed at the same mental constructs did not necessarily produce the same results. In terms of Fig. 18.5, the analysis within the representational system (performance on a set of tasks) was just fine, but the mapping back to the conceptual framework (the attribution of certain logico-deductive structures on the basis of the analyses) was questionable. The issues are hardly more straightforward when the constructs involved are things such as “power relationships” and “self-concept.”

Another example where construct validity is problematic is that of IQ. If IQ is defined by performance on various IQ tests, one obtains (relatively) consistent scores. But when one thinks of such scores as reflecting “intelligence,” one opens a can of worms. (Historically speaking, Binet thought of his tests as identifying places where people needed remediation, a somewhat questionable but defensible position. Later on, people took scores on IQ tests to represent the measure of an inherent (and immutable) capacity. That overextension has been the cause of unending problems.
Along the Fifth Arrow: How Well Do the Constructs and Relationships in the Conceptual Model Map Back into the Corresponding Attributes of the Original Situation?

It must be stressed that constructs that seem important in the representational system may or may not have much explanatory power—or even be meaningful—in the conceptual model (or the situation from which the model was abstracted). This can easily occur when the constructs in the model are arrived at statistically (e.g., when they are produced by methods such as factor analysis). “Verbal ability” in mathematical performance is one case in point.

As we complete the circuit in Fig. 18.5, it is worth recalling that the last arrow represents the completion of the representation and analysis process, and that the process involves working with selected and abstracted features of the situations they represent. Any use of a model or representation idealizes and represents a subset of the objects and relationships of the situation being characterized. The conceptual model may cohere, and analyses within it may be clear and precise, but the whole process is no better than any of the mappings involved, especially the mapping back into the original situation. All results must be interpreted with due caution, for they reflect the assumptions made throughout the entire process.

One quantitative case in point was the use, in the 1960s and 1970s, of factor analyses to determine components of mathematical ability. Various tests were constructed to assess students’ “verbal ability,” “spatial ability,” and more; then studies were done correlating such abilities with problem-solving performance. Over time, however, it became clear that most such “abilities” were almost tautologically defined, that is, you had “verbal ability” to the degree that you scored well on tests of verbal ability. Researchers were unable to explain how these abilities might actually contribute to competent performance, however.

If a whole field could delude itself in this way, imagine how easy it is for a single researcher to do the same! Much qualitative research consists of the construction of categories to represent perceived patterns of data. The analytic perspective that one brings to one’s work may well shape what one sees or attends to and thus which categories are constructed.

As an indication of the universe of possibilities, LeCompte and Preissle (1993, pp. 128–133) distill “major theoretical perspectives in the social sciences” into a table that covers six pages of small-sized print. Those perspectives, accompanied by a few of their major theoretical constructs, are as follows:

- **Functionalism** (systems, functions, goals, latent and manifest functions, adaptation integration, values, cultural rules)
- **Conflict theory** (many of the same concepts as functionalism, plus legitimacy, consciousness, domination, coercion)
- **Symbolic interactionism and ethnomethodology** (self, self-concept, mind, symbols, meaning, interaction, role, actor, role taking)
- **Critical theory** (resistance, human agency, repression, hegemony, subjectivity, political economy, consciousness [false and true])
- **Ethnoscience or cognitive anthropology** (cultural knowledge, cognitive processes, cognitive models)
- **Exchange theory** (cost, benefit, rationality, fair exchange, rewards, norms of reciprocity, satiation)
- **Psychodynamic theory** (id, ego, superego, culture and personality, neurosis, psychosis)
- **Behaviorism** (individual differences, stimulus, response, conditioning)
Given this extraordinary diversity of perspectives and constructs, one must ask: how can the field sort out which ones make sense; which perspectives are relevant and appropriate to apply in which conditions; and, how much faith can one put in any perspective or claim? These questions are the focus of section V.

V. STANDARDS FOR JUDGING THEORIES, MODELS, AND RESULTS

Given the wide range of perspectives, methods, and results in educational research, the following questions are essential to address. What grounds should be offered in favor of a general theory, or a model of a particular phenomenon? How much faith should one have in any particular result? What constitutes solid reason, what constitutes “evidence beyond a reasonable doubt”?

The following list puts forth a set of criteria that can be used for evaluating models and theories (and more generally, any empirical or theoretical work) in mathematics education:

- Descriptive power
- Explanatory power
- Scope
- Predictive power
- Rigor and specificity
- Falsifiability
- Replicability, generality, and trustworthiness
- Multiple sources of evidence (triangulation)

In this section, each is briefly described. In the next section, these criteria are invoked when various types of research are considered.

Descriptive Power

*Descriptive power* denotes the capacity of theories or models to capture “what counts” in ways that seem faithful to the phenomena being described. As Gaea Leinhardt (1998) pointed out, the phrase “consider a spherical cow” might be appropriate when physicists are considering the cow in terms of its gravitational mass, but not when one is exploring some of the cow’s physiological properties.

Simply put: Theories of mind, problem solving, or teaching (for example) should include relevant and important aspects of thinking, problem solving, and teaching, respectively; they should capture things that “count” in reasonable ways. At a very broad level, it is fair to ask: Do the elements of the theory correspond to things that seem reasonable? Is anything missing? For example, in the 1970s and 1980s researchers designed a fair number of data coding schemes to “capture” the actions taken by people as they tried to solve mathematics problems (see Lucas et al., 1980, for one such example) or to capture classroom actions (see, e.g., Beeby, Burkhardt, & Fraser, 1979). Here is one test of its descriptive power. Suppose you study the coding scheme and become proficient at its use. Suppose further that someone else proficient in the use of the scheme makes a videotape of the relevant phenomenon and then codes it according to the scheme. You are given the coding, which you examine. Then, when you look at the videotape, are there any “surprises,” relevant behaviors or actions that

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6This discussion is expanded from Schoenfeld (2000).
the coding scheme did not prepare you to see? If so, there is reason to question the
descriptive adequacy of the scheme.

More broadly, there is the question of whether an analytic scheme or representa-
tion takes the right factors into account. Suppose someone analyzes a problem-solving
session, an interview, or a classroom lesson. Would another person who read the anal-
ysis and then saw the videotape reasonably be surprised by things that were missing
from the analysis? This might call into question the theoretical underpinnings of the
approach. To take a historical example, consider the “process–product” approach, a
once-dominant paradigm in studies of teaching (Brophy & Good, 1986). Researchers
coded classroom behaviors (amount of time on task, frequency of direct questions
asked of students, etc.) and then explored correlations between the extent of those
behaviors and measures of student success, such as scores on standardized tests. Cur-
iously absent from such studies (and easy to see in hindsight, although not at all
apparent at the time) were what we now consider relevant cognitive considerations:
What did it really mean to understand the mathematics? How was it explained? What
content did the teacher and students focus on? How did the study of the relevant math-
ematical processes play out in the classroom, and how was it represented on the tests?
With 20–20 hindsight we can see such omissions in methods of the (recent) past. We
need to keep our eyes open for similar lapses in our current work.

Explanatory Power

Explanatory power denotes the degree of explanation provided regarding how and why
things work. It is one thing to say that people will or will not be able to do certain
kinds of tasks, or even to describe what they do on a blow-by-blow basis; it is quite an-
other thing to explain why. Consider, for example, the kinds of finely detailed coding
schemes for problem-solving behavior (Lucas et al., 1980) discussed above. They pro-
vided a wealth of detail regarding what the subjects did (along specific dimensions) but
little relevant information regarding how and why the subjects were ultimately suc-
cessful (or not) at solving the problems. Likewise for the process–product paradigm:
“classroom processes” were hypothesized to be related to “learning” and “performance outcomes,” but the mechanisms by which they were related went unexamined.

There are, at the current point, many alternative forms of explanation and de-
scriptions of mechanism; the field will need to sort these out, over time. Cognitive
explanations tend to focus on “what goes on in the head,” at some level of detail.
It is one thing, for example, to say that people will have difficulty multiplying two
three-digit numbers in their heads. But that does not provide information about how
and why the difficulties occur. A typical cognitive explanation would focus on a de-
scription of working memory. It would provide a description of memory buffers, a
detailed explanation of the mechanism of “chunking,” and the careful delineation of
how the components of memory interact with each other. Such explanation works
at a level of mechanism: It says in reasonably precise terms what the objects in the
theory are, how they are related, and why some things will be possible and some not.
Similarly, socioculturally and anthropologically oriented research aimed at explaining
what takes place in classrooms focuses on describing how and why things happen the
way they do. There are, of course, myriad ways to do this. For example, Bauersfeld’s
(1980) article “Hidden dimensions in the so-called reality of the classroom” provides
an alternative perspective on classroom events, elaborating on the “hidden agendas”
of students and teachers. Baler’s (1997) study of reform and traditional instruction
traces the impact of alternative classroom practices on students’ performance and
hypothesizes mechanisms to account for the very different patterns of gender-related
performance in the two instructional contexts. Stigler and Hiebert (1999) provided
coherent explanations for what might seem incidental or inexplicable phenomena.
For example, why is it that overhead projectors (OHPs), which are widely used in the United States, are rarely found in Japanese classrooms? After all, such technologies are easily accessible in Japan. The answer has to do with lesson coherence. OHPs are devices for focusing students’ attention. As such, they fit in wonderfully with typical instructional patterns in the United States and support teachers in saying “here is what you should be attending to, now!” A major goal of Japanese lessons, however, is to provide students with a coherent record of an unfolding story, reflecting the evolution of the lesson as a whole. Japanese teachers make careful use of the entire white- or chalkboard, providing a cumulative record of how a whole lesson unfolds. The OHP, with its limited focus and ephemeral nature, is not suitable for this purpose.

Scope

Scope denotes the range of phenomena “covered” by the theory. A theory of equations is not very impressive if it deals only with linear equations. Likewise, a theory of teaching is not very impressive if it covers only straight lectures.

One reason that there is currently so much theoretical confusion is that adherents of one approach or another rarely delineate the set of phenomena to which their theories apply and to which they do not. Buswell made this point a half century ago:

The very reason that there are conflicting theories of learning is that some theories seem to afford a better explanation of certain aspects or types of learning, while other theories stress the application of pertinent evidence or accepted principles to other aspects and types of learning. It should be remembered that the factual data on which all theories must be based are the same and equally accessible to all psychologists. Theories grow and are popularized because of their particular value in explaining the facts, but they are not always applied with equal emphasis to the whole range of facts. (Buswell, 1951, p. 144)

When he wrote, Buswell was explaining that behaviorism explained some things well (and still does) while not being of much use with regard to some other phenomena; likewise for “field theories” such as Gestaltism. But the point is general, and research will only make progress if researchers take care to specify what a theory (or a model, or piece of research) actually does do and does not.

One case in point is the Teacher Model Group’s work studying teachers’ online decision making in the classroom, called a “theory of teaching-in-context” (Schoenfeld, 1998, 1999c). The goal of that research is to provide an explanation of every decision made by a teacher while engaged in the act of teaching, as a function of the teacher’s knowledge, goals, and beliefs. This is ambitious, and the work is carried out at a very fine-grained level of detail. At the same time, the constraints of the theory and its associated models of teachers are carefully spelled out. It is not a theory of teaching (writ large) or a theory of “what happens in the classroom.” For example, the theory provides a view of “classroom reality” only as seen from the teacher’s point of view; each student’s view will certainly differ, and that of an observer focusing on the class as a “dynamic entity” will differ as well. External constraints (e.g., the politics of schooling) are not modeled directly, although the teacher’s perception of them is included as part of the model. Changes in the teacher (i.e., learning as a function of experience) are not modeled: The way the model works is that the teacher’s understandings are modeled at the beginning of a lesson and serve as the basis for the analysis that follows. That is, given what we know about the teacher (including his or her history with the students and understandings of them, understanding of content, etc.) right now, here is how he or she is likely to react to the “next” thing students do. In short, the research group has taken pains to specify what the theory of teaching-in-context
does do and what it does not. It can then be held accountable (according to some of the criteria enunciated in this section) for the adequacy with which it addresses the phenomena it claims to address.

**Predictive Power**

“Prediction” in education and the social sciences is a touchy business. Claiming to have a model of some form of behavior or a model of an individual (for example, modeling someone’s teaching) is likely to raise hackles almost immediately. A typical response is, “People are individuals, they have free will, they make on-the-fly decisions; how can you possibly can predict what they’ll do?” And of course, one can’t in the sense of saying precisely how someone will act in any situation. The idea of suggesting that one can predict someone’s actions seems reductive and dehumanizing, yet prediction is possible and important, if not essential.

For those in the sciences, prediction is a sine qua non of theory. Most theories in mathematics and the sciences allow for predictions of the type, “When X happens in certain circumstances, then Y happens.” Of course, “Y happens” can take various forms. The kinds of predictions that make people nervous when they think about predictions of human behavior are those like the definitive predictions from classical mechanics (specifying the motion of particles subjected to specific forces) and chemistry (specifying the precise amount of radioactive decay, or the precise substances and quantities that will emerge from a chemical reaction). There are many other forms of prediction, however. Consider, for example, models of predator–prey relationships. Once the initial assumptions are fed into a model, the model predicts the change of the populations relative to each other. Such models predict specific trends (with numbers attached), and the accuracy of the predictions can be measured against the actual populations of predators and prey. Predictions may be in the form of statistical distributions, as in the case of Mendelian genetics. In this case, evaluation of the predictions is easy: Does the population of offspring have the distribution that the theory predicts? In other cases, predictions can be converted into statistical or probability distributions. Weather forecasting also gives rise to statistical distributions. The question is, over time, what percent of the time did it rain, on those days when the forecaster said there was a (e.g.) 30% chance of rain? Also, predictions may be in the form of constraints, statements of what is possible or impossible. Evolutionary theory is a case in point. Whatever evolutionary theory is proposed must apply not only to known data but to previously unexamined fossil records as well. That is, the theory predicts what properties sequences of fossils in geological strata can or cannot have. A cumulative fossil record consistent with the theory is taken as substantiation for the theory, and any discrepant fossil record that is discovered will be considered very problematic for it. In short, even theories such as evolution, which are anything but deterministic, support strong predictions. The question for educational studies is, What kinds of predictions does a proposed theory support?

Sometimes it is possible to make precise predictions. For example, Brown and Burton (1978) studied the kinds of incorrect understandings that students develop when learning the standard U.S. algorithm for base 10 subtraction. They hypothesized very specific mental constructions on the part of students, the idea being that students did not simply fail to master the standard algorithm, but rather that students often developed one of a large class of incorrect variants of the algorithm (“bugs”) and applied it consistently. Brown and Burton developed a simple diagnostic test with the property that a student’s pattern of incorrect answers suggested the false algorithm he or she might be using. About half of the time, they were then able to predict the specific incorrect answer that a student would obtain to a new problem, before the student worked the problem.
Such fine-grained and consistent predictions on the basis of something as simple as a diagnostic test are extremely rare, of course. For example, no theory of teaching can predict precisely what a teacher will do in various circumstances; human behavior is simply not that predictable. A theory of teaching or a model of a particular teacher, however, can make specific predictions of the kinds just discussed. It can suggest constraints (“in these circumstances, this teacher will not do the following . . .”), and it can suggest likely events (“Given this chain of events, there is a 70% chance the teacher will respond in the following way, and a 30% chance the teacher will respond this way instead”). Such predictions can be made without being either reductive or dehumanizing.

It should also be noted that making predictions is a powerful tool in theory refinement. When something is claimed to be impossible and it happens, or when a theory makes repeated claims that something is very likely and it does not occur, then the theory has serious problems! Thus, engaging in such predictions is an important methodological tool, even when it is understood that precise prediction is impossible.

**Rigor and Specificity**

Constructing a theory or a model involves the specification of a set of objects and relationships among them. This set of abstract objects and relationships supposedly corresponds to some set of objects and relationships in the “real world.” The relevant questions are the following: How well defined are the terms? Would you know one if you saw one? In real life, in the model? How well defined are the relationships among them? And, how well do the objects and relations in the model correspond to the things they are supposed to represent? Of course, one cannot necessarily expect the same kinds of correspondences between parts of the model and real-world objects as in the case of simple physical models. Mental and social constructs such as “memory buffers” and the “didactical contract” (the idea that teachers and students enter a classroom with implicit understandings regarding the norms for their interactions and that these understandings shape the ways they act) are not inspectable or measurable in the ways physical objects are. But we can ask for detail, both in what the objects are and in how they fit together. Are the relationships and changes among them carefully defined, or does “magic happen” somewhere along the way? Here is a rough analogy. For much of the 18th century, the phlogiston theory of combustion, which posited that in all flammable materials there is a colorless, odorless, weightless, tasteless substance called “phlogiston” liberated during combustion, was widely accepted. (Lavoisier’s work on combustion ultimately refuted the theory.) With a little hand waving, the phlogiston theory explained a reasonable range of phenomena. One might have continued using it, just as theorists might have continued building epicycles upon epicycles in a theory of circular orbits.7 The theory might have continued to produce some useful results, good enough “for all practical purposes.” That may be fine for practice, but it is problematic with regard to theory. Just as in the physical sciences, researchers in education have an intellectual obligation to push for greater clarity and specificity and to look for limiting cases or counterexamples to see where the theoretical ideas break down.

Here are some quick examples. The model of the teaching process constructed by the Teacher Model Group (Schoenfeld, 1998, 1999c) includes components that represent aspects of the teacher’s knowledge, goals, beliefs, and decision making. Skeptics

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7This example points to another important criterion, *simplicity*. When a theory requires multiple “fixes” such as epicycles upon epicycles, that is a symptom that something is not right.
(including the authors) should ask, how clear is the representation? Once terms are defined in the model (i.e., once a teacher’s knowledge, goals, and beliefs are described) is there hand waving when claims are made regarding what the teacher might do in specific circumstances, or is the model well defined enough so that others could “run” it and make the same predictions? These criteria—are the objects and relations in the model well specified and is the correspondence between those entities and the entities they are supposed to represent clearly delineated?—should be applied whenever researchers claim to have a model of some phenomenon. For example, Lesh and Kelly (2000) claimed that there are three levels of models in their three-tiered teaching experiments: models created by students, teachers, and researchers. Are the models specified and inspectable? Similarly, APOS theory (see Asiala, Brown, de Vries, Dubinsky, Mathews, & Thomas, 1996) uses terms such as action, process, object, and schema. Would you know one if you met one? Are they well defined? Are the ways in which they interact or become transformed well specified? In all these cases, the bottom line issues are, “What are the odds that the so-called theory or model is a phlogiston-like theory? Are the people employing the theory constantly testing it to find out?” Similar questions should be asked about all of the terms used in educational research (e.g., the “didactical contract,” “metacognition,” “concept image,” and “epistemological obstacles”). They should be applied to all of the theoretical constructs in the long list that ended section IV. (In the biased view of this author, many if not most of the constructs fail the test. We have our work cut out for us.)

Falsifiability

The need for falsifiability, for making nontautological claims or predictions whose accuracy can be tested empirically, should be clear at this point. Simply put: If you can’t be proven wrong, you don’t have a theory. A field makes progress (and guards against tautologies) by putting its ideas on the line.

Replicability, Generality, and Trustworthiness

Replicability, like prediction, is controversial. It should be, if one takes the spirit and meaning of replicability from the experimental sciences: if one does “exactly the same thing,” will the same results occur? Given the variability of people and contexts, that strict notion of replicability is rarely appropriate for educational research. Moreover, one should not expect many educational studies to be replicable: There is a wide range of studies that deepen our understandings without making general claims. For example, biographical studies may help readers understand how certain forces shaped the lives of certain individuals without claiming that others would necessarily act in the same way. Studies of how attempts at “reform” played out in various school districts are similarly not replicable; readers may derive important lessons from them, but there is no expectation that similar attempts at reform in similar school districts will necessarily play out in the same ways. Likewise, some studies of teaching may have the primary value of enhancing readers’ understandings of the subtleties and complexities of teachers’ classroom actions and what drive them. Cooney’s (1985) study of “Fred” shows how a teacher can avow the importance of teaching for problem solving but come to teach in a very traditional way. Cooney presented evidence that Fred, despite his rhetorical homage to Pólya, understood problem solving to be a motivational

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8“Folk wisdom” is a case in point. Everything can be explained (at least post hoc) by folk wisdom. Depending on circumstances, for example, you can invoke the maxim “haste makes waste” to say that things must be done slowly and carefully, or “a stitch in time saves nine” to say that being speedy is essential. A “theory” that explains everything explains nothing.
device rather than a way engaging in mathematics. Hence, when students did not value his use of motivational problems, and he felt pressure to make sure that the students understood core content, he jettisoned “problem solving” to spend more time on “basics.” David Cohen’s (1990) study of “Mrs. Oublier” provides similar insights. Mrs. Oublier claimed to have adopted reform methods, but her understanding of reform was rather shallow, and many of her established teaching habits undermined her attempts to adopt reform practices. From such studies readers learn to look at teaching in more subtle, nuanced ways, but they do not expect other teachers to behave precisely the ways that Fred or Mrs. Oublier did.

Replicability is an issue, however, when theoretical claims are made and also when claims are made regarding the generality of various phenomena. If a theory posits that people have certain mental structures, for example, then other researchers should expect to document the existence of such structures. A paradigmatic case is that of short-term memory. George Miller’s famous 1956 paper, “The Magic Number Seven, Plus or Minus Two: Some Limits on Our Capacity for Processing Information,” makes the claim that the capacity of short-term memory is strictly limited—that people typically have between five and nine short term memory “buffers” that hold information, temporarily, while performing mental actions. Such a limitation would put serious constraints on the capacity of individuals to perform a wide range of mental actions. For example, multiplying two three-digit numbers, say 384 × 673, requires keeping track of more than nine subtotals. Most people will not be able to perform this task with their eyes closed because they will forget some of the numbers involved before they can complete it. This finding can be easily replicated, and the fact that it can be establishes the robustness of the finding.

Claims about other cognitive structures or patterns can be subjected to comparable tests of robustness. For example, much of the early work on aspects of metacognition or on the development and impact of beliefs on students’ mathematical performance has been replicated with students at various age levels.

Similar observations can be made with regard to sociocultural or ethnographic perspectives. Of course, many such studies do not bear replication; they provide insights into particular situations and contexts, which cannot be “duplicated” in any meaningful sense. One can examine the robustness of theoretical constructs, however, by asking about the consistency with which they are applicable and informative in contexts where they are said to apply. For example, the idea of the “didactical contract” (see, e.g., Brousseau, 1997) has been at the foundation of a large body of French educational research for some decades and has provided a consistent and productive orientation to empirical studies.

It should be noted that issues of replicability, generality, and trustworthiness are deeply connected to the issues of rigor and specificity discussed above. The ability to replicate a study or to employ a theoretical construct in the way it was employed by an author depends on the original work being well enough defined that other researchers following in the footsteps of the authors can employ methods or perspectives that are quite close to the original. This should be obvious, but historically it has not been. Consider this case in point from the classical education literature. Ausubel’s (1968)

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9It is possible to perform calculation such as 384 × 673 mentally by rehearsing the subtotals. For example, one can calculate 3 × 384 = 1152 and repeat “1,152” mentally until it becomes a “chunk” which only occupies one short-term memory buffer. “Chunking” is a well-documented mechanism by which people can perform mental tasks.

10The hypothetical limit of the number of short-term memory buffers would be of little interest if it applied only to tasks such as multiplication. Miller’s finding, however, has great scope: There is a wide range of tasks, from many domains, on which people begin to falter badly when the number of things they have to “keep in mind” approaches seven.
theory of “advance organizers” postulates that if students are given an introduction to materials they are to read that orients them to what is to follow, their reading comprehension will improve significantly. After more than a decade and many, many studies, the literature on the topic was inconclusive: About half of the studies showed that advance organizers made a difference, about half did not. A closer look revealed the reason: The very term was ill defined. Various experimenters made up their own advance organizers based on what they thought they should be—and there was huge variation. No wonder the findings were inconclusive!

There are, of course, standard techniques within both the cognitive and anthropological traditions for dealing with such issues. One is the training of multiple observers. A series of trained observer–analysts can work through the same body of data independently and “compare notes” afterward. If all goes well, those observers should “see” pretty much the same things. And, if the constructs are truly well defined and communicated, researchers from outside the original research group should be able to learn the techniques and, when given the data (such as videotapes, etc.) draw essentially the same conclusions. It should be noted that the cognitive/experimental and social/anthropological communities each have their own approaches to these issues but that there is overlap in spirit if not in detail. Within the cognitive community, for example, there is a tradition of computing “intrarater reliability” to identify the degree to which independent researchers assign the same coding to a body of data (say a transcript or videotape). Those who work within anthropological traditions tend to discuss the “trustworthiness” of a study. For a discussion of the relationship between these two traditions, see Moschkovich and Brenner (2000). For more extended discussions of these constructs, see LeCompte et al.

One source of trustworthiness is “multiple eyes” on the same data. Another, to which we now turn, is having multiple lines of evidence or argument that point to the same interpretations or conclusions.

Multiple Sources of Evidence (“Triangulation”)

Argumentation in education is much more complex than in mathematics and the physical sciences. In mathematics, one compelling line of argument (a proof) is enough; validity is established. In education (more broadly, in the social sciences), we are generally in the business of looking for compelling evidence. The fact is, evidence can be misleading. What we think is general may in fact be an artifact or a function of circumstances rather than a general phenomenon.

Here is one example. Some years ago, I made a series of videotapes of college students working on the problem, “How many cells are there in an average-sized human adult body?” Their behavior was striking. A number of students made wild guesses about the order of magnitude of the dimensions of a cell—from “let’s say a cell is an angstrom unit on a side” to “say a cell is a cube that’s 1/100 of an inch wide.” Then, having dispatched with cell size in seconds, they spent a very long time on body size, often breaking the body into a collection of cylinders, cones, and spheres and computing the volume of each with some care. This was very odd.

Some time later, I started videotaping students working problems in pairs rather than by themselves. I never again saw the kind of behavior described above. It turns out that when they were working alone, the students felt they were under tremendous pressure. They knew that a mathematics professor would be looking over their work. Under the circumstances, they felt they needed to do something mathematical, and volume computations at least made it look as if they were doing mathematics! When students worked in pairs, they started off by saying something like, “This sure is a weird problem.” That was enough to dissipate some of the pressure, with the result being that there was no need for them to engage in volume computations to relieve
it. In short, some very consistent behavior was actually a function of circumstances rather than being inherent in the problem or the students.

One way to check for artifactual behavior is to vary the circumstances, to ask, Do you see the same thing at different times, in different places? Another is to seek as many sources of information as possible about the phenomenon in question and to see whether they portray a consistent “message.” In modeling teaching, for example, the Teacher Model Group draws inferences about the teacher’s behavior from videotapes of the teacher in action, but it also conducts interviews with the teacher, reviews his or her lesson plans and class notes, and discusses tentative findings with the teacher. In this way the group deliberately seeks convergence of the data. The more independent sources of confirmation there are, the more robust a finding is likely to be.

For additional discussions of the issues discussed in this section of this chapter, see Clement (2000), Cobb (2000), and Moschkovich and Brenner (2000). Clement’s comments are grounded in his experience using clinical interviews to build models of students’ understandings of a range of science concepts. A key concept for Clement is the viability of a model. He offered (p. 560) a set of criteria for evaluating the viability of models and theories that overlaps significantly with those discussed here. Cobb, in a discussion grounded in his experience with teaching experiments, focuses on the generalizability and trustworthiness of analyses. Moschkovich and Brenner provided an overview of both traditional and naturalistic approaches to these issues.

VI. A HEURISTIC FRAMEWORK FOR SITUATING RESEARCH STUDIES, AND A SET OF ISSUES IT RAISES

Prologue: A Structural Dilemma

Researchers in mathematics education now have access to an extraordinarily wide array of methods. They confront enduring questions regarding which kinds of methods are appropriate in which circumstances, a problem exacerbated by the variety of methods currently available. My original intention for this section of this chapter was to provide a selective overview of some relevant categories of research methods and to raise some issues about their use. This is by no means a straightforward task. Indeed, as I worked to organize this section, I came to realize that the very notion of an “overview of methods” is likely to be a fruitless endeavor. More central, and more to the point, are questions regarding the purposes of the research undertaken and the kinds of information that various research methods can yield; when those are understood, the selection of methods and their application should follow. Thus, rather than offering a taxonomy of methods, this section offers what might be considered a heuristic guide to thinking about different kinds of claims that are made in educational research and the warrants researchers might produce to justify those claims.

Because the idea of a taxonomy of methods has clear face validity and might seem natural to the reader, it is worth explaining why that approach was abandoned. When I constructed the outline of this chapter, it seemed logical that at some point I would discuss what might be considered “rough equivalence classes” of approaches to research, raising some issues concerning the character of each. If one decides to take that approach, it is hardly necessary to reinvent the wheel; others have produced state-of-the-art categorizations. It seemed reasonable, therefore, to base the taxonomy on recent categorizations of current research. An obvious candidate for a starting point was the Handbook of Research Design in Mathematics and Science Education (Kelly & Lesh, 2000). Its editors
chose to emphasize research designs that are intended to radically increase the relevance of research to practice. Examples of such research designs include:

- Teaching experiments
- Clinical interviews
- Analyses of videotapes
- Action research studies
- Ethnographic observations
- Software development studies
- Computer modeling studies (Kelly & Lesh, 2000, p. 18)

My expectation was that I would supplement this categorization of designs (many of which reside in “Pasteur’s quadrant”) with discussions of more traditional approaches to educational research, such as experimental designs and statistical studies.

To my dismay, such an approach turned out to be impossible. The reason is that on closer examination the set of categories given above turns out to be fundamentally incoherent. This incoherence is on at least two dimensions: ill-definedness and categorical overlap. Regarding the former, consider, for example, the term “teaching experiment.”

In general, teaching experiments focus on development that occurs within conceptually rich environments that are explicitly designed to optimize the chances that relevant developments will occur in forms that can be observed. The time periods that are involved may range from a few hours, to a week, to a semester or an academic year. Furthermore the environment being observed may range from small laboratory-like interview rooms, to full classrooms, to even larger learning environments. (Kelly & Lesh, p. 192)

Such an all-encompassing definition allows for studies that bear little resemblance to each other—with regard to any of context, focus, or investigatory method(s)—to be considered members of the same category. In the handbook’s section on teaching experiments, for example, Lesh and Kelly (2000) described “multitiered” teaching experiments in which teams of students “work on a series of model-eliciting activities,” participating teachers “construct and refine models to make sense of students’ modeling activities, and researchers ”develop models to make sense of teachers’ and students’ modeling activities.” In a chapter titled “Teaching Experiment Methodology: Underlying Principles and Essential Elements,” Steffe and Thompson (2000) focused on developing an understanding of “students’ mathematics. . . whatever might constitute students’ mathematical realities” (p. 268). Their goals were in many ways consonant with the goals of traditional experimental studies, although their methods were radically different. Their teaching experiment was, in essence, a form of hypothesis testing as follows: “Suppose we identify two groups of students who (as far as we can tell) have developed for themselves different understandings of the counting process.\footnote{In simplest terms, one group of students used what is called the “count all” strategy for addition, whereas the second group had developed the “counting on” strategy.} We hypothesize that these two groups of students will respond differentially to a particular kind of instruction, with the gaps between the two groups increasing as a result of instruction.” In short, Steffe and Thompson were conducting an experiment with the expectation that students’ specific (attributed) cognitive structures would interact with instruction in particular ways. They noted the following:

We use experiment in “teaching experiment” in a scientific sense. The hypotheses in the teaching experiment [described immediately above] were that the differences between...
children of different groups would become quite large over the 2-year period and that
the children within a group would remain essentially alike. That the hypotheses were
confirmed is important, but only incidental to our purposes here. What is important is
that teaching experiments are done to test hypotheses as well as to generate them. One does not
embark on the intensive work of a teaching experiment without having major research hypotheses
to test. (Steffe & Thompson, 2000, p. 277; emphasis added.)

There are indeed some similarities between the studies reported by Lesh and Kelly
and by Steffe and Thompson, including the iterative and reflective character of the
studies. But the differences far outweigh the similarities. Lesh and Kelly characterized
their work as “a longitudinal development study in a conceptually rich environment”
(p. 197), while Steffe and Thompson characterized their work as a (very rich) form of
hypothesis-testing experiment.

These two examples alone point to the fundamental incoherence of the category,
and the problem gets worse when one considers other examples of the category given
in the Handbook or classical examples such as the found in the Soviet Studies in School
Mathematics (Kilpatrick & Wirszup, 1975).

This problem was exacerbated by the significant overlap of the various categories
listed above. For example “analyses of videotapes” are employed in all of the cat-
egories listed above—in teaching experiments, clinical interviews, action research,
ethnographies, and computer-based development and modeling studies. A large
number of “action research studies” are teaching experiments (in the sense defined by
Lesh and Kelly, above) and vice versa. Software development studies often involve
teaching experiments as part of their design. And so on.

In short, the kind of taxonomy offered by the handbook—the kind of taxonomy
I had hoped to use as the basis for this section of this chapter—is fundamentally
incoherent. It is based on surface structure rather than deep structure. The problem
for this chapter, then—and for the field—becomes: What is an appropriate deep structure
for conceptualizing and organizing research in (mathematics) education?

Would that there were a straightforward or clear answer to this question. This
section offers one tentative approach, which can be considered preliminary at best.
In keeping with much of the qualitative literature (see, e.g., Cobb, 2000), I argue that,
whether one is discussing quantitative or qualitative research, generality (or scope)
and trustworthiness are two fundamental dimensions of research findings; and that
importance is a third. In what follows, I briefly elaborate on this perspective. I then use
this frame to structure the discussion of a number of illustrative examples. The goal
is to provide a way of thinking about the implications of various findings—how well
they are warranted, and how widely they apply. The discussion will proceed along
the “generality” dimension of the framework. I start with examples of little generality
and discuss their properties (specifically, their trustworthiness and importance). The
discussion then proceeds through a series of examples of increasing generality.

A Provisional Organizational Frame

Figure 18.6 provides a schematic representation of a three-dimensional framework
for considering the character of research studies in education. As suggested in the
previous paragraph, three main attributes by which a study can be judged are the
following:

- Generality, or Scope. The claimed generality of a study is the set of circumstances
  in which the author(s) of a study claim that the findings of the study apply. The
  potential generality of a study is the set of circumstances in which the results of
  the study (if trustworthy) might reasonably be expected to apply.
FIG. 18.6. Three important dimensions along which studies can be characterized.

- Trustworthiness. The issue is, how well substantiated (according to many of the criteria elaborated in section V) is the claimed generality of the study? How solid are the warrants for the claims? Do they truly apply in the circumstances in which the authors assert that the results hold?
- Importance: To put things bluntly, how much should readers care about the results?

Here are a few examples to clarify the way these constructs are used here.

One classic study, conducted by Harold Fawcett, is reported in the 1938 yearbook of the U.S. National Council of Teachers of Mathematics, *The Nature of Proof*. Fawcett provided a richly textured description of a course in geometry that he developed and taught. The fundamental goals of the course were to (a) help students develop a deep understanding of the concept of proof in mathematics through the study of geometry and (b) to link those understandings, and the reasoning processes involved, to “real-world” deductive reasoning. Fawcett describes the nature of instruction with some care. Readers get a sense of classroom discourse, of the kinds of questions the class debated, and of the flow of argument. Fawcett provided instructional artifacts, such as the forms students used to analyze arguments in the media, and he described classroom discussions concerning those arguments. He provided a list of geometric results derived by the class, so readers can develop a good sense of the curriculum. And, he offered multiple forms of evaluations of student performance in the course: Student scores on a statewide test of plane geometry; a “transfer” test of reasoning in nonmathematical situations; data from students regarding their reasoning outside the course; comments from parents regarding their children’s abilities to think critically; comments from six external observers; and student testimonials.

The body of evidence offered by Fawcett is compelling and convincing, hence his research report scores quite high on the *trustworthiness* dimension. Its score on the *generality* dimension, in contrast, is quite low: Fawcett offered compelling detail in the description of one class, and a rather unusual one at that. His report is, in essence, an existence proof. Fawcett showed that it is possible to offer instruction from which students can develop a deep understanding of geometry and that his students were able to apply the reasoning skills that they learned in the course to “everyday” arguments as well. From my perspective, that makes his findings quite important; it shows that such goals can be achieved. (Consider by analogy another existence proof, Orville & Wilbur Wright’s flights at Kitty Hawk in 1903. The Wright brothers made four flights in one day, the last of which lasted 59 seconds and covered 852 feet. The evidence of engine-powered heavier-than-air flight was trustworthy. There was no generalization at that point, but the very fact that flight was achieved
ultimately opened up a world of possibilities.) As I describe in the paragraphs that follow, a fair number of important studies are of this type. (And, of course, pioneering studies are often followed by replications and extensions, which serve to establish the generality of the findings.)

As a second example, in the 1970s and 1980s there were a large number of studies (e.g., Clement, 1982; Clement, Lochhead, & Monk, 1981; Rosnick & Clement, 1980) that dealt with the “students and professors” problem:

Using the letter \( S \) to represent the number of students at this university and the letter \( P \) to represent the number professors, write an equation that summarizes the following sentence: “There are six times as many students as professors at this university.”

Numerous replications of the original studies in a wide range of contexts indicated that more than half of the undergraduate students not majoring in mathematics, science, or engineering produce the equation \( P = 6S \) instead of \( S = 6P \). A wide variety of explanations for this phenomenon were offered, and some attempts at remediation were made on the basis of those explanations. After some time, however, the well ran dry. Compelling explanations of the phenomenon did not emerge, and attempts at remediation were not demonstrably successful. The field’s attention turned elsewhere.

In terms of the criteria discussed above, this body of research is trustworthy, at least in the sense that the phenomenon was robust and easily replicated. Generality is relatively low, in that the phenomenon, although robust, was never tied to any theoretical ideas that had significant scope. And ultimately, the findings, although “hot” for some time, were relatively unimportant.

Note that there can be very different sources of trustworthiness, depending on the nature of the claims and the methods involved. Fawcett’s work, although “small \( n \),” was trustworthy because of the richness of the analyses and consistency of the data. The “students and professors” data were trustworthy because of the replicability of the phenomenon and the large number of data points involved. Note also that large \( n \) in the latter example did not imply generality. Indeed, significant generality may be suggested by small \( n \) studies. For example, early studies of monitoring and self-regulation in mathematics suggested that the absence of effective metacognitive skills could be a cause of problem solving failure in any domain. Large \( n \) is no guarantee of either trustworthiness or generality. For example, early studies regarding the effects of caffeine consumption turned out to be invalid because researchers neglected to take into account the correlation between coffee drinking and smoking and were thus unknowingly conflating the effects of smoking and caffeine consumption. And the findings of decades’ worth of medical studies conducted with solely male samples were assumed—incorrectly, it turns out—to apply to females as well. The studies were far less general than originally thought. (Recall Fig. 18.5: The choices of conceptual model and of focal variables, whether consciously or unconsciously made, have a fundamental impact of both the generality and trustworthiness of a study.)

The discussion that follows examines a series of research studies, ordered roughly by the generality of the claims made for them. For each study, methodological issues related to trustworthiness are discussed, the question being, “For studies of this type, what does it take to provide adequate warrants for the claims being made?”

Two points should be made regarding the choice of generality as the dimension along which studies will be ordered. First, this approach explicitly renders irrelevant the “qualitative–quantitative” distinction that has bedeviled the literature. The issues that count are the following: What kinds of claims are being made? What methods are appropriate for making those claims? What warrants are offered in defense of those claims? Providing trustworthy documentation of any particular kind of claim may
Another possible bifurcation is to separate research studies into the following two classes: research that tries to describe “things as they are” and research that documents attempts at change.12 “Descriptions of things as they are” consist of attempts to describe objects, events, structures, and relationships as they occur. One obvious set of such studies consists of “naturalistic” observations. However, this class of studies is much broader: various probes, experimental or otherwise, are often used to discover what things are and how they work. For example, Piaget’s claim that “object permanence” is learned rather than innate was established through a series of clinical interventions with young children. (Piaget obscured an infant’s view of a key ring just as the child was in the middle of reaching for it, and the child stopped in midreach.) The same holds for almost all Piagetian clinical interviews, such as those regarding conservation of volume, the child’s sense of time and space, and more. Piaget’s goal was to develop an understanding of underlying cognitive structures and their development. He did so by confronting his interview subjects with interesting (and very carefully designed) situations and then drawing inferences about the interviewees’ underlying cognitive structures from their responses to the situations. Similarly, laboratory studies aimed at determining how many “buffers” people have in short-term memory are attempts to describe stable cognitive structures, and, large-scale testing often aims at descriptions of how things are. One example is Artigue’s (1999) statement that “[m]ore than 40% of students entering French universities consider that if two numbers A and B are closer than 1/N for every positive N, then they are not necessarily equal, just infinitely close.” In sum, the category “descriptions of how things are” is quite broad, and the methods used extraordinarily diverse.

On the surface, descriptions of action research or “attempts to make change and document it” might seem to be different. Much such work (e.g., Fawcett’s, discussed above; my problem-solving courses, or various “design experiments”) consists of attempts to establish existence proofs—attempts to show that something can be done. Other studies are comparative: The claims regarding the implementation of various kinds of software, or other instructional practices, are that students do “better” under certain conditions than others. Yet the methods used to document the claims often overlap with those used for descriptions of things as they are. More important, the underlying issues concern questions such as, what kinds of claims are being made, and why should one believe them?” Like those mentioned above, these claims can be ordered by generality; then, given the nature of the claims, one can examine their trustworthiness. For these reasons, descriptions of things as they are and things as they might-be will be conjoined in the discussion that follows.

A Spectrum of Studies, Ordered by Generality

Category 1: Limited generality, but . . . (if properly done) . . . “here is something worth paying attention to.” A large number of studies are important not because they provide documentation of phenomena that are widespread, but because they bring readers’ attention to an issue worth considering (and worth further exploration). The studies themselves may have very limited generality, but they may have heuristic value; they may point to issues that are important to consider and may turn out to be

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12 These categories are not crisply defined, of course; the character of the event is a function of the perspective of the researcher. For example, from the perspective of teacher-researchers involved in implementing a new curriculum, their work is an attempt at change. From the perspective of anthropologists examining the “cultures” of their classrooms, the observations may be “descriptions of the reality of a school in flux.” Both perspectives on the same set of events are possible.
For example, various studies have suggested that the “lessons learned” in classrooms can be different than intended. One case in point is Wertheimer’s argument, quoted in section II, that instruction that focuses on drill and practice “is dangerous because it easily induces habits of sheer mechanized action, blindness, tendencies to perform slavishly instead of thinking, instead of facing a problem freely.” On the one hand, the reader may well resonate with Wertheimer’s claim on the basis of personal experience or classroom observations. On the other hand, one has to recognize that by contemporary standards, the evidence he offers in support of his claim is no more than anecdotal. The observations are not fleshed out in detail. One knows little about the background and classroom experiences of the students. There is little sense of how prevalent the phenomenon might be; of how deep it is and resistant to change. (Were the results perhaps artifacts of his interactions with the students? Might they have acted differently if he had structured the conversations somewhat differently?) The point here is not to chaitise Wertheimer—those were different times, with different standards—but to point out that his observations, no matter how intriguing and important (and they were), were not trustworthy in the sense of meeting the criteria elaborated in section V. That trustworthiness can be compared with, say, the descriptions of “making sense of linear functions” and “Hawaiian children’s understanding of money” found in Moschkovich and Brenner (1999). In those studies (explicitly chosen as cases illustrating the integration of “a naturalistic paradigm” into research on mathematical cognition and learning), the authors explicitly addressed the questions one would expect the skeptical reader to pose: How credible are the claims? How broadly might they apply, and why should one believe that they do? How rich are the descriptions of events? What kinds of sampling was done? What kinds of triangulation? Did the researchers create an “audit trail” and make it accessible? When the answers to such questions are available and inspectable, readers can assess the degree to which the findings are trustworthy.

A very large percentage of educational studies are of the type, “here is a perspective, phenomenon, or interpretation worth attending to.” The ultimate value of such papers is both heuristic (“one should pay attention to this aspect of reality”) and because such papers can serve as catalysts for further investigation. As a case in point, consider Bauersfeld’s 1980 paper, “Hidden Dimensions in the So-Called Reality of a Mathematics Classroom.” Bauersfeld reinterpreted a teaching episode that had been the subject of another scholar’s analysis. His “text” was a dissertation by G. B. Shirk (1972) at the University of Illinois, in which Shirk focused largely on the content and pedagogical goals of beginning teachers. Bauersfeld wanted to highlight a metatheoretical point: that teaching is a social activity as well as a cognitive one, and that viewing teaching as such can yield powerful insights into what happens in classrooms. His reanalysis “is used to identify four hidden dimensions in the classroom process and thus deficient areas of research: the constitution of meaning through human interaction, the impact of institutional settings, the development of personality, and the process of reducing classroom complexity” (Bauersfeld, 1980, p. 109). The phenomena were not (yet) claimed to be general but were declared to be worthy of investigation. Similarly, various studies of discourse in classrooms, illustrating analyses from a "situative perspective" (see, e.g., Greeno & the Middle-School Mathematics Through Applications Project Group, 1997, 1998), serve the joint purpose of illuminating a set of particular classroom events and highlighting the potential value of emerging theoretical approaches.

Many other studies do not make such claims overtly but in essence have similar intentions. Consider three teaching studies, which are in some ways similar and in some ways very different. Cooney’s (1985) study of a beginning teacher showed how a
beginning teacher’s professed instructional goals could be undermined by his deeply held beliefs and his interactions with students. Cohen’s (1990) study of a teacher undertaking “reform” showed how the teacher’s well-established classroom routines could result in the perception but not the substance of reform:

In the mid 1980s, California state officials launched an ambitious effort to revise mathematics teaching and learning. The aim was to replace mechanical memorization with mathematical understanding. This essay considers one teacher’s response . . . she sees herself as a success for the policy: she believes that she has revolutionized her mathematics teaching. But observations of her classroom reveal that the innovations in her teaching have been filtered through a very traditional approach to instruction. The result is a remarkable melange of novel and traditional material. Policy has affected practice in this case, but practice has had an even greater effect on policy. (Cohen, 1990, p. 311)

In a third, richly detailed study, Eisenhart et al. (1993) portrayed the myriad factors that shape a student teacher’s decision making:

We reveal a pattern in which . . . there were a variety of strong commitments to teaching for both procedural and conceptual knowledge; but . . . the student teacher taught, learned to teach, and had opportunities to learn to teach for procedural knowledge more often than and more consistently than she did for conceptual knowledge. We find that the actual teaching pattern (what was done) was the product of unresolved tensions within the student teacher, the other key actors in her environment, and the learning-to-teach environment itself. (p. 8)

In all of these studies, there are suggestions of generalizable findings: Teacher goals can be subverted if they are not tied to meaningful, implementable ideas (Cooney, 1985); some instructional goals are sufficiently nebulous that teachers can believe they are attaining them when they are not (Cohen); and conflicting pressures and mixed messages from the school district and state, along with shaky subject matter knowledge, can undermine the intention to teach for concepts as well as skills (Eisenhart et al., 1993). The suggested generality of these findings, and the fact that attempts at teaching for understanding might be undermined if they are not taken into account, is what makes them important. The implications are heuristic, however: “We believe there are many cases like this in the world, and it would be good to keep the implications of these studies in mind.” The claims themselves are not about generality: The evidence offered is about the cases at hand. The standard for judging these papers, given their claims, is the following: Are the cases compelling, and the analyses trustworthy? Making that decision entails, of course, judging whether the methods employed provide adequate evidence for claims made (and evidence that counters alternative explanations).

As noted above, various “existence proofs” also fall into this category. Fawcett’s (1938) study demonstrated that students could, under appropriate circumstances, learn aspects of formal mathematical arguments and apply their understandings in real-world contexts. The same is the case for various design experiments (e.g., Brown, 1992; Brown & Campione, 1996) and fine-grained research on various other instructional interventions (see, e.g., Cognition and Technology Group at Vanderbilt, 1997; Schauble & Glaser, 1996).

A word about “design experiments” is in order here. The term was invented to justify the idea that scientific work could be done in the context of real-world interventions and to offer an alternative to the standard model of experimentation, where “treatments” and outcome measures are designed in advance. The underlying idea is that a complex intervention is planned and implemented, and huge amounts of data (including videotapes, class logs, student work, etc.) are gathered. If interesting or important events appear to take place, the data are analyzed (depending on the
nature of the events) to document their existence or explain their occurrence. Some of these explanations are post hoc: The events are noted, and the record is combed for relevant evidence. But the order of data gathering is not essential. What is essential is the following: Once claims are made, how do they stack up against the criteria identified in section V? Do the methods employed provide some substantial degree of trustworthiness regarding the findings? As such, the methodological issues concerning such studies are similar in kind to those concerning other studies described in this category. (If broader claims are made regarding design experiments or other interventions, then the studies fall into the next category.)

**Category 2: Some Generality Is Claimed.** One case in point here is the quotation from Artigue 1999 given earlier: “[m]ore than 40% of students entering French universities consider that if two numbers A and B are closer than 1/N for every positive N, then they are not necessarily equal, just infinitely close.” Similar kinds of statements are made regarding various national assessments of mathematical competency, cross-national comparisons, and so on. When such statistical statements are made, there are issues of sampling, of construct validity (does the question warrant the interpretation given?), and more—recall Fig. 18.5.

Other statements concerning the typicality of various phenomena—especially phenomena not amenable to testing of the type just described—may come with different kinds of warrants. Here are two examples that claim some degree of generality but do not quantify it.

In her studies of mathematics teachers’ knowledge, Liping Ma (1999) analyzed Chinese and U.S. teachers’ responses to a series of questions regarding topics or problems in elementary mathematics and how they might teach them. Ma’s sample of teachers included 23 “above average” teachers from the United States, and 72 teachers of a wide range of ability from China. Her research documents, with care and detail, the fact that the sample of Chinese teachers had a deeper knowledge of mathematics and how to teach it than did their U.S. counterparts. Specifically, eight of the Chinese teachers had developed a form of understanding that Ma called “profound understanding of fundamental mathematics,” a rich and connected view of the content and ways to promote student learning of it. None of the U.S. teachers had developed comparable knowledge.

Ma did not focus on the statistics. Hers were not random samples, and there is no claim that her statistics to represent the percentages of the populations of U.S. and Chinese teachers who have developed a profound understanding of fundamental mathematics. Nonetheless, the differences in percentages are dramatic. They suggest strongly that a nontrivial percentage of teachers in China develop this deep form of knowledge, and that it is relatively rare among teachers in the United States. Indeed, the way that Ma’s samples were constructed lends additional credence to those findings: Her sample included a spectrum of Chinese teachers, whereas the teachers from the United States were considered “above average.” Hence, in addition to the trustworthiness of her analysis, there is a plausible degree of generality to her findings. The richness of the analysis lends plausibility to the generality of the findings, even though no claim is made for it.

A similar suggestion of generality, without precise statistics, could be seen in a series of studies I conducted regarding student beliefs about learning and doing mathematics. In a series of observations in one focal classroom school, I documented instructional practices, including the fact that a typical test contained 25 problems to be worked in 54 minutes, and that in a typical class period, students would work more than a dozen problems. The documentation included statements from the teacher to the effect that students would not have time to think through problem solutions on
tests; they would have to enter the exam knowing how to solve the problems they would face. Students were interviewed, and they were videotaped as they worked on problems. I also administered a questionnaire to more than 200 students (including the focal class) at various grade levels in the metropolitan area containing the school. Among the data gathered were the following:

The 206 responses to the question “how long should it take to solve a typical homework problem” averaged just under 2 minutes, and not a single response allotted more than five minutes. The largest of the 215 responses to “What is a reasonable amount of time to work on a problem before you know it’s impossible?” was twenty minutes; the average was twelve minutes. The following responses to both questions were typical. “Up to 2 or 3 minutes. I would work on a problem for about 10 minutes before deciding it’s impossible.” “A typical homework problem would take about 45 seconds. About 10 minutes for the impossible problem.” “It would probably take from 30 seconds to 2 minutes. I usually give up after 3 or 4 minutes if I can’t do it.” “It should only take a few minutes if you understand it. No more than 10–15 minutes should be spent on a problem. (Schoenfeld, 1989, p. 340)

This kind of analysis led to the following conclusions.

The data from this study help to provide a link between the fine-grained but small-scale observations in the [focal] study and the coarse-grained but nationwide data gathered in surveys such as the [U.S.] National Assessment. The questionnaire was administered in highly regarded schools with good graduation and placement rates. . . . The rhetoric of problem solving . . . was frequently heard in the classes we observed—but the reality of those classrooms is that real problems were few and far between, if they were seen at all. Virtually all of the problems the students were asked to solve were bite-size exercises designed to achieve subject matter mastery; the exceptions were clearly peripheral tasks that the students found enjoyable, but that they considered to be recreations or rewards rather than the substance of what they were expected to learn. This kind of experience, year after year, has predictable consequences. Students come to expect typical homework and test problems to yield to their efforts in a minute or two, and most of them come to believe that any problem that fails to yield to their efforts in twelve minutes of work will turn out to be impossible.” (Schoenfeld, 1989, p. 348)

At issue here is the set of warrants for generality. Fine-grained studies of a focal class provided a possible explanation of mechanism, and a description of classroom practices allowed readers to decide whether these practices seemed typical. Statistical analyses of the focal classroom revealed no differences between their responses to the questionnaire and those of the larger group of 230 students from a number of schools (which used statewide curricula). The student responses on questions that overlapped with national assessments suggested that their responses were typical of responses nationwide. This web of connections at least lends credence to the claim that the pattern of activities seen in the focal classroom, and their consequences, were anything but anomalous.

Other such broad notions of (typically unquantified) generality can be seen in research on aspects of thinking and learning such as metacognition. The general claim, broadly substantiated in the literature, is that the absence of effective self-monitoring and self-regulatory behavior is a significant cause of student failure in problem solving. Understanding this statement depends among other things, on one’s definition of “problem solving.” The circumstances in which it applies are those in which the problem solver is confronted with a task for which there is no obvious solution path, and decisions about how to approach the problem must be made. The claim is unlikely to apply to any significant degree in contexts in which problem solvers know
the relevant techniques. The methodological point here is that the “operating conditions” for many general claims need to be specified. Saying “X is important” implies across-the-board generality, and appropriate warrants should be produced. Saying “X is important and is likely to manifest itself in these particular circumstances” calls for a different set of warrants.

One final example of not-quite-specified-but-important generality deals with claims about attributes of particular groups. A paradigmatic example is the claim, in Stigler and Hiebert (1999) that “teaching is a cultural activity.” As a generalization, this kind of statement is of important heuristic value—and the authors make a good case for it. The warrant is that along certain dimensions, there is much more across-nation variation than there is within-nation variation. The devil is in the details, which in this case concern dimensions such as lesson coherence, time spent on individual exercises or problems, underlying conceptions of subject matter understanding, the use of instructional artifacts, and so on. The question for readers is, how solid are the warrants along the particular dimensions identified, and how well do differences in performance along those dimensions justify the general claims? (Here, as with all of the other studies discussed, the criteria discussed in section V can be applied to claims and the warrants provided for them.)

Subcategory 3: Significant Generality, If not Universality, Is Claimed. Some years ago Henry Pollak, in discussing differences between research in mathematics education and in mathematics, said, “there are no theorems in mathematics education.” By that he meant that there are no abstract proofs that something must be the case; instead, evidence is offered until the conclusion seems established to the legal criterion “beyond a reasonable doubt.”

The fact is that certain claims in education are universals, typically, claims about underlying cognitive mechanisms or structures. Here are two familiar examples, mentioned earlier in this chapter.

As noted above, Piaget documented the fact that children are not born with “object permanence” but that such understandings develop over time—“out of sight, out of mind” may be a description of cognitive reality for infants. And, theories of memory including constructs such as short-term memory buffers are grounded in reliable data that people have major difficulty handling more than “the magic number 7 ± 2” pieces of information in short-term memory. More broadly, general notions such as “schemata” are universal components of theories of memory. The initial findings, often obtained with very small n, have been replicated and extended numerous times.

It is, of course, impossible to prove that such claims actually hold for everyone. However, with precise enough definitions and operationalization of the research, replications of the studies can document the near-universality of the claims.

Beyond such cases, caveat emptor is probably the best attitude. A large number of claims appear to be universal, but they may need unpacking in various ways. Consider, for example, a generic claim for the effectiveness of instructional software: “One-on-one human tutoring is two standard deviations more effective than whole-class instruction. Our computer-based tutors are not yet that effective, but they are one standard deviation more effective than whole-class instruction.” One can (and should) ask,

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13This is not a hypothetical issue. In my book Mathematical Problem Solving (Schoenfeld, 1985), I described a scheme for analyzing transcripts of problem-solving sessions that focuses on “make or break” decisions during problem solving. Following the book’s publication I received a substantial number of communications from colleagues who said the scheme had not helped them analyze transcripts of students solving “problems” such as finding the product of two three-digit numbers. It should have been no surprise that strategic decisions are few and far between when one is working on problems that are purely procedural.
effective according to what criteria? With what populations? Compared with what, under what circumstances? Absent compelling answers to such questions, there is reason to doubt the generality of the claims. Similarly, linguistic inflation or the desire for scientific prestige result in various claims regarding researchers having various theories or models. As discussed in section IV, various theories (functionalism, conflict theory, symbolic interactionism, ethnomethodology, critical theory, ethnoscience, cognitive anthropology, exchange theory, psychodynamic theory, behaviorism, APOS theory, etc.) all have their applicability conditions. It is the responsibility of theorists to specify those conditions, to define the relevant constructs, and to address the limits (as well as the strengths) of what the theories can actually explain.

The same is true for the use of the term “model.” A model is more than a picture with a collection of objects and arrows. Claiming to have a model of a particular phenomenon means that one has specified particular objects and the relationships among them in the model and that these entities correspond in some well-defined way to the objects and relationships in the phenomenon being modeled. That is a high standard. A cursory glance at any handbook related to mathematics education (e.g., Berliner & Calfee, 1996; Grouws, 1992; Kelly & Lesh, 2000; Sikula, 1996) reveals models galore. Let us examine a random example from each of these handbooks.

In the Berliner and Calfee handbook, Mayer and Wittrock (1996) offer a model of the human information processing system in a schematic diagram (their figure 3.1, p. 54) that includes inputs, outputs, and various kinds of memory. Various arrows go from one box to another. A key question (which the literature may well address, but which has to be asked of any such figure) is: Just what goes along the arrows? What are these processes called selecting, organizing, integrating, and storing, and how do they work?

In the Grouws handbook, Romberg’s (1992) chapter, “perspectives on scholarship and research methods,” provides a “model for research and curriculum development” (Romberg’s figure 3.3, p. 52). Here too there are boxes and arrows, with arrows coming from the boxes labeled “classroom instruction” and “students’ behaviors” to the box labeled “students’ cognitions.” Once again, the same question needs to be asked: Just what goes along the arrows? And, what do the boxes really represent?

In the Kelly and Lesh handbook, these authors (see their table 9.1, p. 198) described a project in which (a) the goals for students include “constructing and refining models,” (b) the teachers “construct and refine models to make sense of students’ modeling activities,” and (c) the “researchers develop models to make sense of the teachers’ and students’ modeling activities.” Now, just what are the models in this case? What are the objects and relationships among them, and how do they correspond to the objects being modeled?

In the Sikula handbook, Christensen (1996) reproduced two “teacher education design models” (figures 3.1 and 3.2) used by institutions of higher education to describe their teacher education programs. These almost defy description. The first is a Venn diagram (no arrows) in which the outer ring appears to be a “diverse global society,” the next ring inward is labeled “private university/school of education/Christian environment,” and the next ring contains “facilitator/lifelong scholar/professional/decisionmaker,” inside of which are four interlocking rings. The second model appears in the outline of a tree, with “applied research,” “professional societies,” “world of practice,” and “state guidelines” at its roots, and a series of arrows that ultimately arrive (via “program goals and objectives,” “general education,” and more) to the “practicing professional.” It seems, alas, that the seductions of scientism that led to the adoption of experimental paradigms in the 20th century live on in the field’s wish to claim “theories” and “models” as part of its working apparatus. The aspiration is admirable if and only if it is matched with a concomitant commitment to rigor.
Section III of this chapter discussed a series of assertions regarding desiderata for high quality research, among them the following:

- One must guard against the dangers of compartmentalization. Educators need a sense of the “big picture” and of how things fit together.
- One must guard against the dangers of being superficial. Generally speaking, high-quality research comes when one has a deep and focused understanding of the area being examined.
- Researchers should be self-consciously aware of their theoretical perspectives and the entailments thereof. The methods they choose to employ should be selected on the basis of their appropriateness to address the questions that are considered important.
- Researchers must develop a deep understanding of what it means to make and justify claims about educational phenomena. What is a defensible claim? What is the scope of that claim? What kinds of evidence can be taken as a legitimate warrant for that claim?

Much of the substance of this chapter has been devoted to addressing the substance of these last two points. The issues are by no means straightforward, even for established professionals. The question, then, is what can beginning researchers do to bootstrap some of the relevant knowledge? I continue with additional assertions and some justifications for them.

- Students should have the opportunity to engage in research as early as possible in their careers, and they should be continually involved in various aspects of research—problem definition, methods selection, data gathering, and data analysis. Students should be encouraged, early on, to formulate problems and try to solve them (even if their first attempts are as awkward as a baby’s first steps).

The reason is simple: Research is not a spectator sport, and people will not develop a feel for doing research until they start doing it. This is the case even when one is learning to master standard techniques. It is especially the case when the research calls for the kinds of problem framing and methods development that are now part and parcel of our ongoing work. One colleague has summarized the issue succinctly as follows: “The best way to succeed is to fail early and often—with the appropriate support and guidance, of course.” This chapter has emphasized the fact that there are myriad places where one can go wrong when doing research. Fundamental errors can occur in the ways one conceptualizes a problem, selects data, or analyzes them (to name just a few). Everyone will make mistakes; they are unavoidable. With the proper feedback and reflectiveness, one will learn from those mistakes. It makes sense,
I believe, to start this process as early as possible. (To put things in very direct terms: Would we rather have a student make a major conceptual error in a course project or in pilot work for a thesis?)

- Multiple perspectives and multiple sources of feedback are good things. Students are likely to learn more if their work is commented on by more than one faculty member, especially if the faculty’s expertise overlap and complement each other’s.

This is, I hope, self-evident.

- Living in a research culture makes a difference; it is where habits of mind get shaped. Living in a research culture helps develop the kinds of breadth, depth, and multiple perspectives that are essential for the conduct of good research. It also provides important opportunities for the refinement of one’s work.

There may well be “independent scholars” (in the sense of those whose ideas have sprung almost completely from within), but I suspect they are relatively few in number and that most scholars profit from sustained membership in a congenial intellectual community. My experience has been that there is no better way to have one’s ideas shaped than to be a member of a community in which your ideas and ideas related to them are discussed. Sometimes the shaping is obvious: One walks out of a discussion with new or different thoughts as a result of the exchange. Sometimes the shaping is extremely subtle: I have realized, after the fact, that some of my ideas were, in important ways, the product of my environment. That is, I was most unlikely to have come up with some of the ideas I’ve come up with had I not been engaged in long-term conversations with particular colleagues and influenced by their thinking.

Moreover, students can pick up many skills through discussions of others’ work, before they are ready to grapple with big problems on their own. (See the discussion of research groups, below, for more detail.)

An active research culture also serves as a crucible for the refinement of work in progress. This can be the case for student papers or student presentations at meetings, but it is also the case for my own work, this chapter being a case in point. I bring drafts of all of my papers to my research group, which does me the favor of questioning the work in careful detail. I profit every bit as much from these exchanges as my students do when their work is being discussed.

- Passion helps (when appropriately harnessed, of course). People do their best work when they care about what they do. A program that allows students to pursue their interests is likely to result in a higher degree of commitment and higher quality work than a program that does not.

This, too, should be self-evident.

If one accepts these assertions, there are various pragmatic ways to ensure that students have such experiences early in their careers as developing researchers. Some such mechanisms are described in the balance of this section.

**Project-Based Courses**

Roughly half of the courses in our program (Education in Mathematics, Science, and Technology Program at the University of California, Berkeley) require students to conduct an empirical project of some significant scope. In such courses, there is usually
a heavy reading load “up front.” In the middle of the term, the reading load lightens as students design and implement projects that are related to the course content. At the end of the course, the projects are used as vehicles to reflect on that content. (In addition, students can negotiate projects that meet the requirements for more than one course. This allows them to work on projects of substantial scope and to get feedback from more than one faculty member.)

Course projects come in all shapes and sizes. The default option, which is actually exercised by very few students, is to replicate a study discussed in the course. Another option is to make a minor modification or extension of a study examined in the course. A third, and the one most frequently taken, is to find a phenomenon of interest and try to make sense of it. The data examined might be videotapes of a classroom or school, students’ work on particular instructional materials or a computer program, people’s “out-loud” thoughts as they try to solve problems, or just about anything else. Almost anything related to the general content of a course is considered fair game.

Here are examples of student projects in a recent first-year course.

One student had been working for some time as part of a team developing a test of “mathematical ability” that was being administered to thousands of students and analyzed using statistical measures. For her course project, she selected some people at various points on the spectrum from “ordinary beginner” to “talented expert,” the latter being a faculty member in mathematics. She hypothesized the kinds of performance that people with different levels of mathematical ability would display when they worked the problems and then videotaped the people solving the problems. The reality of people’s performance was an eye-opener: Some novices displayed much more effective problem-solving practices on some of the problems than she expected, and her “expert” engaged in rather sloppy reasoning in places. This experience led her to question some of the assumptions she had been making about the problems and about what people’s test scores really meant.

A second student hypothesized that girls and boys would act differently in same-sex problem-solving groups than they would if all the other students in the group (of four or five) were of the opposite sex. As it happened, the people she chose (somewhat randomly) as the main subjects in her study tended to have robust character traits (shyness in some cases, aggressiveness in others), and there wasn’t much apparent difference in their performance. In reviewing the tapes, however, the student became interested in how collaborative the various groups were. She began to develop a coding scheme that looked at comments from students that invited reaction versus those that were neutral or closed off conversation. This was a legitimate first step toward the quantification of collaborativeness and a good example of problem definition and method creation (with help, of course).

A third student analyzed some videotapes of an experimental course on mathematical representations that had been taught as part of a colleague’s research and development work the previous summer. As part of the project, the student in my course examined the beliefs of a student in the experimental course regarding what “counts” as being mathematical; he then tried to correlate the second student’s beliefs with her behavior. The student in the experimental course tended to disparage successful qualitative reasoning as “mere” common sense while giving high praise to mathematical behavior that included writing and solving equations, even though the equations she praised were (from our perspective) pretty much gobbledygook. Such beliefs seemed to play out in her actions during the course as well. (The evidence that my student offered in support of this claim was rather tenuous. That fact catalyzed for some productive discussions about what it takes to justify such claims.)

Other projects dealt with student and teacher perceptions of a “reform” mathematics course, an attempt to analyze the teaching of a master teacher, the use of artifacts
such as white boards (instead of individual sheets of paper) to catalyze interactions during group problem solving, and more.

How good were the projects? The truth is that when beginning students try to carry out such projects, their attempts tend to be half-baked. Students come to realize that they didn’t see the things they expected to see, that they can’t make the arguments they thought they’d be able to make—and sometimes, that there are interesting and unexpected leads in what they did see, which provide pointers to issues they’d like to pursue. Almost all of the papers were flawed in some way or other. That is no surprise; the students didn’t have the background to design or carry out near-perfect studies at that point. Indeed, I think what happened is quite healthy. What the course offered was an institutionally supported way to make mistakes in the process of trying to define and work on a nontrivial research problem. Of course, this is only profitable if the students have the opportunity to learn from their mistakes. As part of their projects, students are asked to say how they would do things differently if they had them to do over again. (And—see the discussion of first and second year projects—they often have the opportunity to do them over again.) Then, in class discussion of the projects (which are presented formally as though at professional meetings) and in faculty evaluations of them, there is extended discussion of what worked, what didn’t, and what might be done about it.

Typically students will take a number of courses each year that have such projects. In that way, the program offers an institutionalized mechanism for failing early and often—and for learning from those failures. They have the opportunity to develop their own perspectives on issues, refine their ideas and their methods, and try them out on critical but sympathetic audiences.

First- and Second-Year Projects

The scope and quality of course projects are usually limited by the obvious constraint: Things have to be done in the midst of one semester. For this reason, course projects often have the character of pilot studies—an idea has been explored, but there was insufficient time to work it out. To provide such opportunities, we also require much more substantial project work.

In the summer following the first year of the program, and again in the summer following the second, students are required to conduct and write up more extensive studies. Typically, these first- and second-year projects are extensions of course projects: A course project may have yielded some tantalizing results, so the student goes back to gather more (or better) data to explore the issue in greater depth. With some frequency, project work is cumulative. A second-year project is an outgrowth or modification of a first-year project and may itself evolve into a dissertation project. These projects are expected to meet rather stringent standards. They are to be written up as though for publication and are judged accordingly. Each project report is read by two faculty members, and the discussion of the student’s project is a major component of our annual student evaluation.

Even though they come on the heels of course work, first-year projects can turn out to be seriously flawed, in which case the students are told to revise them and try again. Many are respectable, however, and only need minor revisions. Either way, it is healthy to establish the standard for judgment and provide rigorous feedback. Second-year projects tend to be of uniformly high quality, and a fair number of them have been published; a significant number are presented at professional meetings. The acceptance rate for student proposals and papers is quite high, and I have no doubt that the students’ success is attributable to the fact that we provide them with consistent opportunities to do independent work and to receive solid critical feedback on it.
Research Groups

As noted above, I believe that students are more likely to become productive researchers, and to develop useful habits and perspectives more rapidly, if they are members of a research community. When you are constantly engaging with people who live and breathe research issues, participating in the development of their ideas and in their successes and failures, you are much more likely to pick up “what counts” than you would be if you were working in isolation.

In our program, each faculty member has at least one research group. Every student in the program is expected to participate regularly in one or more research groups. Many students attend two or more groups because they find the complementary perspectives and expertise of value.

Although there is tremendous variation, certain properties tend to be present if a research group or community is functioning well. Three of those properties are as follows:

- There is a sense of purpose and meaningfulness: Much of the work done really matters to the people involved. (It is not seen as “busy work,” but as part of what needs to get done to advance the enterprise.)
- Much of the work being done is visible—the processes of doing research, including mulling through problems, are public property in the sense that dilemmas are shared, and community input is valued as a way of solving them. There is a culture of reflectiveness, where the expectation is that problematic issues will be raised and that members of the community will consider contributing to their solution (even when doing so does not contribute to their own progress) as one of their communal responsibilities. The culture is such that there is room for the work and contributions of all members to be taken seriously.
- The work and interactions of the group provide a series of “handholds” that allow individuals at various levels of knowledge and expertise to contribute meaningfully to the enterprise, and to make parts of it their own. Newcomers’ contributions may consist of routine work in the service of the cause (e.g., first-year students in a research group that has an ongoing development project might play a small role in the development process, help field test some materials, or help videotape lessons). At the same time, those students are present for the theoretical discussions and are invited to contribute whenever they felt comfortable doing so. Typically, early contributions consist of occasional comments or questions, as beginners try to sort out the spirit or the details of what is being done. As they become more central members of the community, the character of their questions and contributions tends to evolve. They are likely to take on larger tasks, individually or in collaboration, and they increasingly take on ownership of tasks and ideas.

Another way to describe this process in somewhat more theoretical terms is that a functioning research community provides multiple opportunities for legitimate peripheral participation. As once-peripheral members become more central to the enterprise, they find more means of achieving centrality, and there is room and access for new members at the periphery. The detailed examination of this process would be a most welcome study.
There is, it should be stressed, no single model of a productive research group or community. Such communities may be small, consisting of one senior researcher and a few students, or they may be large, including a substantial number of people with varied levels of skill and expertise. Moreover, no such group is static: Depending on the people involved and the tasks at hand (is a major focus of the group conceptualizing a new project, building a collaboration, “engineering change,” designing or implementing materials, gathering data, analyzing data, writing or revising papers or proposals, etc.), the day-to-day transactions of the group and its level of activity will vary. Among the activities that research groups in our environment have supported are the following:

**Participation (Whether As Central Player Or Legitimate Peripheral Participant) In a Major Ongoing Project.** The benefits of this kind of engagement were discussed immediately above.

**Providing Group Members Feedback on Issues of Importance to Them.** One function of a research group is to serve as a critically supportive environment for discussions of student work. What is brought to the group can vary substantially. A student may have a vague idea for a project and ask for the group’s help in honing that idea. He or she may have some data and want to see the group’s reactions or may want to see the group’s reaction to a tentative explanation of those data. The student may have a draft piece of work—a course project, a master’s thesis, a dissertation proposal, a chapter of a dissertation, a proposal for a conference presentation, or a paper for submission—and want feedback. Sessions are scheduled with enough lead time so that group members are expected to go through the relevant materials, and to serve as colleagues in providing help to the presenter.

It is not at all necessary for the students to be working on the “same thing” in order for them to take each other’s work seriously. For example, students in one research group were working simultaneously on transfer, teacher knowledge, cultural forces shaping the effectiveness of instruction, and issues of reflection on professional growth and integration. Yet discussions of these students’ ongoing work, from the early stages of problem formulation through the stage of selecting data, agonizing over what the data meant, and then writing things up in ways that were cogent and compelling, all proceeded in parallel and in comfort.

What made the group function effectively was a common interest in helping each other work things through and an understanding that at some fundamental level everyone was grappling with the same issues. No matter whose work was being discussed, conversations were all grounded in the same kinds of questions: What are you trying to say (What are the “punch lines”? Why would anyone think this is important? What kind of evidence will convince people that what you are saying is justified? What are counterinterpretations? What position will you be in if the data don’t tell the story you’d like? What are the implications of your expected results, and why should anyone believe them? Of course, the group tried not only to raise questions, but also to help answer them.

In these conversations, everyone profits. The presenter gets useful feedback; the others hone their skills in understanding and critiquing research and in learning to ask others the kinds of questions they will have to ask themselves as independent researchers.

**Dealing With Topics Or Readings of Interest.** Research groups often serve as a reading or discussion groups. This provides a way to delve deeply into issues as a community. Groups have, at various times, decided to “go to school” on various theoretical perspectives (constructivism, situated cognition), to explore the strengths
and limitations of particular research methods or to discuss papers on topics that just plain seemed interesting.

**Providing a Critical But Friendly Audience for Practice Talks.** Before major professional meetings, research groups often provide forums for practice presentations. In most groups, students and faculty rehearse their presentations before the group before they go “public.” It is much better to learn to deal with tough questions in the comfort of a research group than to hear them for the first time when at the podium in a public presentation.

What really matters in all of the above? What counts from my perspective is to provide a supportive environment that lives and breathes research issues, that is open and reflective, that allows people to pursue ideas that they really care about, and that provides them with many opportunities to learn, early on, from the mistakes they will inevitably make.

In closing this section, I would like to address an issue that Frank Lester raised when he reviewed a draft of this chapter:

“I would like to read about what an overall program might look like at three types of institutions: (1) those few that expect students to begin thinking seriously about research from the beginning, (2) those that are preparing math educators who might also do some research, but who surely will be (primarily) consumers of research, and (3) those that simply require students to write a dissertation as a final requirement for the terminal degree. My fear is that institutions in the third category are preparing most of our future math educators. Even if this is not the case, it surely is true that there are relatively few Category 1 institutions. In fact, I would like to see him discuss the type of program appropriate for Category 2 institutions and to engage in some speculation about how to prepare math educators to be good consumers and interpreters of research.”

Elsewhere (Schoenfeld, 1999b) I have discussed ways to think about core content for a doctoral program in mathematics education; what follows are “headlines” of that discussion. First, content. There is no solution to the “content problem,” but one can satisﬁce; the goal is to give students a sense of the many flavors of educational work and their contributions. There are, I think, reasonable approaches grounded in the structure of any institution. On the one hand, one needs to provide “disciplinary” information. This can be done via core courses that reﬂect the disciplinary organization of the institution. For example, Berkeley’s School of Education is organized into three overarching academic units called Areas. Faculty in each Area are encouraged to propose core courses that give a “taste” of mainstream issues, perspectives, and methods in that area, while highlighting connections with and perspectives and methods from the other areas. (If a school has $n$ areas where $n$ is large, courses bridging such areas can be cotaught.) On the other hand, students should come to understand how educational issues transcend disciplinary boundaries. One way to do this is to offer a series of courses on “cross-cutting topics,” where faculty from different units bring varied disciplinary perspectives to the study of such issues. Any of a number of topics—teacher preparation, assessment, or diversity, to name three—can serve as topics for discussion.

Second, there is the issue of research methods. It is impossible in a few introductory courses to provide the depth and breadth of coverage that will result in students being adequately prepared for the research they will do. We can prepare students to be knowledgeable and skeptical consumers, and we can help engender in them an understanding of the fundamental issues. We can, in the best of all possible worlds, help them develop the right kinds of questioning attitudes, asking as they proceed with their work what they can say with justification and how best to approach issues so they can make strongly warranted claims. But we should not make the mistake of thinking
that methods courses will prepare students adequately for their research. If students emerge from their methods courses with a sense of how to approach a problem, of how to select methods that seem reasonable, and of where to go for help when they realize the limits of those methods, then the core has been quite successful. My bias is that “less is more”: A small number of cases carefully studied will be more productive in the long run than an “encyclopedic” treatment (with each topic studied in the depth of a typical encyclopedia article). Really learning about methods should come when students try to use them—in courses, in projects, and as members of a research community.

Above and beyond core content, I would strongly recommend a program that contains a large number of project-based courses and that has projects similar in kind to the first- and second-year projects discussed above. Research groups are extremely valuable, but if the environment does not support them, some of their functions can be achieved through research seminars or through modifications of the core courses.

This discussion may not seem fully responsive to the issues Frank Lester raised in that I do not separate out three different kinds of institutions as he suggests. My failure to do so is deliberate. Frank is right that at present there are relatively few institutions at which students begin conducting research early in their careers. From my perspective, that is most unfortunate, even if most students at a particular institution intend to become consumers rather than producers of research. One learns a great deal about how research is done, about what to believe and what not to believe, by trying to conduct it and learning from the experience.

A case in point is the first student whose work was discussed in the section on “project-based courses.” Her course project consisted of taking a close look at what people of (ostensibly) different mathematical abilities actually did when they worked mathematics problems. I indicated above that by virtue of having conducted the project, the student developed a much more nuanced view of problem-solving abilities and of what tests reveal about such competencies.

As it happens, that student did not intend to have a career as a researcher. She was enrolled in our teacher preparation program, and she went on, as planned, to become a teacher. Her presence in the project-based course was no accident. Our teacher preparation program is designed so that student teachers and beginning doctoral students are enrolled in many of the same courses. Student teachers are also enrolled in faculty’s research groups. The idea is that this kind of “hands-on” experience with research will enable them to develop a much better understanding of thinking, teaching, and learning than they would by merely reading about it. Even if they never go on to do formal research on their own, they will be much better consumers of research, and they will be better at understanding student thinking for having explored it in detail. I think all mathematics educators should have such experiences, no matter what their intended careers.

VIII. A FINAL COMMENT

As one reflects on the state of the field, it is worth recalling the historical data with which this chapter began. Mathematics education began to coalesce as a discipline only a few decades ago, with its first professional meetings and journals appearing in the late 1960s and early 1970s. Its growth since then has been nothing short of phenomenal. Once held tightly in the stranglehold of a reductive epistemology and scientistic methods, the field has blossomed. Important phenomena as diverse as “metacognition” and “communities of practice” have been uncovered and elaborated in substantial detail. Important theoretical frameworks as diverse as cognitivism, ethnomethodology, and critical theory (to name just a few) have been developed.
Methods as diverse as cognitive modeling and discourse analysis have been crafted. The results have been myriad thought-provoking interpretations of mathematical thinking and learning. During one scholar’s lifetime, the field has progressed from the point where controlled laboratory studies were necessary to explore simple cognitive phenomena to the point where the detailed modeling of thinking and learning in complex social environments is possible.

At the same time, much of the growth has been chaotic. As is absolutely characteristic of young fields experiencing rapid growth, much of the early work has been revealed to be seriously flawed. As discussed above, unarticulated theoretical biases or unrecognized methodological difficulties undermined the trustworthiness of a good deal of work that seemed perfectly reasonable at the time it was done. This should not cause hand wringing—such is the nature of the enterprise—but it should serve as a stimulus for devoting seriously increased attention to issues of theory and method. As the field matures, it should develop and impose the highest standards for its own conduct. The tools are within its reach. My hope is that the framework discussed in section VI and the criteria discussed in section V will prove useful catalysts for improvement.

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Why and for whom is research in mathematics education conducted? Is our research, as some cynically insist, simply an activity pursued by “ivory tower” academics intent on publishing articles read only by other academics? Or, as others believe, is its purpose to promote the development of robust theories about the teaching and learning of mathematics? Some hold yet another view, namely, that research should focus on the pursuit of knowledge that causes real, lasting changes not only in the way people think about learning and teaching, but also in how they act. In this chapter, we discuss these and related questions and propose a way to think about mathematics education research that can serve to move us toward making productive contributions to both policy and practice. The first part of the chapter deals with how (and for whom) research in mathematics education has been carried out; the second discusses what counts as evidence in mathematics education research.

HOW AND FOR WHOM HAS RESEARCH IN MATHEMATICS EDUCATION BEEN CONDUCTED?

In the 1990s, at least three carefully researched, rather comprehensive, English-language compendia were written on the state of the field’s knowledge about mathematics teaching and learning (viz., Bishop, Clements, Keitel, Kilpatrick, & Laborde, 1996; Grouws, 1992; Sierpinska & Kilpatrick, 1998). Will any of these volumes have
any real impact on the practices of teachers? Will any of them have any direct influence on educational policy? We think it unlikely and believe that the failure of publications such as these to resonate with the interests, needs, and concerns of practitioners is because the research presented in them is concerned primarily with the pursuit of "knowledge" (in the sense of collections of items of generally agreed on information) and developing theories, rather than focusing on actually moving people—teachers, teacher educators, school administrators, policymakers, and so forth—to action.

How Has the Research Been Conducted?

For most of the history of research in mathematics education, the predominant way of learning about and understanding phenomena related to the teaching and learning of mathematics has been based in the tradition of scientific rationalism—we have wanted to emulate the successes of the physical sciences (Lester & Lambdin, 2002), successes that have stemmed largely from the fact that the meanings of the results of experimental data are likely to be agreed on across a wide range of contexts and by a large proportion of the research community. In the writing up of such research, the authors assume that the text produced has the same meaning to the vast majority of readers and that it applies across a wide range of contexts. Only recently have mathematics educators come to realize that these "objective" methods are often not appropriate to address educational research problems (Wiliam, 1998).

For some, this lack of success can be attributed to the fact that educational research has not yet developed into a fully mature science. Thomas Kuhn, in his classic book The Structure of Scientific Revolutions, described natural philosophy (i.e., the study of nature) before the Renaissance as a "pre-science" (Kuhn, 1962), indicating a period in which there was no agreement about basic principles and ways of working. During the Renaissance, however, there was increasing agreement about methods of inquiry, leading to a period of stability that Kuhn termed "normal science." Currently, educational research shares many (although not all) of the features of a "pre-science," but what this means for the future of educational research is not clear. For some, only when educational researchers agree about how one goes about creating knowledge in educational research will education start producing "reliable knowledge" (Zinman, 1978) and become a "proper science." For others, however, the very nature of the educational activity—the complexity of the objects of study—means that educational research can never become a "science" in the traditional (and narrow) sense.

One of the most salient differences between research in mathematics education and research in the physical sciences is the importance accorded to "context." In the physical sciences, issues of context rarely arise. This is because differences between situations that are not represented in our theoretical models rarely make a difference. For example, physical scientists know that they need to record the current passing through an ammeter but do not need to record the color of the ammeter. In this sense, theorization's in the physical sciences are relatively complete. In contrast, research in mathematics education frequently generates results that hold in some settings but not in others. For example, studies of the effectiveness of feedback to learners have produced mixed results. In some studies, giving feedback was found to be effective in improving learning, whereas in other studies, there was no clear effect. One way to interpret this is to attribute the differences to the effects of "context" (much in the same way that statisticians describe anything not accounted for in their models as "error"). Such differences can also be interpreted as pointing to the need for further

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1Zinman (1978) insisted that "the primary foundation for belief in science is the widespread impression that it is objective." By objective he meant "knowledge without a knower: it is knowledge without a knowing subject" (p. 107, emphasis in original).
theory building, however. When Kluger and De Nisi (1996) found huge differences in
the effectiveness of feedback in improving performance, they looked carefully at the
kinds of feedback involved in the different studies. They found that feedback focusing
on how well the individual was doing (ego-involving feedback) produced very small
(and sometimes slightly negative) effects. Feedback that focused on what the individ-
ual needed to do to improve (task-involving feedback) on the other hand produced
significant gains. Furthermore, they found that feedback was even more effective
when, as well as focusing on what needed improvement, the feedback indicated how
to go about such improvement.

Given the complexity of classrooms and other learning environments, we do not
believe we will ever reach a situation in which our theorization of educational settings
will approach the completeness of that in the physical sciences. We do suggest, how-
ever, that progress in research in mathematics education will benefit from regarding
the effects of “context” as opportunities for the further development of theory.

A distinct but related difficulty in research in mathematics education is that even
when we collect the “right” data, phenomena interact with each other in ways that
are likely to be impossible to predict. In this context, we note that an increasing num-
ber of philosophers of science and others have recognized, through work in chaos
and complexity theory, that many physical systems show the kind of unpredictability
prevalent in the social sciences. In particular, the sensitivity of educational phenom-
ena to small changes in detail means that it is literally impossible to put the same
innovation into practice in the same way in different classrooms. The difficulty of
putting into practice the fruits of educational research suggests we need a different
way of thinking about educational inquiry.

Typically in the physical sciences, reliable knowledge is produced by searching for
patterns that are common across a range of contexts and by looking at broad trends.
For example, the behavior of individual molecules in a gas is impossible to predict,
but the aggregate behavior of large numbers of such molecules can be predicted quite
well. Similarly, it is impossible to predict which people will die in a given year, but the
actuarial sciences have developed sophisticated and accurate methods for predicting
the numbers of people dying of particular causes.

In education, however, because of weak theorization, increasing sample sizes to
“average out” the effects of context often ends up producing only bland platitudes,
which seem only to “tell us what we already knew.”

Nonetheless, the fact that educational research cannot (or at least at the moment
does not) produce transcendent truths does not mean that such research cannot be
useful, and we suggest that a focus on the usefulness of educational research—and
in particular, its fertility in suggesting more appropriate courses of action—is a more
relevant criterion than the reliability and transcendence of the knowledge produced.

In the 1960s and 1970s, partly as a response to the failure of educational research to
have much impact on practice, there was a surge in interest in different ways of
finding out about and understanding educational processes, principally derived from the
more qualitative approaches that had been developed in sociology and anthropology.
In particular, many studies addressed the problem of context by looking in detail at
a single educational site (see, for example, Lacey, 1970). In such a study, the problem
of context is tackled not by trying to average out across all contexts, but by attending
to the details of the particular context. The actual setting for the research is laid out in

2Concern about the ineffectual nature of mathematics education research is anything but new. In a
1971 article in the American Mathematical Monthly, Walbesser and Eisenberg argued against the emphasis
in doctoral programs on experimental research: “The consequences of such training are obvious when
one examines mathematics education dissertations. Too often the dissertation concerns the investigation
of a trivial problem cloaked in an elegant statistical design” (p. 668).
considerable detail, and readers can make up their own minds about how convincing is the evidence and the conclusions drawn from it. This does not absolve the researcher from any responsibility about how the research is conducted, however. Indeed, it is still incumbent on researchers to conduct their inquiry and to report their findings in such a way that the meanings of such findings will be shared, to a greater or lesser extent, by various readers.

For Whom Is the Research Intended?

The Journal for Research in Mathematics Education, an official journal of the National Council of Teachers of Mathematics, is devoted to the interests of teachers of mathematics and mathematics education at all levels—preschool through adult. (inside front cover of every issue of the journal)

It is an unfortunate fact that too few teachers and other education practitioners pay attention to the research that is so carefully and thoughtfully reported in research journals such as the Journal for Research in Mathematics Education (JRME), but this is not a new development. Writing on the occasion of the publication of the journal’s inaugural issue in 1970, then-president of the National Council of Teachers of Mathematics (NCTM) Julius Hlavaty suggested that the purpose of the new journal was “to give the teacher in the classroom, the administrator and curriculum consultant at the planning level, and even the man in the street [sic], the information, guidance, and help that research can provide” (Hlavaty, 1970, p. 7).

Despite Hlavaty’s vision, opinions about the relevance of JRME for teachers remained mixed for several years after the establishment of journal. For example, in the May 1978 issue, then-NCTM president John Egsgard, himself a classroom teacher, insisted that

Until the mathematics-education research community can come up with results that will affect the classroom teacher, be it an elementary school teacher, a junior high teacher, a secondary school teacher, a community college teacher, or a teacher of mathematics education, I do not believe that the Council would be justified in providing additional resources for research. (p. 241)

The sentiment among many practitioners is not so different today than it was in 1978. Among the many explanations proposed for the failure of our research to resonate with teachers, one that has not been given adequate attention by mathematics educators, is that researchers and teachers have different ways of validating what they know and believe about mathematics teaching and learning. They also accept different ways to frame their discourse about what they know and believe. Many researchers tend to seek validation for knowledge claims by means of formal research that adheres to certain rules of procedure, including such matters as reliability and validity. By contrast, teachers often rely on personal judgments and social (dialogical) discourse to determine “what works for me” (Hargreaves, 1998). Glaser, Abelson, and Garrison (1983), in discussing how research results are put into practice, summarized the distinctions between researchers and teachers as follows:

The differences include: a tendency to live in two different professional communities, or “worlds”; distinctive cognitive styles; responsiveness to divergent rewards; and different beliefs about how knowledge can best contribute to human welfare. (p. 395)

If researchers and teachers live in two different worlds, it seems natural that they would also communicate differently about phenomena occurring in those worlds. Indeed, Schwandt (1995, 1996) suggested that the reason for the lack of perceived
relevance of most educational research for teachers and other practitioners can be attributed to how members of these communities communicate their ideas. He insisted that many researchers communicate their ideas in terms of (monological) scientific rationalism, whereas teachers—and some researchers—tend to communicate their ideas through, “the lens of dialogical, communicative rationalism” (Schwandt, 1995, p. 1). In the following sections, we elaborate on these two ways to communicate.

(Monological) Scientific Rationalism

According to Schwandt (1995), scientific rationalism is a style of inquiry shaped by six principles:

1. True knowledge begins in doubt and distrust.
2. Engaging in this process of methodical doubting is a solitary, monological activity.
3. Proper knowledge is found by following rules and method (rules permit the systematic extension of knowledge and ensure that nothing will be admitted as knowledge unless it satisfies the requirements of specified rules).
4. Proper (i.e., scientifically respectable) knowledge requires justification, or proof.
5. Knowledge is a possession and an individual knower is in an ownership relation to that knowledge.
6. In justifying claims to knowledge there can be no appeal other than to reason.

(pp. 1–2)

Of special concern for scientific rationalists are the nature of the claims that are made and how these claims should be justified. Furthermore, all the ways deemed acceptable for justifying a claim are regarded as uncertain or unreliable in one way or another. Historically, scientific rationalists typically employ four basic types of argument to justify claims: (a) argument by example to arrive at some sort of generalization, (b) argument by analogy (because phenomenon A is like phenomenon B in certain ways, the researcher argues that they are also alike in another specific way of interest), (c) argument from authority (the use of existing literature to support a position or help make a case), and (d) argument from statistical inference. Examples of the use of each of these types of argument are easy to identify in issues of our mathematics education research journals. Adherents of scientific rationalism accept that each of these methods of justification is readily subject to the error of reaching a conclusion with insufficient evidence or to the error of overlooking alternative explanations.

(Dialogical) Communicative Rationalism

As explained by Shotter (1993, p. 166), communicative rationalism opposes scientific rationalism in three fundamental ways. First, rather than regarding the social world as “out there waiting to be discovered,” the communicative rationalist insists that the world can only be studied from a position of involvement within it. Second, “knowledge of [the] world is practical-moral knowledge and does not depend upon justification or proof for its practical efficacy.” Third, “we are not in an ‘ownership’ relation to such knowledge, but we embody it as part of who and what we are.” Thus, communicative rationalism provides a different way to consider what it means to know. “Instead of simple observational claims about objects, knowing other people is offered as a paradigm for knowledge” (Schwandt, 1995, p. 7). When we adopt a

3Such an approach does not assert that there is no such thing as the physical world, but merely that the world is not “knowable” in any absolute sense. As Roger Shattock remarked, “Words do not reflect the world, not because there is no world, but because words are not mirrors” (quoted in Burgess, 1992, p. 119).
communicative rationalistic approach to research, “we come to understand that the apparently orderly, accountable, self-evidently knowable and controllable characteristics of both ourselves and our social forms of life are constructed upon a set of disorderly, contested, conversational forms of interaction” (Schwandt, 1996, p. 14). These “conversational forms of interaction” help us to develop knowledge of our practices and ourselves. Shotter suggested that to Ryle’s (1949) two kinds of knowledge—knowing that and knowing how—we should add a third type: knowing from. This type is characterized as knowledge “one has from within a situation, a group, a social institution, or society” (Shotter, 1993, p. 19).

To accept communicative rationalism involves accepting that reason is dialogical in nature: “It is concerned with the construction and maintenance of conversational reality in terms of which people in influence each other not just in their ideas but in their being” (Schwandt, 1995, p. 7).

The implications of communicative rationalism for mathematics education research may not be immediately apparent, but they at least involve how we make and justify claims in our research, how we go about convincing others of the claims we make as a result of our research, and how we defend our claims on ethical and practical grounds. In particular, communicative rationalism attempts to avoid treating students and teachers as objects of thought to make claims about them that will guide future deliberative actions. Instead, it aims to include researchers and teachers (and students) in dialogical conversations to generate practical knowledge in specific situations. Thus, claims are made only after the various perspectives (or worldviews, background assumptions, beliefs, etc.) of all those engaged in the dialogue have been openly considered and negotiated. Schwandt and Shotter believe that it is this process of open negotiation of claims (and of what is regarded as evidence) among all participants in the discourse that ultimately moves people to take action to change.

Scientific rationalism therefore differs from communicative rationalism not only in how knowledge is warranted, but also in what is to count as knowledge. Within communicative rationalism, the practical knowledge that teachers possess in the contexts of their classrooms—how to make complex, nuanced judgments in the face of considerable complexity—is to be counted as knowledge just as much as the decontextualized, transcendent, but often difficult-to-apply “truths” of scientific rationalism. From the perspective of scientific rationalism, the failure of teachers to “take on board” the findings of educational research may be viewed as inexplicable (or at least “irrational”). From the perspective of communicative rationalism, however, the reasons for the failure of “center-to-periphery” models of dissemination are all too clear: A huge part of the “knowledge”—specifically, how to make it work in practice—is missing. This knowledge is missing because, as remarked above, the relatively incomplete theorization in mathematics education research means that the explicit knowledge available from the research literature does not tell teachers what to do. There is a whole range of choices for the teacher to make in a given situation, all of which are consistent with the findings of research, leaving the teacher to choose among the alternatives, using his or her knowledge of the students, the school context, and a range of other variables. The sheer complexity of classroom and school life, and the speed with which decisions often have to be made, means that the knowledge brought into play by teachers in making decisions is largely implicit rather than explicit.

The complementary roles of tacit and explicit knowledge are brought out clearly in the model of knowledge creation in organizations developed by Nonaka and Tageuchi (1995). They began by observing that although some of the knowledge that individuals in organizations possess is explicit, much of it is tacit, and the extent of this tacit knowledge is often unrecognized. Indeed, organizations frequently discover what an individual knows only after that person has left the organization.
The existence of two types of knowledge—explicit and tacit—results in four modes of knowledge conversion, as shown in Fig. 19.1 (the distinction between explicit and tacit in reality is, of course, a continuum, but for reasons of clarity it is presented as a dichotomy in the figure). The process of socialization can be viewed as one of passing on existing tacit knowledge to others, whereas externalization involves making tacit knowledge explicit. Developing new explicit knowledge from existing explicit knowledge is a process of combination, and internalization consists of making explicit knowledge “one’s own.”

Nonaka and Tageuchi (1995) then proposed that these four processes typically occur in the following sequence:

First, the socialization mode usually starts with building a “field” of interaction. This field facilitates the sharing of members’ experiences and mental models. Second, the externalization mode is triggered by meaningful “dialogue or collective reflection,” in which using appropriate metaphor or analogy helps team members to articulate hidden tacit knowledge that is otherwise hard to communicate. Third, the combination mode is triggered by “networking” newly created knowledge and existing knowledge from other sections of the organization, thereby crystallizing them into a new product, service or managerial system. Finally, “learning by doing” triggers internalization. (pp. 70–71)

What this analysis makes clear is that scientific rationalism is concerned only with those situations in which one person’s explicit knowledge is transmitted to others as explicit knowledge (bottom right cell of Fig. 19.1). Communicative rationalism, on the other hand, involves all the kinds of knowledge creation shown in Fig. 19.1.

Implications for Research Practice

From the foregoing, it should be clear that we believe scientific rationalism has a place in educational inquiry. Particularly for the kinds of phenomena studied in educational research, however, other kinds of knowledge-building processes are also absolutely necessary if educational research is to inform educational practice.

It should also be clear that we do not advocate abandoning concern for careful argument and evidence in favor of some sort of political rhetoric devoid of reason.
Instead, we promote a renewal of a sense of purpose for our research activity that seems to be disappearing, namely, a concern for making real, positive, lasting changes in what goes on in classrooms.\(^4\) We suggest that such changes will occur only when we become more aware of and concerned with sharing of meanings across researchers and practitioners.

Communicative rationalism, then, is intended to move people to action, in addition to giving them good ideas. That is, it aims to cause people to sit up and take notice, to do something as a result of the dialogue in which they have engaged. To move others to action, the claims that researchers argue for must involve careful attention to what researchers share with their intended audiences and what distinguishes the researchers from them. By so doing, researchers become more familiar with other ways of thinking about their data (i.e., they are able to consider how defensible their claims are in comparison with those of others), and they become better prepared to consider the ethical consequences of their claims.

**WHAT ROLE DOES (SHOULD) EVIDENCE PLAY IN MOVING PEOPLE TO ACTION?**

The relationship between different approaches to research can be illuminated by using ideas from hermeneutics, the name given to the study of interpretation (named after Hermes, the messenger god of classical Greek mythology). Originally developed in theology for the interpretation of Biblical texts, hermeneutics was applied to philosophy more widely by Thomas Dithey in the 19th century.

It is often assumed that an utterance, picture, piece of writing, and so forth. (collectively referred to as text) has a single absolute meaning and that if we only stare at the text long enough, the one true meaning will emerge. The meaning of a piece of text can vary according to its context, however, and even in the same context, a piece of text might have different meanings for different readers. For example, if a teacher praises a student’s work, the student might interpret this as indicating that the work is a significant achievement of which he or she should be proud. But if the student’s experience of the teacher is that praise is used routinely and without sincerity, then the interpretation of exactly the same words might be quite different (Brophy, 1981). The text (in this case, the praise) will be interpreted differently in different contexts and by different readers (e.g., students). These three key ideas—text, context, and reader—are said to form the hermeneutic circle.

In educational research the “text” is usually just “data.” Harding (1987) suggested that “one could reasonably argue that all evidence-gathering techniques fall into one of the following three categories: listening to (or interrogating) informants, observing behavior, or examining historical traces and records” (p. 2).

Sometimes the fact that the data have to be elicited is obvious, as when we sit down with people and ask them some questions and tape record the responses. At other times, this elicitation process is less obvious. If we are in a classroom observing and making notes on a teacher’s or students’ actions, it does not feel as if we are eliciting evidence. It feels much more like a process in which the evidence presents itself to us.

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\(^4\)Ken Ruthven, current editor of *Educational Studies in Mathematics*, has proposed that in the United States “internally-focused—to the research community—concerns of epistemology and methodology [have seemed] predominant. . . . [Whereas] in the UK . . . the burning questions are externally-focused—touching on the credibility of the research community and its capacity not just to influence but to make a productive contribution to policy and practice (K. Ruthven, personal e-mail communication with F. Lester, December 3, 1998). We suggest that the situation in the United Kingdom may not be as sanguine as Ruthven would have us believe.
Nonetheless, the things we choose to make notes about, and even the things that we observe (as opposed to those we see), depend on our personal theories about what is important. In other words, all data are, in some sense, elicited. This is true even in the physical sciences where, as the physicist Werner von Heisenberg remarked, “What we learn about is not nature itself, but nature exposed to our methods of questioning” (quoted in Johnson, 1996, p. 147).

For some forms of evidence, the process of elicitation is the same as the process of recording. If we ask a school for copies of its policy documents in a particular area, all the evidence we elicit comes to us in permanent form. Often much of the evidence that is elicited is ephemeral, however, and only some of it gets recorded. We might be interviewing someone who is uncomfortable with the idea of speaking into a tape recorder, and so we have to rely on note taking. Even if we do audio tape an interview, this will not record changes in the interviewee’s posture that might suggest a different interpretation of what is being said from that which might be made without visual evidence. The important point here is that what is taken as evidence is relative to the researcher’s interests and perspectives and necessarily involves interpretation.

Research based on approaches derived from the physical sciences (often called positivistic approaches, named after a school of philosophy of science that was popular in the second quarter of the 20th century), emphasized text at the expense of context and reader. The same educational experiment would be assumed to yield substantially the same results were it to be repeated elsewhere (e.g., in another school); different people reading the results would be in substantial agreement about the meaning of the results. Other approaches will give more or less weight to the role played by context and reader. For example, an ethnography will place much greater weight on the context in which the evidence is generated than would be the case for more positivistic approaches to educational research but would build in safeguards so that different readers would share, as far as possible, the same interpretations. In contrast, a teacher researching her own classroom might pay relatively little attention to the need for the meanings of her findings either to be applicable elsewhere (so that generalizability across context is not a concern) or to whether others agree with her interpretations (so that generalization across readers is not a concern). For her, the meaning of the evidence in her own classroom might well be paramount.

In what sense, then, can the results of research in mathematics education—and particularly, those results emerging from communicative, rather than scientific, rationalist epistemologies—be regarded as “knowledge”? The traditional definition of knowledge is that it is simply “justified true belief” (Griffiths, 1967). In other words, we can be said to know something if we believe it, if it is true, and if we have a justification for our belief. There are at least two difficulties with applying this definition to research in mathematics education.

The first is that there are severe difficulties in establishing what constitutes a justification or a “warrant” for belief (Kitcher, 1984). The second is that these problems are compounded in the social sciences because the chain of inference might have to be probabilistic rather than deterministic. In this case, our inference may be justified but not true.5

An alternative view of knowledge, based on Goldman’s (1976) proposals for the basis of perceptual knowledge, offers a partial solution to the problem. The central

5We wish not to engage in a discussion of what it means for something to be true. We do find, however, von Glasersfeld’s notion of viability an appropriate alternative to the notion that only truth describes the world. For him, a thing (theory, model, concept, etc.) is viable if it proves to be adequate in the context in which it was developed (von Glasersfeld, 1995).
feature of his approach is that knowing something is, in essence, the ability to eliminate other rival possibilities. For example, if a person (let us call her Diana) sees what she believes to be a book in a school, then we are likely to say that Diana knows it is a book. However, if we know (but Diana does not) that students at this school are expert in making replica books that, to all external appearances, look like books but are solid and cannot be opened, then with a justified-true-belief view of knowledge, we would say that Diana does not know it is a book, even if it happens to be one because her belief is not warranted. With such a view of knowledge, it is almost impossible for anyone to know anything.

Goldman’s solution to this dilemma is that Diana knows that the object she is looking at is a book if she can distinguish it from a relevant possible state of affairs in which it is not a book. In most cases, the possibility that the booklike object in front of Diana might not be a book is not a relevant state of affairs (because not many schools go around making such replicas), and so we would say that Diana does know it is a book. In our particular case, however, there is a relevant alternative state of affairs—the book might be a dummy or it might be genuine. Because Diana’s current state of knowledge (i.e., before she picks up the book and tries to open it) does not allow her to distinguish between these two possibilities, we would say that Diana does not know.

Applying this to research in mathematics education, we would say that we know something when we have evidence that supports our inference and that we have ways of discounting relevant alternative interpretations of our data. Aspects of this are built into traditional experimental designs—for example, in trying out new educational treatments, we might randomize assignment to treatment and control groups to “head off” the rival interpretation that the higher test scores of the treatment group were due to factors unrelated to the treatment. However, much of the debate in research in mathematics education concerns what is to count as a relevant alternative interpretation. For example, some researchers claim that the poorer performance of female students on some mathematics tests indicates a lower level of ability in mathematics, whereas others would attribute these differences to the gendered nature of the particular definition of mathematics underlying the tests or teaching styles that were more suited to male than to female students (Boaler, 1997).

In practice, we suggest that this is determined not by any absolute criteria of what interpretations should and should not be counted as “relevant” but by the consensus of some community of practice (Lave & Wenger, 1991), be it teachers, researchers, or politicians (and of course different communities will come to different conclusions about what is relevant).

This is true in the physical sciences as much as in the social sciences. For example, Collins and Pinch (1993) described the investigations following Joseph Weber’s claim in 1969 to have discovered gravitational radiation. The traditional view of the philosophy of science would have us believe that the claim was subjected rigorously to investigation and refutation, but the question of the existence of gravitational radiation was not settled by empirical means.

Between 1969 and 1975, there were six major attempts to replicate the original findings each of which was unsuccessful. Weber then pointed out methodological flaws in each of the unsuccessful attempts providing plausible rival interpretations—that is, that the results were due to defects in the experimental procedure. In fact, Weber’s critics also found flaws in five out of the six experiments. A scientific rationalist’s perspective would require at this point that the experiments be repeated and correcting the previous flaws to see whether Weber’s results could be replicated. But this didn’t happen. Weber’s rival interpretations of the experimental results have been rejected by the community not on rationalist grounds but because Weber’s interpretations of the results are not considered relevant or plausible (Collins & Pinch, 1993, p. 107).
Sometimes what is and is not to be regarded as a plausible rival interpretation is made absolutely explicit, in the form of a theoretical stance. In other words, a researcher might say “because I am working from this theoretical basis, I interpret these results in the following way, and I do not consider that alternative interpretation to be plausible.” A good example of this is the convention that any interpretation of an experimental result that has a probability of less than 1 in 20 is rejected in the logic of statistical significance testing. More often, however, communities of researchers operate within a shared discourse that rules out some alternative hypotheses, even though the assumptions are implicit and are often unrecognized.

Such a process can never be finally completed, and therefore knowledge can only be provisional rather than absolute. With this view of knowledge it is clear that there can never be a “recipe” for generating knowledge, and knowledge is more or less reliable according to the strengths of warrants for the preferred interpretation and the assiduousness with which alternative interpretations have been pursued.

To sum up our discussion so far, we argue that solutions to educational questions require the consideration of both the traditional, decontextualized knowledge produced by approaches espoused within scientific rationalism and also a knowledge of the contextual and human factors that are required if potential courses of future action are to be realized in classrooms. The prior beliefs and previous experiences of those involved influence both the amount and the kind of evidence that must be marshaled in support of the claim being made and also the extent and the nature of alternative interpretations that must be explored.

The foregoing analysis has demonstrated that what might count as evidence in the production of knowledge is far more complex and varied than is usually acknowledged, and this multiplicity of forms of evidence creates its own difficulties. For this reason, the next section of this chapter deals with a typology of forms of evidence developed by C. West Churchman that leads to a systematization of different ways of building knowledge.

**CHURCHMAN’S CLASSIFICATION OF SYSTEMS OF INQUIRY**

Churchman (1971) classified all systems of inquiry into five broad categories, each of which he labeled with the name of a philosopher (viz., Leibniz, Locke, Kant, Hegel, & Singer) who he felt best exemplified the stance involved in adopting the system. He gave particular attention in his classification to what is to be regarded as the primary or most salient form of evidence, as summarized in Table 19.1. For detailed accounts of Churchman’s classification scheme see, Churchman (1971), Messick (1989), Mitroff and Kilmann (1978), and Mitroff and Sagasti (1973).

Churchman’s framework is particularly useful in thinking about how to conduct research that makes a difference and, specifically, whether the research moves

<table>
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<tr>
<th>Inquiry System</th>
<th>Source of Evidence</th>
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<td>Leibnizian</td>
<td>Reasoning</td>
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<td>Lockean</td>
<td>Observation</td>
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<td>Kantian</td>
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<td>Hegelian</td>
<td>Dialectic</td>
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<tr>
<td>Singerian</td>
<td>Ethical values and practical consequences</td>
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people toward appropriate action. The framework poses three questions that we should attempt to answer about our research efforts:

1. Are the claims we make about our research based on inferences that are warranted on the basis of the evidence we have assembled?
2. Are the claims we make based on convincing arguments that are more warranted than plausible rival claims? and
3. Are the consequences of our claims ethically and practically defensible?

In the following discussion, we describe Churchman’s framework by considering how it might be applied to a real research question in mathematics education. The current controversy over reform versus traditional mathematics curricula has attracted a great deal of attention in the United States and elsewhere among educators, professional mathematicians, politicians, and parents and can serve to illustrate how these three questions might be used.

For some, the issue of whether the traditional or reform curricula provide the most appropriate means of developing mathematical competence is an issue that can be settled on the basis of logical argument. On one side, the proponents of reform curricula might argue that a school mathematics curriculum should resemble the activities of mathematicians, with a focus on the processes of mathematics. On the other side, the antireform movement might argue that the best preparation in mathematics is one based on skills and procedures. For example, a report by the London Mathematical Society, the Institute of Mathematics and Its Applications and the Royal Statistical Society (1995) argues that “To gain a genuine understanding of any process it is necessary first to achieve a robust technical fluency with the relevant content” (p. 9).

Despite their opposing views, both these points of view rely on rhetorical methods to establish their position, in an example of what Churchman called a Leibnizian inquiry system. In such a system, certain fundamental assumptions are made, from which deductions are then made through the use of formal reasoning rather than by using empirical data. In a Leibnizian system, reason and rationality are held to be the most important sources of evidence. Although there are occasions in educational research when such methods might be appropriate, they usually are in sufficient. In fact, typically the educational research community requires some sort of evidence from the situation under study (usually called empirical data).

The most common use of data in inquiry in both the physical and social sciences is via what Churchman called a Lockean inquiry system. In such an inquiry, evidence is derived principally from observations of the physical world. Empirical data are collected, and then an attempt is made to build a theory that accounts for the data. This corresponds to what is sometimes called a naive inductivist paradigm in the physical sciences. Consider the following scenario.

A team of researchers, composed of the authors of a reform-minded mathematics curriculum and classroom teachers interested in using that curriculum, decide after considerable discussion and reflection to design a study in which ninth-grade students are randomly assigned either to classrooms that will use the new curriculum or to those that will use the traditional curriculum. The research team’s goal is to investigate the effectiveness (with respect to student learning) of the two curricula over the course of the entire school year. Suppose further that the research design they developed is appropriate for the sort of research they are intending to conduct.

6The notion of “effectiveness” raises a thorny issue because effectiveness is determined by what is valued. Thus, it is possible that each curriculum might be judged the more effective depending on the research team’s value judgments.
From the data the team will gather, they hope to be able to develop a reasonable account of the effectiveness of the two curricula, relative to whatever criteria are agreed on, and this account could lead them to draw certain conclusions (i.e., inferences). Were they to stop there and write a report, they would essentially be following a scientific rationalist approach situated in a Lockean perspective.

The major difficulty with a Lockean approach is that because observations are regarded as evidence, it is necessary for all observers to agree on what they have observed. Because what we observe is based on the theories we have, different people will observe different things, even in the same classroom.

For less well-structured questions, or those on which people are likely to disagree about what precisely is the problem, a Kantian inquiry system is more appropriate. This involves the deliberate framing of multiple alternative perspectives, on both theory and data (thus subsuming Leibnizian and Lockean systems). One way of doing this is by building different theories on the basis of the same set of data. Alternatively, we could build two (or more) theories related to the problem, and then for each theory, generate appropriate data (different kinds of data might be collected for each theory).

For our inquiry into the relative merits of traditional and reform curricula, our researchers might not stop with the “crucial experiment” described above but instead would consider as many alternative perspectives as possible (and plausible) about both their underlying assumptions and their data. They might, for example, challenge one or more of their assumptions and construct competing explanations on the basis of the same set of data. These perspectives would result in part from their engagement in serious reflection about their underlying assumptions and in part from submitting their data to the scrutiny of other persons who might have a stake in the research, for example, teachers who have used the traditional curriculum. An even better approach would be to consider two or more rival perspectives (or theories) while designing the study, thereby possibly leading to the generation of different sets of data. For example, a study designed with a situated cognition (or situated learning) perspective in mind might result in a very different set of data being collected than a study based on contemporary cognitive theory (see Anderson, Reder, & Simon, 1996; Greeno, 1997). These two perspectives probably would also lead the researchers to very different explanations for the results (Boaler, 2000). For example, the partisans of the situated cognition perspective might attribute results favoring the reform curriculum to certain aspects of the social interactions that took place in the small groups (an important feature of the reform curriculum), whereas cognitivists might claim that it was the increased level of individual reflection afforded by the new curriculum materials, rather than the social interaction, that caused the higher performance among students who were in the reform classrooms.

The different representations of traditional and reform classrooms developed within a Kantian inquiry system may not be reconcilable in any straightforward sense. It may not be immediately apparent where these theories overlap and where they conflict, and indeed, these questions may not be meaningful in that the enquiries might be incommensurable (Kuhn, 1962). Nonetheless, by analyzing these enquiries in more detail, it may be possible to begin a process of theory building that incorporates the different representations of the situation under study.

This idea of reconciling rival theories is more fully developed in a Hegelian inquiry system, in which antithetical and mutually inconsistent theories are developed. Not content with building plausible theories, the Hegelian inquirer takes a plausible theory...
and then investigates what would have to be different about the world for the exact opposite of the most plausible theory itself to be plausible. The tension produced by confrontation between conflicting theories forces the assumptions of each theory to be questioned, thus possibly creating a coordination of the rival theories.

In our example, the researchers should attempt to answer two questions: What would have to be true about the instruction that took place for the opposite of the situated learning explanation to be plausible? and What would have to be true about the instruction that took place for the opposite of the cognitivist explanation to be plausible? If the answers to both these questions are “not very much,” then this suggests that the available data underdetermine the interpretations that are made of them. This might then result in sufficient clarification of the issues to make possible a coordination, or even a synthesis, of the different perspectives, at a higher level of abstraction.

The differences between Lockean, Kantian, and Hegelian inquiry systems were summed up by Churchman as follows:

The Lockean inquirer displays the “fundamental” data that all experts agree are accurate and relevant, and then builds a consistent story out of these. The Kantian inquirer displays the same story from different points of view, emphasizing thereby that what is put into the story by the internal mode of representation is not given from the outside. But the Hegelian inquirer, using the same data, tells two stories, one supporting the most prominent policy on one side, the other supporting the most promising story on the other side. (Churchman, 1971, p. 177)

Perhaps the most important feature of Churchman’s typology is that we can inquire about inquiry systems, questioning the values and ethical assumptions that these inquiry systems embody. This inquiry of inquiry systems is itself, of course, an inquiry system, termed Singerian by Churchman after the philosopher E. A. Singer (see Singer, 1957). Such an approach entails a constant questioning of the assumptions of inquiry systems. Tenets, no matter how fundamental they appear to be, are to be challenged to cast a new light on the situation under investigation. This leads directly and naturally to examination of the values and ethical considerations inherent in theory building.

In a Singerian inquiry, there is no solid foundation. Instead, everything is “permanently tentative”; instead of asking what “is,” we ask what are the implications and consequences of different assumptions about what “is taken to be”:

The “is taken to be” is a self-imposed imperative of the community. Taken in the context of the whole Singerian theory of inquiry and progress, the imperative has the status of an ethical judgment. That is, the community judges that to accept its instruction is to bring about a suitable tactic or strategy. . . . The acceptance may lead to social actions outside of inquiry, or to new kinds of inquiry, or whatever. Part of the community’s judgement is concerned with the appropriateness of these actions from an ethical point of view. Hence the linguistic puzzle which bothered some empiricists—how the inquiring system can pass linguistically from “is” statements to “ought” statements—is no puzzle at all in the Singerian inquirer: the inquiring system speaks exclusively in the “ought,” the “is” being only a convenient façon de parler when one wants to block out the uncertainty in the discourse. (Churchman, 1971, p. 202; our emphasis in fourth sentence)

An important consequence of adopting a Singerian perspective is that with such an inquiry system, one can never absolve oneself from the consequences of one’s research. Educational research is a process of modeling educational processes, and the models are never right or wrong, merely more or less appropriate for a particular purpose, and the appropriateness of the models has to be defended. It is only within a Singerian perspective that the third of our key questions (Are the consequence of
our claims ethically and practically defensible?) is fully incorporated. Consider the following scenario.

After studying the evidence obtained from the study, the research team has concluded that the reform curriculum is more effective for ninth-grade students. Furthermore, this conclusion has resulted from a consideration of various rival perspectives. However, a sizable group of parents strongly opposes the new curriculum. Their concerns stem from beliefs that the new curriculum engenders low expectations among students, deemphasizes basic skills, and places little attention on getting correct answers to problems. The views of this group of parents, who happen to be very active in school-related affairs, have been influenced by newspaper and news magazine reports raising questions about the new curricula, called “fuzzy math” by some pundits. To complicate matters further, although the teachers in the study were “true believers” in the new curriculum, many of the other mathematics teachers in the school district have little or no enthusiasm about changing their traditional instructional practices or using different materials, and only a few teachers have had any professional development training in the implementation of the new curriculum.

Before they begin to publicize their claims, the research team is obliged to consider both the ethical and practical issues raised by concerns and realities such as those presented above. Is it sensible to ask teachers to implement an instructional approach that some parents and perhaps others will be challenge vigorously? Can they really claim, as the school district superintendent desires, that student performance on state mathematics tests will improve if the new curriculum is adopted? Are they confident enough in their conclusions about the merits of the new curriculum to recommend its use to inexperienced teachers? Should they encourage reluctant or resistant teachers to use this approach in their own classrooms if they may do so halfheartedly or superficially? Can these reluctant teachers be expected to implement this new curriculum in a manner consistent with reform principles? Such ethical and practical questions are rarely addressed in research in mathematics education but must be considered if the researchers really care about moving the school district to act on their conclusions. Answers to questions of this nature will necessitate prolonged dialogue with various groups—among them teachers, school administrators, parents, and students.

Implicit in the Singerian system of inquiry is consideration of the practical consequences of one’s research, in addition to the ethical positions. Greeno (1997) suggested that educational researchers should assess the relative worth of competing (plausible) perspectives by determining which perspective will contribute most to the improvement of educational practice; we would add that this assessment must take into account the constraints of the available resources (both human and financial), the political and social contexts in which education takes place, and the likelihood of success. Whereas the Lockean, Kantian and Hegelian inquirer can claim to be producing knowledge for its own sake, Singerian inquirers are required to defend to the community not just their methods of research, but which research they choose to undertake.

Singerian inquiry provides a framework within which we can conduct a debate about what kinds of research ought to be conducted. Should researchers work with individual teachers supporting them to undertake research primarily directed at transforming their own classrooms, or should researchers instead concentrate on producing studies that are designed from the outset to be widely generalizable? Within a Singerian framework, both are defensible, but the researchers should be prepared to defend their decisions. The fact that the results of action research are often limited to the classrooms in which the studies are conducted is often regarded as a weakness in traditional studies. Within a Singerian framework, however, radical improvements on a small scale may be regarded as a greater benefit than a more widely distributed, but less substantial, improvement.
We introduced this chapter by stating that the first part considered how (and for whom) research in mathematics is undertaken, whereas the second focused on what counted as evidence. As should be clear from the foregoing analysis, we do not believe that such a distinction is, in fact, tenable. Research in mathematics education, as in any other field, is an integrated, and ultimately moral, activity that can be characterized as a never-ending process of assembling evidence that particular inferences (i.e., claims) are warranted on the basis of the available evidence, that such inferences are more warranted than plausible rival inferences, and that the consequences of such inferences are ethically and practically defensible (cf., Wiliam, 1998).

Furthermore, the basis for warrants, other plausible interpretations, and the ethical and practical bases for defending the consequences are constantly open to scrutiny and question.

Unfortunately, in our experience, only rarely has any of the published mathematics education research included significant attention to a discussion of rival inferences, and even more rarely have researchers addressed in their reports issues related to the ethical and practical defensibility of the claims they make.

**CLOSING THOUGHTS**

Philosopher Richard Rorty (1979) offered a point of departure for conceptualizing the dialogues that take place within the research community, within the community of practitioners, and between these two groups. Specifically, Rorty embraced postmodern philosophy as one voice in the ongoing conversation about what it means to be human. Within this conversation, he distinguished between analytical philosophy and hermeneutic philosophy. In an analytic endeavor, the participants are seeking to extend a scientific rationalistic account of some phenomenon and may indeed conceive of themselves as producing eternal knowledge. In hermeneutic activity, the conversants seek only to steer the conversation in ways that enable people to cope better with some phenomenon in the present, not to establish an eternal body of knowledge. This form of discourse is essential to the development of ethically informed, reasoned conversation between researchers and practitioners about issues that are fundamental to teaching and learning mathematics in contemporary society.

Anthropologist Mary Catherine Bateson (1994) presented a moving vision of learning to which we might also turn for inspiration. No single framework anchors learning in her account. She found discourse based solely on abstract concepts inadequate to the challenge of understanding specific lived experience. Drawing on several cases in which multiple diverse perspectives on shared experiences led her to deeper insights, she argued convincingly that “insight . . . refers to that depth of understanding that comes by setting experiences, yours and mine, familiar and exotic, new and old, side by side, learning by letting them speak to one another” (p. 14; emphasis in original). For Bateson, it is in the boundaries between what two or more people have to say about a common experience that real learning takes place.

In this chapter, we have outlined why we believe that without a radical shift in its orientation, research in mathematics education is unlikely to influence practice, and we have also argued that such an outcome is indefensible. We have suggested some possible ways in which to enhance communication among researchers, teachers, and other practitioners and consequently to do research that will move us—teachers, school administrators, curriculum developers, teacher educators, and others—to action. The likelihood that this will happen will increase if the conversation about

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8The distinction between researchers and practitioners has become increasingly blurred as more and more research is being conducted by teacher-researchers and by teacher–researcher collaborative teams.
the focus of our research is expanded in a rich and complete manner, paying attention to the multiple meanings and interpretations (including beliefs and assumptions) brought to the discussion by each participant in the conversation.

ACKNOWLEDGMENTS

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REFERENCES


This chapter invites the reader to consider the research activity in the field of mathematics education using a lens wider than is normally applied in this regard. It does so by taking as a model the perspective of an educational research funding agency. We hope the reader will find in this chapter alternative ways of framing the efforts of the field not with the goal of agreeing with the arguments made by the authors necessarily, but rather with the goal of beginning conversations about the direction and teleology of the entire research effort. It goes without saying that in a chapter of this length, we cannot touch on all the important work in mathematics education or in other fields; furthermore, there is insufficient space to explicate many of the authors’ claims, particularly with regard to complexity theory. Nonetheless, we hope that by the end of the chapter readers will find themselves not at the point of frustration but ready to engage in and to entertain new views of the subject matter.

We begin the chapter by characterizing or providing one view of the current state of research in mathematics education. Here we highlight the maturing of the field as a research enterprise by illuminating recent developments in methodologies used to conduct this research. The critical change here is the adaptation of multiple methodologies to resolve the issues of mathematics education researchers. When contrasted with the simple methodological approaches used in other fields, the maturation becomes more apparent.

The chapter continues with a brief reflection of the National Science Foundation’s (United States) investment in mathematics education. This reflection is in effect no
different than the reflections of many other national, or state, funding agencies worldwide. The United Kingdom’s Teaching and Learning Research Programme (TLRP), for example, explicitly seeks to blend advances in the educational research with “authentic conditions of learning.” Another example includes the efforts by the state of Victoria in Australia (Stacey, Asp, & McCrae, 2000). The state has asked, recently “what algebra should be taught if technology (particularly that of the handheld calculator) is truly embraced?” Yet another example is a major priority program effort, BIQUA (Bildungsqualität von Schule), instituted the German national research funding agency Deutsche Forschungsgemeinschaft (DFG) to “evaluate theoretically well-grounded interventions aimed at contributing to the increase of the quality of education in German schools.” The BIQUA initiative comprises a series of approximately two dozen school-quality research and evaluation efforts throughout Germany to elucidate effective mechanisms of innovation in educational systems. Put differently, the program considers what should be taught; how this material might be learned; and what impact will it have on teachers, curriculum developers, children, and assessment techniques both formal and less formal.

These examples serve to highlight the common needs and challenges of many funding agencies across the world. In the context of this chapter, we pose this “reflection” more broadly as three distinct but overlapping challenges. The challenges described in the body of the chapter relate to the developing science of learning, the need for system change, and the challenge of informing and affecting school and classroom practice.

The chapter continues by describing one national U.S. agency’s programmatic response to these overriding challenges. Simply put, the response comprises a continuum of research across four overlapping quadrants that include the following:

- Brain research as a foundation for research in human learning
- Fundamental research on behavioral, cognitive, affective, and social aspects of human learning
- Research on the teaching and learning of mathematics in formal and informal settings
- Research on mathematics learning in complex educational systems

This continuum approach as response to a national educational research challenge, and the complex system it is likely to generate, serves as the central unifying focus of the chapter. The complexity metaphor is used to provide coherence in framing the working of the research program and the anticipated products of the program in terms of its initial conditions and the possible resulting dynamics derived from program participation. Finally, we offer the metaphor for the illumination it provides and the debate it will likely generate.

**CURRENT STATUS**

A review of the various chapters in the recent *Handbook of Research Methods in Mathematics and Science Education* (Kelly & Lesh, 2000) suggests that the methods that mathematics education researchers are currently applying (and those in education research more generally) are designed to give ever more refined insight into the learning of subject matter. Although not all the authors in that volume would welcome the label constructivist, it may be fair to state that many of these researchers are motivated to better understand students’ understanding of mathematics. In other words, they direct their investigations, by and large, to individual learning of fairly routine classroom mathematics. The payoff, as it were, of this line of work is rooted in psychological views of mathematics learning in formal educational groupings of students.
The reader may expand on this approach by considering the learning of mathematics in informal settings also. These settings may include learning in museums, from the mass media, and from other sources (e.g., the Internet and related technologies). Within this strand of research, important topics include individual mathematics learning, teaching and professional development, curriculum and instruction, and assessment issues that map well to broad themes of mathematics education research.

This topic, research on the teaching and learning of mathematics in formal and informal educational settings, has been the core interest of the Education and Human Resources (EHR) Directorate of the National Science Foundation (NSF) in the United States. The NSF is the sole U.S. civilian federal agency with a breadth of research investment responsibilities that cut across the domains of science, mathematics, technology, and engineering (for further information on the NSF and EHR, see http://www.nsf.gov).

In the recent past, the NSF began to reflect on the focus of its investment on this type of educational research. The agency has begun to consider a broader continuum of learning in which informal and formal mathematics learning is seen as its dominant, yet not sole, emphasis. In doing so, it has encountered three challenges.

THE THREE CHALLENGES

The Science of Learning Challenge

The NSF is comprised of a number of different science directorates, which in special cases collaborate to produce research synergies. One example of this multidisciplinary synergy is the Learning and Intelligent Systems (LIS) program (NSF, 1999). The LIS program provided an important experiment in framing a research agenda. Its purpose, beginning in 1997, was to support inquiry into learning and intelligence based on experimental and theoretical frameworks from a variety of disciplines. These include, but are not limited to, computer science and education, biological sciences, cognitive science, neuroscience, mathematics, among others. LIS awards provide an important methodological foundation for research on learning and have led to significant breakthroughs (for example, in intelligent tutors, simulation agents, and semantic analysis) for classroom computer technologies of the future (for a review, see Hamilton, 2000a).

The LIS program produced synergies and “findings at the boundaries” of several previously disparate disciplines. Indeed, the burgeoning and converging developments in learning, cognitive science, and cognitive neuroscience partly stimulated by LIS and highlighted, perhaps most prominently, by the National Academy of Sciences Report *How People Learn*, (Bransford, Brown, & Cocking, 1999) created a critical challenge to EHR to take advantage of and respond to the development of a multidisciplinary science of learning. Where in this growing body of research did the current research in mathematics education reside? What was its relationship to this body of work? What impact might it have on the research questions entertained by these new sciences of learning?

THE SYSTEM CHALLENGE

A second challenge may seem, at first blush, quite removed from the first. The EHR invests approximately $100 million per year in the education system reform (ESR) activity of urban, rural, and statewide systemic initiatives, or approximately 20% of its primary and secondary school (K–12) budget. The systemic initiatives, although framed in available research at the time of their initiation, clearly needed to be
informed in an ongoing manner by sound educational research. Consequently, the EHR’s Research in Educational Policy and Practice (REPP; NSF 96-13) program explicitly sought to build a research base in system reform and to develop models that could inform the ESR investment. Regrettably, the REPP program did not succeed in generating a critical mass, or community mobilized, around important system reform issues, in part because of the research community’s specific focus on components of the system (e.g., classrooms). The second challenge was thus to stimulate educational systems research effort more forcefully and expansively.

The Challenge of Impacting and Informing Practice

The REPP program funded a large number of projects that conducted research in situ in classrooms (employing a variety of qualitative methods). These methods and projects (see Kelly & Lesh, 2000; Suter & Frechtling, 2000) successfully bridged the theory–practice divide and produced results more readily usable by teachers compared with results from more traditional quantitative studies. When considered in the light of the other two challenges, however, this success appeared somewhat limited. Again, where did this work fit in a larger science of learning? Was it sensitive to and informed by work on educational systems writ large?

The Combined Challenges

In looking at the three challenges together, the NSF set out to devise a program that would attempt to bridge the advances in funded research on teaching and learning with the unfolding developments in the science of learning and with a greater knowledge about how to bring (or “scale”) advances in various learning educational settings to the system structures that would either sustain or suffocate the advances.

THE CONTINUUM: RESPONDING TO THE CHALLENGES

The Research on Learning (ROLE; NSF, 2000) program, initiated in 2000, attempts to spur educational researchers to respond to these three (and other) challenges. It explicitly seeks to connect a core interest of research in the teaching and learning of science and mathematics both to research in learning sciences and to research on educational system change. The ROLE program extends the purview of the REPP program to include the learning and cognitive sciences in a more micro direction while, at the same time, extending its reach in the macro direction by including learning about systems and system behavior. The resulting continuum involves four sections or levels, the third of which represents the core NSF interest in education research, and characterizes much of the current research effort in mathematics education. The levels (referred to as quadrants in the ROLE program guidelines) comprise the following:

1. Brain research as a foundation for research on human learning. This level includes areas such as neurophysiology and cognitive neuroscience. Connections with other levels may include comparative neural and behavioral effects of different training strategies and pedagogical sequences; functional imaging of neural activity during different types of knowledge and skill acquisition; and neural plasticity at the micro level, its variability, and its relation to learning at the macro level.

2. Fundamental research on behavioral, cognitive, affective, and social aspects of human learning. This level includes areas such as cognition and perception in learning, including concept formation, acquisition, and change; informal learning and attention mechanisms; memory architectures and their substrates; spatial representations and
manipulations; reasoning; and development of increasingly complex models and representations.

3. Research on the teaching and learning of mathematics in formal and informal educational settings. This is the core interest of the EHR directorate. The ROLE Guidelines outline a series of topics in mathematics learning, teaching and professional development, curriculum and instruction, and assessment that map well to broad themes of mathematics education research. Most of the types of learning technology development that NSF sponsors fall into this quadrant of investment.

4. Research on mathematics learning in complex educational systems. The term systems refers to traditional entities (e.g., pre-K–12 school systems, postsecondary organizations and authorities) and to broader views of educational stakeholders, including research scientists and policymakers, and the ways in which stakeholders interact. The ROLE guidelines (NSF, 2000, p. 11) state that systemic studies may include uncovering the mechanisms for the transfer of fundamental research findings in scientific disciplines to innovation-based SMET curriculum reform, the adoption of experimental SMET learning technology prototypes into scaled and sustained educational practice, or the conditions for widespread increases of the participation of learners in scientific research. Other questions for which research findings are sought include core issues in systemic reform at all levels of education, and systemic reform issues that require better theoretical specification than is currently available.

The quadrant view recognizes both the boundaries that separate aspects of learning and the permeability of these boundaries. Because some research topics in the quadrants significantly overlap and inform one another, ROLE seeks gains at the intersections of the quadrants, where issues arising from research and educational practice can be reconciled and where hypotheses generated in one area may be tested and refined in others. In this way, ROLE attempts to emulate the success of the LIS program, described earlier.

This chapter does not discuss extensively Quadrants 1 and 2, focusing instead on Quadrants 3 and 4. First, it attempts to provide some examples of multilevel research threads that can operate across the continuum and to show how, when viewed as part of a continuum of effort, current approaches to mathematics education research can be enriched and extended. Second, this chapter introduces by way of analogy the notion that the design of the research activity supported by EHR can be viewed as behaving as a “complex system.” From use of this analogy, we hope to expand the shared vocabulary (and perhaps the conceptual tools) available to funding agency officers and by extension to the educational research field generally.

The Core Interest of the Research Program: Extending Quadrant 3 Research

The first two challenges form a type of symmetry of levels, from micro level to macro level. The first takes advantage of a consequential and emerging multidisciplinary science of learning (for which we solicit participation through the first two quadrants). The second considers the system variables that confound and mitigate classroom-level effects, as explicit objects of policy and organizational research (which we address in Quadrant 4). Our thesis is that building these levels into the research program and mapping the core mathematics education research that NSF has previously supported both to converging advances in biological and cognitive functions and to complex system effects will benefit immeasurably research on teaching and learning mathematics. We now briefly describe examples of what we consider to be research that
falls into Quadrant 3 and expand two examples, fleshing out the larger continuum of research described in the RÖLE guidelines.

**Teaching Experiments in Mathematics Education**

According to Steffe and Thompson (2000),

A primary purpose for using teaching experiment methodology is for the researchers to experience, firsthand, students’ mathematical learning and reasoning. Without the experiences afforded by teaching, there would be no basis for coming to understand the powerful mathematical concepts and operations students’ construct nor for even suspecting that these concepts and operations may be distinctly different from those of the researchers. (p. 267)

Notice in this selection the recognition that the world of mathematical understanding of the researcher is seen as importantly different from that constructed by the student. Steffe and Thompson set about identifying ways and means by which children operate that they called essential. Essential modes of operation are those that were not taught directly to the students and that appeared resistant to attempts on the part of the researchers to change them. The cited examples involve children’s counting routines. Knowledge of such essential ways of operating provides the basis for further exploration and research for Steffe and Thompson.

By contrast, Confrey promoted a transformational and proactive approach to teaching experiments (e.g., Confrey & Lachance, 2000). Confrey sought to understand the effect on learning of conjectures about what mathematics content should be taught and how that content should be taught. Conjectures emerge from substantial reviews of the literature and reflections on prior experiences of teachers, teacher-researchers, and students. One such conjecture is whether multiplication is better taught as repeated addition or from some other vantage point, say, making multiple copies of something (i.e., splitting; Confrey & Lachance, 2000).

The work of Steffe and of Confrey echo a general shift from laboratory to classroom research that led Ann Brown (1992) and others to develop design experiment approaches discussed by Lagemann (2000). At the heart of the development of design experiment methodology, generally, is the recurring issue of bridging the tension between research and practice in the study of teaching. Although there are many examples of educational research successfully applied to practice, Collins (1999) argued that there remains a “great divide” or what Shulman (1998) called a competition between research and practice in teaching, which the design method seeks to remedy.

The design approach focuses on iterative experimental interventions; the approach’s goal of proximity and fidelity to real-world learning environments also animates the narrative case-study approach of teacher-turned-researchers such as Ball and Lampert (1999), who extensively documented via video, audio, and paper libraries various mathematics teaching and learning sequences. The documentation process has created opportunities to peel away and to scrutinize many of the layers of classroom dynamics, such as the nature of the mathematics in a classroom, and, à la Steffe and Cobb, the different mathematical models that a teacher and students form of the same mathematical activity or the sociological dimensions of student–student and teacher–student interactions. The importance of the narrative case study is not simply the grounding of personal teaching experience that the researchers bring to the investigation, but the emergence of a body of data captured in situ that is available to a community of researchers. Shulman (1998) wrote, “When we seek a pedagogy that can reside between the universal principles of theory and the narrative of lived practice, we invent approaches, such as varieties of case methods—capable of capturing of experience for subsequent analysis and review. We render individual experiential
learning into ‘community property’ when we transform those lessons from personal experience into a literature of shared narratives. Such connections between theoretical principles and practical narratives, between the universal and the accidental, forge professional knowledge.” Such commonly forged professional knowledge helps to generate what Lagemann (1999) called a “distinct knowledge community” that not only deepens but stabilizes a body of findings about teaching.

Despite their similarities, an important distinction may be drawn between the approaches to learning about mathematics education espoused by Steffe and Thompson and by Confrey and Lachance. The former approach grows out of a Piagetian orientation to learning and epistemology and places central importance on individual’s constructions of knowledge. The latter approach is primarily concerned with equity (and, implicitly, excellence) with the larger scale impact on student learning: “In our work, we seek continually to advance the kind of research that entails active attempts to ensure equal opportunities for all students to participate in and succeed at mathematics” (Confrey & Lachance, 2000). In other words, Confrey seeks not only how a particular child (or group of children) learns but also considers how the results of research impact the educational system as a whole.

In the quadrant structure of ROLE, one might argue that Steffe’s method, while situated in Quadrant 3, gravitates toward the individual psychology of Quadrant 2, whereas Confrey’s method, again situated in Quadrant 3, gravitates toward the concerns of systemic reform, more characteristic of Quadrant 4. In both cases, an explicit advance of the approach is to bring greater integration to research and practice. In the case of Steffe and Thompson, the bridge may be viewed as practice influencing research via in situ explorations that value the observed mathematical constructions of the student and the teacher, over that observed in clinical settings. In Confrey’s case, the bridge may be viewed as research influencing practice, to test theories in situ, to inform reform efforts that impact all students.

**Multitiered Teaching Experiments**

A final example of Quadrant 3 research methods that advance knowledge about mathematics learning while exploring new devices to bring research and practice into greater proximity involves the multitiered teaching experiment advanced by Lesh (Lesh & Kelly, 1997; Kelly & Lesh, 2000). The tiers represented in the multitiered teaching experiment are those of students, teachers, and researchers.

Students are typically given specifically designed tasks called “model-eliciting” (or “thought-revealing”) problems. The problems, as posed, do not simply state givens and goals in formulaic mathematical terms. The students must bring mathematics to the problem. (For examples of such problems, see Lesh, 1999). Teachers, in turn, model how students solve these problems over a protracted period of time. Teachers may predict students’ solution paths and make tutoring suggestions, which are then borne out (or not) in students’ learning. Researchers, in turn, model both the students’ and the teachers’ learning over the same time period. Self-reflexivity is encouraged for all participants at all levels, including the researcher level. Participants at all levels are viewed as learners and common assumptions are held about the development of comprehension across all levels: Students learn deeper mathematics, teachers learn about students’ learning of mathematics and about tutoring strategies, and researchers learn about these phenomena. Although not explicitly addressed in the multitiered teaching experiment, at a meta-level the local team of researchers is changing through interactions with the field.

Each of these study types indicative of work in Quadrant 3 more tightly knits research and practice, opening avenues of discourse, understanding, and proximity between research and practice. Collectively, these approaches, represent advances
that are bridging and blending research and practice in fundamental ways. But the nature of this bridging is itself quite confined; furthermore, it does not satisfy lay misconceptions that single research advances, sequentially applied, inexorably lead to improvement in student learning experiences.

Issues of large, system-scale, research-stimulated improvement of student learning involve a different lens or level of analysis (Blumenfeld, Fishman, Krajcik, Marx, & Soloway, 2000; Elmore, 1996). The investment in reform of highly dysfunctional educational systems faces an even more daunting challenge. One of NSF’s reform investment principles is that a functional educational system requires functional components across a range of scales or entities. A competent instructional workforce, coherent curriculum and assessment practices, effective policy apparatus, strong leadership and resource management, and an array of community partners are among constituent elements of functional educational systems. The agency’s investment, in fact, requires each educational system to report annually on their progress in improving the operational quality of these specific components. Webb (1997) and Clune (1998) used the construct of “alignment” of system variables such as these to define system reform prerequisites. Alignment of variables is a necessary condition to effective reform, and, in the NSF’s experience, transitioning to alignment has proven to be the most daunting challenge. Reform models that do not capture complex interactions between variables (e.g., shifts to more effective curriculum requires resource availability and on-site school leadership), reciprocal interactions (transition to coherent resource allocation enables and is enabled by research-informed policy shifts), and system feedback and coupling (e.g., communication networks, use of assessment analyses) invariably miss critical transformative developments. These dynamics suggest that complexity theory may provide descriptive and possibly predictive tools for modeling educational systems.

Three Features of Complex Systems

Complexity theorists (e.g., Bar-Yam, 2000; Holland, 1999) generally describe rather than define features of complex systems. We follow the same logical style in our presentation, citing two examples before mapping the description to educational systems. Among typical characteristics of complex systems, three are of interest here. These fall, for purposes of this discussion, into two categories: initial conditions and system dynamics.

Initial Conditions.

- The system requires a collection of individual agents, sometimes functioning in different tiers or levels, whose actions change the context of the other agents within and across levels.
- The system functions derive from different feedback circuits and rules. Often relatively simple rules generate the interactions between agents that mediate large-scale system dynamics.
- System interactions are seeded by initial or baseline condition variables that establish the trajectory of the system’s behavior.

Dynamic Features.

- In a complex system, the initial conditions give rise, without a centralized control, to evolving dynamics, often unanticipated, that adapt to the changing system context and further modify the context.
TABLE 20.1
Maxwell-Boltzmann and Traffic Jams in System Simulation Software

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Students see a depiction of molecules randomly distributed in a box.</td>
<td>Students see a depiction of automobiles in traffic.</td>
<td>Three rules are applied: (a) if a car is in front of yours, slow down; (b) if a car is not in front of you, accelerate up to but not exceeding the speed limit; (c) if you are in a radar trap, slow down.</td>
</tr>
<tr>
<td>Feedback loops and rules</td>
<td>Basic Newtonian principles drive the system. Molecules collide producing three types of action in the simulation: (a) molecules move forward, (b) molecules bounce off of a wall, or (c) molecules collide. The simulation software executes the next action with every clock tick.</td>
<td>Three rules are applied: (a) if a car is in front of yours, slow down; (b) if a car is not in front of you, accelerate up to but not exceeding the speed limit; (c) if you are in a radar trap, slow down.</td>
</tr>
<tr>
<td>Initial or baseline condition variables</td>
<td>One initial variable: number of molecules in the closed system.</td>
<td>Three initial variables: (a) number of cars; (b) initial initial distances between the cars; (c) number of radar traps.</td>
</tr>
<tr>
<td>System patterns</td>
<td>Maxwell-Boltzmann distribution of molecular speeds in clock ticks.</td>
<td>Traffic jams moving through the traffic network. Traffic jams move backward, for example, while cars that make up the traffic jams move forward.</td>
</tr>
</tbody>
</table>

Resnick (1998) and Wilensky and Resnick (1999) provided several interesting examples of what one might call "simple" complex systems that are generated in software such as StarLogo (Resnick, 1996). The two specific examples (GasLab and Traffic Jams) that we use are drawn from their work (Wilensky & Resnick, 1999) and appear in Table 20.1. We use them to introduce complexity as a model for educational systems and later as a metaphor for the cross-continuum research program that we have implemented in the form of ROLE.

Interestingly, the research in and development of tools such as GasLab or the Traffic Jams simulations falls into what we have described as Quadrant 3 activities; they provide innovative curriculum opportunities and enable—indeed require—different pedagogical models to exploit those opportunities. Furthermore, the tools themselves further provide a descriptive metaphor that enables the conceptualization of much needed research frames for understanding the large system change embraced by Quadrant 4 activities.

One value of these examples is that they highlight system patterns that reside beyond the specific level of interactions from which they emerge. Additionally, the two examples illuminate how the initial conditions of variables that seed the system alter the dynamics of the system. As suggested above, in these examples the agents, and interactions between them that give rise to the system, are all relatively simple in themselves, yet they generate quite complex system dynamics.

These examples illustrate simple dynamics at one interactional level that yield collective behavior or patterns at a different level. The molecular speeds fall onto a distribution in the first example, and traffic bottlenecks migrate around the network in the second. Now consider how we might apply these concepts to educational systems...
TABLE 20.2

Selected Features of Educational Systems Mapped to Complex System Characteristics

<table>
<thead>
<tr>
<th>Agents</th>
<th>Students, Teachers, Administrators, Technical and Support Staff</th>
</tr>
</thead>
</table>
| Sample levels | • Human agent levels: students; instructional and organizational leadership, technical staff, community, teachers union, financial, school board  
• Idea Sets: curriculum levels or other intellectual currency in the system  
• “Intellectual capital formation”: learning |
| Selected initial or baseline condition variables | Each of the levels can be assessed or measured at a given time point |
| Selected interactions, feedback loops, and rules | Within classroom and within school interactions between teachers and students; the interaction patterns between and within groups and levels of agents; the premise of reform is that coordinated changes in several levels of agents will change organizational practices |
| Sample collective properties | Adaptive behavior of a system (including a collective capacity to withstand and circumvent significant change); relational trust as a driver for reform |

with many more components (Table 20.2). We offer examples within each category and elaborate briefly on the last category.

Two sample properties illustrate the potential usefulness of a complexity metaphor for the analysis of educational systems and the potential usefulness of models that extend beyond additive approaches to variable alignment. The weared encomium of school personnel about the “next big thing,” that “this too shall pass,” is nicely summarized by Labaree’s (1999) observation of the “genius” for “incorporating curriculum change without fundamental reorganization.” The adaptive propensity of complex systems to absorb and respond to a modified context may be one transcendent reason why school systems withstand so many “makeover” attempts.

A second emergent phenomenon is social trust. Recent work by Bryk, Schneider, and Kochanek (2000) suggests that relational trust resident within a school system effectively predicts readiness and capacity for system reform. Phenomena such as social relational trust and adaptivity of systems to resist change (and the example of system intelligence below) are examples of layers of a system that reside outside yet emerge from the level of operational (traditional reform) activities.

Models for Educational System Change

These are the sort of mappings that may frame whether complexity theory proves to be a promising approach to research of educational system transformation. The ROLE research program explicitly seeks to support investigations that make productive connections between complex systems theory and educational system reform. Research in and implementation of system reform have proved promising in making this connection, in particular by focusing on multilevel feedback loops. A number of investigators have pursued similar lines of inquiry in system reform. Gomez (1998) developed a “preliminary theory of system reform” after seven years of directing a statewide system reform in the commonwealth of Puerto Rico. The fact that the theory was developed after 7 years and that Gomez labeled it preliminary are both
telling indicators of how challenging it is to develop system change theories that effectively model and predict educational system behavior.

The theory schematizes feedback loops across levels of the commonwealth’s educational system, including students, teachers, school administrators, parents, vendors, assessment systems, universities, and government education authorities. Gomez (1998) organized functionalities and dysfunctions of the entire system’s behaviors based on bottlenecks and flowthroughs of the feedback loops, examining the system as a participant in it. Hirsch (1998) attempted to start mapping educational system components to complex system simulation modeling. Weisbender and Takemoto (1999), similarly involved in a systemic initiative (the Los Angeles Systemic Initiative), developed a theory for urban educational system “intelligence,” attempting to model system adaptivity and capacity to learn based on the robustness of feedback loops across multiple nodes and levels of the educational system. Under this model, system intelligence is another example of a collective system property that emerges but also drives complex interactions at constituent levels of activity.

Gomez (1998) and Weisbender and Takemoto (1999) entered system theory building inductively, as leaders of existing system reform efforts. By contrast, Confrey (1998) took a somewhat different entry point of building a theory for educational system reform from general system theories and observations across the NSF’s national portfolio of systemic reforms and then testing the theory in the reform efforts of the Austin (Texas) Independent School District (AISD). Her model also relies heavily on exploration of, intervention in, and mediation of the feedback loops across agents of the educational system. From the NSF’s standpoint, the responsibility of the Puerto Rico and Los Angeles investigators is successful execution of the reform efforts that they are charged with leading. In the Austin example, the investigator did not have responsibility for leading the reform effort, but rather for conducting what she called “implementation research,” the formulation of a tested model for system reform. These differences allow for a qualitative analysis of different research methods and points of entry and leverage in large-scale educational change.

Notice that the language for issues in system change is quite different from the discussion of research on teaching and learning. Each of these domains of interest has its own layers of analysis. In following the premise of Quadrant 3 as the core driver of the program, however, the advance we seek in Quadrant 4—a deeper capacity to catalyze system-level transformations in teaching and learning—requires a deep research base and knowledge about teaching and learning and approaches for improving them. Quadrant 4 advances derive their contribution to the national mathematics education enterprise only insofar as these Quadrant 3 components of system transformation represent research-based improvements in practice. In fact, the components of educational practice lend themselves not only to macro level or system level analysis, but also to smaller grained analysis. Virtually any Quadrant 3 question has analog questions in a research thread across the quadrant continuum, both toward Quadrant 4 and toward Quadrant 2.

Cross-Continuum Research Threads

We use group (or collaborative) problem solving in mathematics instruction as an example of a cross-continuum research thread. Mathematics education researchers use group problem solving to stimulate and elicit mathematical model formation, arguing that rich model formation is less likely to occur when individuals solve problems by themselves. Advocates of this pedagogical device see it as a reasonable and sensible instructional strategy. Others, however, cite it as a strategy emblematic of the ills of mathematics instruction in the United States. Those in the latter group suggest that collaborative problem solving is part of an approach by which teachers functionally
abdicate their professional responsibility to impart knowledge and leave the task to students to manage on their own.

Deeper currents underlie this particular controversy, with important philosophical and public policy differences. Framed as a question, one version is posed here. Is it realistic to sustain expectations of success in mathematics by all students, or should we accept that the only way to secure common achievement levels across student populations is if the achievement levels are diluted? Put more simply, should we advocate equity over educational excellence? Or is it a false choice?

In the eyes of a number of policymakers who argue that excellence and equity are not mutually attainable, the idea that some youngsters will inevitably, and significantly, outperform others in mathematics indicts collaborative problem solving as a “least common denominator” strategy. However, collaborative problem solving has been shown to be one of the pedagogical strategies that may account for important differences in overall mathematics learning between Japanese and American students. That is, it may be a strategy that increments, not decrements, aggregate learning performance.

In short, the use of group or collaborative problem solving is one of many issues from which stark, and consequential, differences at policy levels lend themselves to arbitration by effective and continuously updated research. Consequently, we consider this topic as one that offers a thread of research possibilities that cuts across the type of continuum that ROLE represents. For example:

- Use of the group problem-solving strategy in classroom practice may be considered a learning management/pedagogical issue. Research on curriculum models that mediate collaborative problem-solving strategies, within various constraints including, for example, time and cost, can shed light on their overall effectiveness, and, moreover, on the specific mechanisms that may make them more or less useful in practice (Quadrant 3).

- Research in content-specific pedagogy may suggest that significant teacher retraining in mathematics and mathematical pedagogical content knowledge is critical for a school to execute a school improvement plan that adopts group problem-solving strategies in mathematics, general management issues aside (Quadrant 3).

- At the school or at the district level, large investments in teacher professional development in mathematics may be shaped by research in professional development. Research that asks, for example, whether immersed mathematics exposure in teacher enhancement is essential to building a robust mathematics program of which the use of collaborative learning is but a feature (Quadrant 3, Quadrant 4).

- Immersing teachers in a mathematically and pedagogically rigorous professional development can be prohibitively or at least dauntingly expensive. Research on transformative change in educational systems can help elucidate how such immersive professional development can best leverage transformative change within a school system. Questions of this sort can be also be contrasted with other potent or impactive drivers, such as high-stakes accountability measures (Quadrant 4).

- Comparative national studies such as those sponsored by the International Evaluation Association and the Office of Economic Cooperation and Development can provide evidence that some cultures are organized to produce common outcomes of high achievement (excellence with equity). Social science research can provide explanations of cultural resolutions that allow or inhibit societies from organizing themselves to support high performance for the vast majority of school children (Quadrant 4).

From the research program’s perspective, each of these issues is related to all of the others in a profoundly consequential way. Notice that whereas the sample questions
noted above move outward from the student, the questions are as challenging when moving toward interior processes, a sampling of which includes study of the social, affective, and cognitive systems that interact with each other and with the supporting context of the collaborative problem solving (Quadrant 2); the multiple motivational, attentional, representational, and memory subsystem levels of analysis (Quadrant 2); and the biological and neurophysiological systems that underlie the psychological systems, keeping in mind that these systems shape, and are shaped by, the learning events that are in turn driven by the instructional strategy and the context in which that strategy is implemented (Quadrant 1).

Some may label the latter inward movement as increasingly “reductionist,” but we do not use this term applied to a learning of sciences in any pejorative sense. Byrnes and Fox (1998) and others (e.g., Schunk, 1998; Stanovich, 1998) suggested that analysis of educational and learning processes by cognitive neuroscience be used as an important to building dialogue. In turn, this dialogue can shape research on learning and education while building a “bilingualism” or “multilingualism” among research communities. The result one can expect is a community of researchers who consider mapping constructs and processes in various fields one to another.

Neural processes are at play, of course, during mathematical learning. Recent research (Butterworth, 1999; Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999; Spelke, 2000) suggests that brain function study may be capable of identifying and differentiating between brain activity during different types of mathematical processes (although functional studies are not yet able to probe brain activity specifically during group mathematics problem solving). In summary, it is safe to say that any mathematics learning episode may be considered from many levels of analysis, from the level of brain activity to how the episode relates to policy mandates or cultural patterns. Each level of analysis from the most micro to the most macro, including the eight sampled here, features a different terrain, a different granularity, and different functional contours. All levels, and particularly the intersection of the levels, provide the rich basis for an empirically derived science of learning that can support improved pedagogical practice and increases in student performance.

THE METAPHOR: THE RESEARCH PROGRAM AS A COMPLEX SYSTEM

The “research thread” on collaborative learning suggested above illustrates only one topic across quadrants. In the ROLE research program, we expect many multilevel threads of this type to emerge, and we expect that the full portfolio of projects will begin to affect and reshape the context of each other across levels. We believe that ROLE or programs like it advantage their communities by keeping the developments along these threads in the view of others laboring at other levels of the same research thread.

The term or construct of a “research thread” in itself is somewhat counterintuitive in the sense that researchers at different levels of a research thread would be unlikely to describe their work using the same language, set of problems, or issues. Consequently, said researchers would not necessarily interact with each other, or even read each other’s work. The notion of a research thread, then, especially one that spans the quadrant structure, emerges from a landscape view of the research system. Explicitly recognizing or highlighting these research threads, or potential threads, is part of a strategy to generate connectivity and coherence within a portfolio of projects. This requires normal scholarly interactions through vehicles such as journal articles and conferences, but additionally interactions that are intentionally charged and amplified via the organizational efforts of a funding agency or other research-organizing entity.
We argue that if the interactions between projects and research threads engender sufficiently robust connections and synergies across and within levels, then self-organizing behaviors in the portfolio should emerge; that is, the program portfolio and interactions begin to exhibit the characteristic of a complex system (Hamilton, 2000b). These self-organizing behaviors may be reasonably described by complex systems theory. In the context of this chapter, we ask, “How should a research field be organized to derive some of the benefits of a complex system?”

In Table 20.3 we identify some of the initial conditions, variables, and potential dynamics that we envision in such a research program system. Please note that the entries represent suggested mappings to a simplified outline of a complex system, offered in the previous section. Our interest is stimulating further discussion on the use of complexity as a tool for understanding the behavior and potential for the research program, and we thus expect other, more elaborate mappings and formulations.

The research program agents (research projects, research threads) and at least one type of agent levels (quadrant variables) are fairly transparent, as should be most initial or baseline conditions of the system. In this proposed view of the portfolio (see Table 20.3) of research and research threads as components of a complex system, we consider funding sources (e.g., NSF) as exogenous variables. A broader and more comprehensive treatment of complexity as a conceptual tool for organizing research programs may well treat funding sources as additional agents within the system.

The intensity of interactions between projects and project threads is an issue that the agency needs to consider with a fresh perspective. During the first 6 years of its existence, for example, the NSF’s Division for Research, Evaluation and Communication did not host research investigator meetings for the entire group it supports. It organized few forums through which investigators could develop landscape views of the portfolio or priorities of the funding agency. Feedback from the agency to investigators has typically been on an individual project rather than on a research portfolio basis. Funding officers conduct site visits; they review reports and meet with investigators. In so doing, these officers generally reinforce individual “agent” activities or research and not the collective behavior of the system of projects and research threads. The funding agency has provided little opportunity to encourage or to reinforce a sense of strategic connectedness among its grantees.

The limited agency-supported feedback patterns do little to overcome the perception that education research is a “rural hamlet” rather than “urban center” enterprise (Labaree, 1998), with research pockets distributed over a wide terrain of difficult-to-replicate venues. We hope that recent efforts within the NSF to develop more intense portfolio communication and feedback systems proves to be a positive development and catalyst to generate more robust and connected research threads. Recent efforts include steps by the agency to analyze and identify trends in research methods and findings and to communicate those to the portfolio, and to organize more thematic workshops and to provide venues such as an annual meeting for investigators, by which the cross-continuum landscape of research becomes more visible and consequential to the investigator teams that the agency supports. Ball and Lampert (1999) discussed the “underdeveloped nature of discourse about practice”; we believe that a system view of research, with mathematics learning and teaching as the driver for the system, can help to propel that discourse and help to build within the system a “deep knowledge community” (Lagemann, 1999).

Ball and Lampert (1999) then described some of the approaches behavior that bridging research on practice requires by posing a series of questions:

Some elements of the discourse that we need for investigation of practice transcend particular disciplinary tools, language, structure and syntax. . . . We might ask “How might a borderland” be created within a disciplined and multivocal discourse of practice
### TABLE 20.3
Research Portfolio Functioning as a Complex System

<table>
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<tr>
<th>System agents</th>
<th>Two types of agents: (a) individual research projects in the portfolio and (b) research project threads. In the example of the ROLE program (National Science Foundation, 2000), the portfolio will hold approximately 100 research projects at maturity, spanning the quadrants. Perhaps 40% will fall primarily into one quadrant, 40% will span two quadrants, and 20% may span three quadrants. An appropriate balance of within- and across-quadrant activities is sought.</th>
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<td>Levels</td>
<td>One type of level is associated with the quadrants: the size of the researched phenomenon may range from the neuronal or cellular dimension to the size of the brain to the size of a learning environment to the geographic dimensions of a system. The research thread example involving collaborative problem solving gives examples of projects across the quadrant levels. Another level of the system is time to use. What is the horizon for application of the research to practice? Design experiments or technology research and development may have a relatively short-term horizon; cognitive studies may build foundational understandings that have little impact on teaching and learning for many years. A third type of level is attraction strength of system agents, i.e., of individual research projects and research project threads. Do research projects have a gravitational pull based on a combination of intellectual force, usefulness, tractability, and significance that generates other projects or that helps generate cross-continuum threads?</td>
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<td>Selected initial or baseline condition variables</td>
<td>Research programs do not generally begin “from scratch” but rather may be understood in terms of a trajectory from a given time point, with variables such as the intellectual vibrancy of research community and the community’s human and social capital; quality and scope of merit review by the agency in seeding the future work of the portfolio; and the intensity of interactions between projects and project threads within and across levels of the system.</td>
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<tr>
<td>Interactions, feedback loops, and rules</td>
<td>The most common types of feedback loops involve the generation of findings within one agent or research project in the system, findings that inform, reinforce or converge with, or counterindicate efforts by other agents.</td>
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<tr>
<td>System patterns</td>
<td>The potential for a vibrant system includes the emergence of more robust and connected research threads across the quadrants. We also believe that deeper models for integration of education research and practice will emerge and that the research enterprise will exert a more visible role in a more dynamic and responsive educational enterprise. Patterns that may correspond to the notion of system levels transcending constituent levels include those appearing in the prior discussion of educational systems (adaptivity, social trust, system intelligence.) Dietz (2000) used the construct of social (intellectual) capital to describe a collective property of the research system generated by another National Science Foundation program, the Experimental Program to Stimulate Competitive Research.</td>
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be developed? What would an epistemology of practice look like that could involve people from different worlds and perspectives but would also provide language for a focus on practice, grounded in particular structures? What would such an epistemology of practice entail? It would embody a respect for the complexity of practice. It would seek to illuminate, not solve, its intricacies. (p. 396)

It is our hope that the complex system metaphor and our efforts to support complexity in organizing our agency response to organizational need presented here begin the process of illumination noted earlier. This illumination of practice from the vantage of research, we argue, may prove integral to more effective and higher order and nonadditive solutions to the challenge of impacting and informing practice, self-organizing solutions that transcend traditional bureaucratic or linear approaches to research-to-practice transfer.

Although the complexity metaphor has its limitations, it serves to pull together initial thoughts on program organization that support a broad picture perspective of the field of research in which our individual investigators are involved. Moreover, it provides a coherency for conversation about the value, direction and needs of the field, over and above those of specific projects. We invite this debate and look forward to vigorous discussion and commentary.

REFERENCES


At face value, there are two kinds of time: physical time and inner time. The first is the linear sequence of moments given by the clock we live by, and the other is what we live in. Both are valid as sources of facts and of scientific investigation. The first gives rise to well-developed physical theories; the other, to human temporality, centered on the present and manifesting as a threefold unity of the just-past and the about-to-occur. . . . Perceived temporality is not simply isomorphic to linear time.

—Adapted from Francisco J. Varela (1999a)

1. Time(s) in the Didactics of Mathematics: A Methodological Challenge

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When the focus of a research study is on the processes of classroom activity, time plays an essential role. Consider the following:
Every experiment must occur within school hours, taking periods, weekly schedules, and holidays into account.

Classroom processes, either individual or social, develop over time.

Observation and research is carried out over time.

The above are three instances of “physical time,” the linear sequence of minutes measured by the clock (Varela, 1999a). This physical (external) time, which the observer can read on a clock, is emphasized by the timeline where different media (e.g., video, text, graphics, and audio files) are synchronized, in recent videopapers (i.e., electronic hypertexts which contain embedded video footages with digital subtitling; Carraher, Nemirovsky, Di Mattia, Lara-Meloy, & Earnest, 1999). Yet, Varela calls our attention to the existence of an “inner time,” which gives rise to human temporality, centered on the present and manifest as a threefold unity of the “just-past” and the “about-to-occur.” This inner time is primarily individual and unconscious, although its features may be inferred from external clues (linguistic expressions, gestures, metaphors). Moreover, it may be partially shaped by outside factors (e.g., a teacher), so that the learner becomes conscious of the possibility of molding inner time in the process of problem solving.

The main purpose of this chapter is to show that

1. both physical and inner time are relevant in the mathematics education research and
2. a more refined understanding of time is needed, which requires the introduction of several theoretical constructs related to human temporality and introduces methodological problems concerning the relationships between them.

In the first part of this chapter, we begin by offering examples of a variety of ways to look at time, both physical and inner. To facilitate the discussion, our attention will be focused mostly on single episodes of class sessions that may last from a few minutes to an hour (e.g., a problem-solving session or a classroom discussion). Actually, it is the most common choice in the research literature, yet we will see that this choice is limiting; to grasp the fleeting moment of the learning process in a class, one must consider a wide range of time segments and their mutual and complex relationships, which are created by pupils’ actions, interactions, and thoughts, as well as by the teacher’s (present and past) interventions. To take advantage of an event that is taking place right now, we must switch back and forth from what we tentatively call “the past” to “the future” of our pupils. It is in this construct that the first intriguing methodological problems are rooted, in particular, how do we describe time segments and, specifically, how do we capture such inner phenomena?

The first part of this chapter introduces the reader to the phenomenology of time in the classroom, providing several examples. Each example focuses on a specific aspect of time, which generally encompass the previous aspects as well. The reader will be introduced gradually to the various features of time in the classroom, culminating in a complete picture of our approach. More specifically, Example 1 describes features of external time in the classroom, namely, the didactic memory described by Brousseau and Centeno (1991). In Example 2, the streams and rhythms of a teacher-orchestrated class discussion are considered. This shows another aspect of time in the classroom, namely, the different speeds of several streams of discussion, hence the different “dilations” and “contractions” of various themes.

External time is only one side of the coin. Examples 3 through 7 introduce the different aspects of pupils’ inner time. Inner time can be observed in the written and oral productions of pupils who solve problems (alone or interacting in small groups of 2 to 3 people). Inner time is described as an essential environment, which allows subjects to pace their mental processes. As a typical illustration, in problem-solving activities
and more generally in mathematical activities, subjects connect, dilate, contract, and splice portions of their past experience to predict future productions or to interpret present activities.

Their language productions and functions (written, oral, verbal, and nonverbal) are the major tools through which one can become aware of such performances. Typically, the production of narratives (Nemirovsky, 1996) as well as some metaphoric (Lakoff & Núñez, 2000) or generative action (Radford, 2000) functions are deeply rooted in a subject’s time dynamic. In fact, language is a crucial tool through which pupils, possibly with the support of teachers, elaborate their daily experience toward more sophisticated behaviors: from expressions describing everyday life to sentences that organize experienced relationships into causal, final, hypothetical entities. Through teacher coaching, students’ inner times in these processes are structured according to a double polarity: the past, that is, lived experiences recalled by memory; and the future, namely, the space of anticipation and volition (for a philosophical approach this discussion, see Tooley, 1997). Such mechanisms are described in our examples.

In Examples 3 and 4, the inner times of a single subject are described through his (written) productions while solving problems and elaborating conjectures. In Examples 5 and 6, inner times of more subjects working on the same problem are scrutinized through the analysis of their observed interaction; the examples show two extreme cases of relationships among the inner times of given subjects: In the former, times are diverging, whereas in the latter, they converge and eventually synchronize. In fact, an important issue in time phenomena is synchronization (e.g., between the times of two subjects but also in different rhythms of the same subject (Varela, 1999b). This last issue is fundamental, and it is an object of our current research, although it is not discussed in this paper (for information, see Arzarello, 2000).

In the final example, the teacher influences the inner times of her students. In fact, in Example 7, she organizes a didactic engineering to interpret students’ processes and force them to expand their present time toward their past experience and future explorations. The example shows the relevance of making explicit students’ inner times in the classroom, which allows the teacher to support them in a more flexible manner and, consequently, to improve their thinking processes. The examples are aimed at giving the reader a taste of this new research.

The second part of this chapter is devoted to some refinements of the methodology. The issue of using complementary methods and different grains of observation is discussed, in fact, a variable such as time is complex and cannot be easily described, but only looking at it from a given point of view. Its features with regard to teaching embrace long-term periods (weeks, months), as well brief episodes (microseconds); hence, very different approaches are required within a complex methodological frame. Some problems related to this are described.

### 2. FROM EXTERNAL TO INNER TIME: A GALLERY OF EXAMPLES

**Example 1: The Didactic Memory**

Our first example approaches the relationship between individual events in the classroom and the long-term process of mathematics teaching and learning. Consider the typical situation of a short verbal exchange between teacher and pupils. Even when the exchange is short, many hidden facts may affect one’s interpretation of what is taking place. From a teacher’s trivial comment, an implicit reference to a shared past (if any) or to a possibly shared future might be conveyed, resulting in a positive or negative effect—or perhaps no effect at all. This is, in short, the kind of phenomena
Brousseau and Centeno (1991) studied via the theoretical construct of the teacher’s “didactic memory.” They tried to answer questions such as the following: “Does the ‘didactic memory’ play an important role in teaching? Does it make things easier or more complex? Are there some facts that must be remembered by teachers? And some that must be forgotten? Are there some mathematical concepts for which the ‘didactic memory’ is more important?” (p. 170)

Although the authors’ elaboration takes place within a specific paradigm, most of their findings can be extended.

The authors’ working definition of didactic memory is the following:

The teacher’s memory is that which leads him or her to modify his or her decisions, drawing on the shared school past with pupils, yet without changing the system of decisions. The “didactic” nature of this “memory” depends on the fact that the modified decisions concern the relationships between the pupils (each pupil) and knowledge (the teacher’s own knowledge or the knowledge to be taught) in general or a piece of knowledge in particular. It means that the teacher does not only consider the state of the pupils because of their past. Actually, he or she is supposed to know it because of assessment. The fact is that he or she may mobilize, use, or evoke with them some classroom facts that are not objects of teaching, yet are important for learning. (Brousseau & Centeno, 1991, p. 172 our translation)

In the article, these authors reconstruct some examples “taken from classroom observations, that evidence the influence of either the teacher’s memory or its absence on the pupils’ activity. Sometimes the teacher remembers something that happened with the pupils and evokes it later to produce a didactic effect. The effect is not always positive. Sometimes we can see the effects of the absence of memory. . . . The examples are not on the same plane; some of them happen in the same lesson, others within a longer process” (pp. 173–174).

Among the examples are the following:

1. **An example in which no evocation is possible**: The teacher is a newcomer in the classroom, and, according to her, “she always lacks a reference to the classroom’s past.”

2. **An example in which the evocation is positive**: It allows one to correct a mistake. The pupils are solving an arithmetic exercise about fractions, and the teacher evokes one of the basic situations used to introduce fractions in that classroom. In so doing, he offers the pupils the possibility of overcoming mistakes by giving the proper meaning to those fractions.

3. **An example in which the evocation is negative**: It provokes an automatized answer, one without student understanding. The pupils are constructing a complex figure that is similar to one the teacher has given. When one of the pupil hints at proportional models, the teacher shifts to proportional numbers, inhibiting all the other attempts and focusing attention on the well-known task of finding proportional numbers.

4. **An example in which the teacher intentionally changes the status of the piece of knowledge**: In a classroom where the pupils have explicitly observed that a figure is a square, the teacher says that they are expected to verify and to prove it. Or the teacher selects among different solutions the ones that better fit the expected development of the network of mathematical concepts (Hanna & Jahnke, 1993, call this last phenomenon “appeal to the future”).

In all the above cases (except the first, where no memory exists), the teachers use, consciously or not, something that refers to a part of the shared experience of the classroom to direct the teaching-learning process. In other words, he or she locates
the present activity into the flow of classroom life, between the shared past and the potentially shared future. This goal may be realized by means of communicative strategies that refer to implicit rules of classroom communication. The teacher may evoke, for example, a standard format of a task of proportionality (to fill a table), or the teacher may present explicitly the leading thread of the teaching experiment in which the specific classroom episode gains meaning.

A researcher observing the flow of a classroom episode may be puzzled if he or she doesn’t know the shared past and the potentially shared future. In fact, he or she may be unable to interpret what is happening. This issue is especially relevant in long-term studies in which the students are expected to appropriate, over the long run, a structured piece of knowledge through a sequence of classroom activities. The resulting structure of this knowledge is usually expressed in a timeless way, without any reference to the time-ordered process of appropriation. Think, for instance, of a definition or of the statement of a theorem, which comes after a long exploration of problem situations. Chevallard (1985) studied this phenomenon as the opposition between chronogenesis and topogenesis, the former being related to the student progress through the sequence of problems and the latter to the syntactical organization of the piece of knowledge within the network of mathematical concepts.

In any long-term teaching experiment in which a piece of mathematical knowledge has to be learned, it is difficult to avoid such opposition. Hence, when the focus of a research study is on one single step only, the sense of the whole process might be lost. The theoretical construct of didactic memory (developed by Brousseau & Centeno, 1991, within the French theory of situations) is one of the tools used to model these complex phenomena in which single observable events in short classroom episodes must be related to long-term processes that last weeks, months, or even years. In the elaboration of this construct, terms such as past and future are intermingled with the present of the classroom episode to be observed. In this case, the past, the future, and the present may be described by referring to the observer’s clock—they are moments of the linear sequence of the physical time. Locating events in a time line is only the first tool used to start the analysis of data from the mathematics classroom. Other researchers have expanded this line of research (for a recent review, see Leutenegger, 2000) to study temporal phenomena in standard classrooms.

In the following example, we see other theoretical constructs introduced in the research literature to interpret what is happening in the long-term process of building mathematics objects.

**Example 2. Streams and Rhythms of Discussion**

Let us consider a class discussion orchestrated by the teacher. Such discussions, studied by Bartolini Bussi (1998), are conceived, according to a Vygotskian and a Bachtinian perspective, as “polyphonies of articulated voices on a mathematical object, that is one of the objects–motives of the teaching–learning activity”; yet what follows might be applied, with slight changes, to analyze any episode of classroom interaction. Let us consider the following example (Bartolini Bussi, 1992, p. 141):

The pupils (5th graders) were discussing about negative numbers. The whole of the discussion lasted 91 minutes with 535 interventions. The transcript was analyzed to detect the most relevant features of the teacher’s role, as a moderator of interaction and as a mediator of mathematical meaning as well. In the analysis of the transcript, it was possible to distinguish nearly 50 episodes, a new episode being determined by a change in the content (new problem, new bit of information) or in the form of discourse (e.g., explaining one’s own reasoning; remembering; summarizing; generalizing, and so on). The episodes were classified into 10 areas, according to the focus, of the discussion, as seen in Table 21.1.
TABLE 21.1  
The Turbulent Flow of a Discussion on Negative Numbers

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1. Everyday aspects:
   **Column 1**: contextualized images (e.g., measures of temperature, sea level, before Christ years, debts, and so on)
2. Geometric images, related to number line:
   **Column 2**: focus on symmetry around zero
   **Column 3**: focus on order of integers
3. Algorithmic aspects:
   - **Column 4**: addition
   - **Column 5**: subtraction
   - **Column 6**: multiplication

4. Algebraic aspects:
   - **Column 7**: number sets and inclusion relationships
   - **Column 8**: properties of numbers and operations

5. Geometrical aspects (analytic geometry):
   - **Column 9**: numbers and geometry
   - **Column 10**: from number line to cartesian plane

Having organized the analysis into a table allowed one to distinguish several interesting things such as (details of analysis are omitted):

1. whether a new episode is started by the teacher or by a pupil,
2. whether there are jumps or smooth transitions between foci of the discussion,
3. whether there is a general shift toward a shared focus, and so on.

In the table, there is a missing, yet very important, column on the left: the time line of discussion that allows one to keep note of the time ordering of the emergence of new foci and, if more details were given, also of the different speeds of the episodes (from very fast episodes with pressing and overlapping voices—e.g., column 4—to slow episodes in which silent thinking was interspersed with talk—e.g., columns 1 and 7).

The visual impact of the table is strong: Several subdiscussions or streams of discussion are going on in parallel with interlacements, and then coming back to the main discussion (details in Bartolini Bussi, 1992, p. 141 and ff.).

If we look at the structure of this discussion, as visualized by Table 21.1, we see that it is quite different from the linear structure of a standard teacher-led lesson in which the teacher’s questions and feedback to pupils’ answers are strongly organized toward the intended goal of the teacher.

Seeger (1998) has suggested two interesting metaphors, which may be applied to contrast this kind of discussion with teacher-led conversations: turbulent flow/laminar flow. In a still-unpublished paper (Seeger, submitted), he recalls the distinction between the turbulent and the laminar flow, studied in fluid dynamics:

Imagine you open a water tap. At first, the water jet is smooth, perfectly round and transparent. It forms a kind of tube that seems to stand still. Now, imagine you open the tap further, at a certain point this image changes abruptly. The previously smooth water jet is becoming restless and is forming a couple of strands that give the impression of a muscle. And if you open the tap even further other similar structures will appear. But the beautiful regularity of the first image of the water jet is destroyed and disorder seems to rule.

The physicist calls the first kind of water jet a laminar flow, while the second is called a turbulent flow. From the point of view of the physicist “order” and “disorder” are not assigned in the same way as in our naive perception. From the point of view of the physicist appearances betray truth. It is precisely in the turbulent flow that a higher degree of order rules. Whereas in the laminar flow the movement of the individual water molecules follows a random statistical law, in the turbulent flow the molecules are grouped together in streams, which permit an increase in the amount of water flowing. (Seeger, in press)

Seeger suggested that in teacher-led conversation, the flow of arguments seems laminar, while in teacher-orchestrated discussions, the flow is turbulent (in the same way, a good lecturer shifts from one focus to another in a carefully organized way). Before a detailed analysis with emphasis on educational outcomes, one might be led to
describe the former with adjectives such as “ordered,” “well organized,” “goal oriented,” and the latter with “chaotic,” “turbulent,” and “badly organized.” Yet, as Seeger concluded, the laminar pattern of discourse signals only teacher dominance. Whatever the value system we attribute to the two contrasted structures, it is clear that their comparison and contrast depends on a meaningful organization and clustering of utterances according to the time line.

The first two examples we have presented refer only to physical time as read on the observer’s clock. Yet, as Varela (1999a) reminded us, we also have inner time (i.e., perceived temporality) that is not isomorphic to physical time. In the following examples, we use a different lens to analyze time in individual inner processes as it may be inferred through the observation of external clues. Time here is considered as an intellectual construction, to describe, order, and analyze the flux of external events (Guala & Boero, 1999).

**Example 3: The Freedom of Inner Times in Problem Solving**

Our third example introduces the issue of inner time. The discussion will be enlarged and deepened with other examples that allow us to introduce specifications related to inner time.

We are considering the individual solution of a problem of representation in which the process of measuring and the use of a standard artifact for measuring are in play (adapted from Boero & Scali, 1996; Guala & Boero, 1999). We consider a written protocol. Pupils are accustomed to writing reports about their strategies; hence, writing is not supposed to block the development of the solution strategy. This protocol is, in a sense, static in contrast to the lively record of a “thinking-aloud” process or of a dialogue between pupils (as with transcripts of dialogues from will show other examples). Nonetheless, we claim that focusing on inner time helps us enter more deeply into the solving process and to better describe why this pupil is such a good problem solver.

To discuss the protocol, we borrow terminology from the article by Guala and Boero (1999), who listed several examples of inner time, such as the following:

1. the “time of past experience” traced by means of memory tracks and supported by the ordering activity the mind performs on these tracks; how tracing works (quickly or slowly, in detail or not) can depend on the involuntary perception of the quality of an event and/or on the voluntary reconstruction of past experience.
2. “contemporaneity time”... evaluated by the subject (with estimation slowed down by wishes and accelerated by fears), who may graft on it virtual displacements toward the future or past.
3. “exploration time” in open-ended tasks requiring the subject to find and interlink suitable operations; time projections can be realized from the past onward or in the future and then toward the past.
4. “synchronous connection time” as a perception of coordinated functioning of the component of the real or virtual system under scrutiny.

Different inner times are evoked in this written protocol, which reconstructs the process of problem solving. The pupil is involved in classroom activity (the physical time of the observer’s clock, governed by its own rules). His processes can be influenced and disturbed by external events that he does not control (e.g., the entrance of somebody in the classroom, a noise produced by a classmate). If he is involved in the solution, he might not even be aware of the passing time and acknowledge
Second-grade Classroom
All the pupils know how to use the graded ruler (15 cm) to measure lengths shorter than the ruler.

**PROBLEM.** In the nearby classroom, there is a small plant 23 cm high. How can you draw it with your ruler?

<table>
<thead>
<tr>
<th>Stefano’s (good problem solver) written protocol</th>
<th>The inner times of Stefano</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>If it has to be 23 cm,</em></td>
<td>The <strong>contemporaneity (present)</strong> time: the text of the problem</td>
</tr>
<tr>
<td><em>it means that it is 8 cm more than the ruler,</em></td>
<td>The <strong>time of past experience</strong>: recalling arithmetic problems</td>
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<td><em>that is 15 cm.</em></td>
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<tr>
<td><em>We have already drawn things 15 cm high</em></td>
<td>The <strong>time of past experience:</strong> the solution of a similar (easier) problem</td>
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<tr>
<td><em>and others that are less than 15 cm.</em></td>
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<tr>
<td><em>So I can draw a piece 15 cm long</em></td>
<td>The <strong>exploration (future)</strong> time: the plan to follow the same procedure</td>
</tr>
<tr>
<td><em>and</em></td>
<td>The <strong>exploration (future perfect)</strong> time: the mental experiment and the realization that the old procedure does not work</td>
</tr>
<tr>
<td><em>add to the same line adding a piece 8 cm long</em></td>
<td>The <strong>exploration (future)</strong> time: the design of a modified strategy that incorporates the old one (the past).</td>
</tr>
<tr>
<td><em>and then color everything.</em></td>
<td>The <strong>synchronous connection</strong> time: the intuition of the functioning of the components of the system (a piece 15 cm long and a piece 8 cm long)</td>
</tr>
<tr>
<td>The drawing is produced.</td>
<td>The <strong>contemporaneity (present)</strong> time the production of the drawing that completes the task</td>
</tr>
</tbody>
</table>

Stefano’s inner times are not simple mirrors of this physical time, however. He—a good problem solver—is able to control and mold his inner times according to the needs of an effective process of solution. He can reverse, contract, enlarge, cut, and displace the present time in the past or in the future, realizing goal-oriented strategies (remembering, planning, foreseeing), while language supports these mental acts. In fact, the borders between one inner time and another are fuzzy; shifts (and links) from one time to another are realized by the mastery of language, which allows one to connect in the same utterance expressions related to different inner times. Successful remedial strategies for bad problem solvers have been studied and implemented by Guala and Boero (1999): most of them actually rely on the systematic, coordinated, and repeated teacher intervention in controlling and molding pupils’ inner times through language until self-regulation takes place.
The traces of different inner times in the same protocol (and in the same utterance) reminds one of the theoretical construct of a “mathematical narrative” (Nemirovsky, 1996). Nemirovsky introduced a mathematical narrative as a form of discourse that is (a) embedded in a sequence that is meant to reflect temporal order and (b) articulated with mathematical symbols.

In Stefano’s protocol, one can see the skeleton of his thought process through a sequence of episodes, which determine a sequence of different inner times. The protocol cannot be reduced to this chronology of scattered events, however. It must be interpreted as a whole (the problem solution) in which shifts between different parallel courses of time occur. As we have seen, the time of the narration does not correspond to the times of the events that are narrated.

Example 4: The Freedom of Inner Times in Producing a Conjecture and the Disappearance of Time in Proof

The fourth example, we again discuss an example of a mathematical narrative. The freedom of inner times, observed in problem solving, are seen in producing a conjecture. We follow the transformation of this narrative into the form of a mathematical proof and observe the corresponding transformation of inner times. The following protocol concerns the production of a conjecture in paper-and-pencil geometry (Garuti, personal communication):

Figure 21.2 is a reconstruction of Yari’s drawing. If we interpret the “always” in Yari’s initial statement as meaning “for any point of the plane” or “for any given two circles,” the statement is generally wrong. Actually, there are only two points that solve the problem for two given circles and a given radius; even if the radius of the circle to be drawn is left free, only the points on one of the branches of a suitable hyperbola with foci in the centers H and K of the two given circles can be assumed as centers of the circle to be drawn. “Always” might be interpreted differently, however:

1. to recall the format of the question;
2. to claim that it is possible for any pair of circles; or
3. to emphasize the detimed (namely, time is not any longer present in this part of Yari’s protocol) feature of the solving method (confirmed by the conditional form of the conjecture, if . . . then, given a few lines below).

Actually, Yari does solve the problem correctly in this particular case. The most interesting part of the protocol, however, is the description and the justification of the method used to solve the problem (Fig. 21.3).

When Yari explains his reasoning, he reconstructs an interesting process that shows a flexible use of time: He pretends to have solved the problem (“by trial and error,” as he says) and searches for the procedure that might allow the production of the solution and the construction of an argument to justify it on the basis of what is already known.

When Yari explains his reasoning, he describes the process as developed in time:

1. The mental experiment: I have imagined the three circles already drawn, tangent to each other,
2. The external representation of the mental image: I have drawn them by trial and error.
3. The sudden perception of the triangle: Their centers described a triangle, and I have measured the sides.
4. The measuring: I have noticed that two sides measured 7 and 6 cm respectively.
5. The link to the known property of tangent circles: I have understood that 7 was the sum of 4 + 3 and 6 the sum of 4 + 2.
8th-grade Classroom
The students are familiar with modeling geared, toothed wheels by means of tangent circles. A new construction problem is given.

**PROBLEM.** We have often drawn tangent circles. With the center in any given point of the plane, is it possible to draw a circle with a radius of 4 cm, tangent to the two circles drawn at right. Is it always possible? Never? Sometimes? When? Write your conjecture as a statement. Carefully explain your reasoning.

![Fig. 21.1. The problem of tangent circles.](image)

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<table>
<thead>
<tr>
<th>Yari's (low achiever) written protocol</th>
<th>Yari’s inner times</th>
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<tr>
<td>The text of the problem (above).</td>
<td>The <strong>time of past experience</strong>: the teacher reminds students well-known problems; the teacher uses a jargon (<em>statement</em>) that reminds students of the past experiences.</td>
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<td><strong>STATEMENT:</strong> It is always possible to draw a circle with a 4-cm radius tangent to the two given circles. “To draw a circle with 4-cm radius and tangent to other two circles, one must measure the radii of the two circles already drawn (in this case, 3 and 2 cm), then add to each of them the length of the radius of the circle to be drawn (in this case, 4 cm; hence, they become 7 and 6, because $3 + 4 = 7$ and $2 + 4 = 6$).</td>
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<tr>
<td>If the point $O$, i.e., where I must plant the compass, is 7 cm from the center of the circle with the 3-cm radius and 6 cm far from the center of the circle with the 2-cm radius, then the circle that will be drawn will be exactly tangent to the other two circles.</td>
<td>The <strong>contemporaneity (present) time</strong>: the text of the problem, with an immediate answer to the problem. The <strong>exploration time</strong>: the procedure to be applied by somebody else. <em>Intermingled with</em> The <strong>time of past experience</strong>: the procedure is based on the procedure used in another problem.</td>
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<td><strong>CAREFULLY EXPLAIN YOUR REASONING</strong></td>
<td>A detimed expression (if . . . then), where the procedure is justified by recalling a justification already given in the past (the <strong>time of past experience</strong>).</td>
</tr>
<tr>
<td>To solve it, I have imagined the three circles already drawn, tangent to each other.</td>
<td>The <strong>time of past experience</strong>: a narrative with a reconstruction of the process that requires continuous shifts in time.</td>
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<tr>
<td>I have drawn them by trial and error.</td>
<td>The <strong>exploration time</strong>: imagining the solution.</td>
</tr>
<tr>
<td>Their centers describe a triangle, and I have measured the sides.</td>
<td>The <strong>contemporaneity time (in the past)</strong>: drawing by trial and error.</td>
</tr>
<tr>
<td>I have noticed that two sides measured 7 and 6 cm respectively</td>
<td>The <strong>time of past experience</strong>: recognize a known figure, the triangle, and recall a known action, measuring.</td>
</tr>
<tr>
<td>I have understood that 7 was the sum of $4 + 3$ and 6 the sum of $4 + 2$.</td>
<td>The <strong>contemporaneity time (in the past)</strong>: action.</td>
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<td>The <strong>time of past experience</strong>: recalling an arithmetical result.</td>
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Yet, what is interesting is that in the written presentation, he has given first the method obtained by reversing the process, expressed in impersonal form:

1. Measuring: *One must measure the radii of the two circles already drawn*
2. Adding: *Then add to each of them the length of the radius of the circle to be drawn*
3. Finding a point with the needed distance from the other two
4. Justifying the result (a tangent circle) by recalling a known property of tangent circles

Even if Yari’s arguments are not the standard form of mathematical proof, his solving process echoes the well-known method of *analysis and synthesis* used and theorized.
(Otte & Panza, 1997; Polya, 1954) in the classical age to produce the solution of the construction problems. In analysis, the problem is supposed to be solved: The configuration is transformed according to the allowed geometric rules until a known figure is obtained, then (synthesis) the process is reversed.

Yari molds time as in the analysis and synthesis method, allowing himself the freedom to reverse it in the report of the process and transforming the time order of the process into a logical chain. Anecdotically, we can say that, the whole classroom assumed this way of attacking the problem under the teacher’s guide. It was a powerful tool to attack difficult construction problems and to construct the proof of the validity of the method.

The transformation of sense (from a time sequence to a conditional detimed chain) has been studied in the genesis of the conditionality of statements (i.e., the fact that statements of theorems are implicitly or explicitly shaped according to the “if A then B” clause) by Boero, Garuti, and Lemut (1999). The authors have reported an impressive set of findings on the mental dynamics that underlie the production of conditional statements. They have found four models of the “process of generation of conditionality,” described as follows (adapted from Boero et al., 1999): “PGC1: A time section is created in a dynamic exploration of the problem situation: during the exploration one identifies a configuration inside which B happens, then the analysis of that configuration suggests the condition A, hence “if A then B” (p. 140).

To better understand this model, we refer to a typical exploration conducted with a dynamic geometry software in which two sets of configurations are “separated” by a special one. Consider the following example:

A triangle ABC is given with the three heights and their point of intersection, H. Dragging one of the points A, B, C, it is evident that H moves. We can see H going from inside to outside and vice versa. If we fix the photogram when H is exactly crossing the border, we have a special configuration that suggests the following statements:

- If ABC is acute-angled H is inside.
- If ABC is right-angled H coincides with A or B or C.
- In the other cases H is outside the triangle.

The second model is the following (Boero et al., 1999, p. 141): “PGC2: A regularity in B is noticed in a given situation: then a condition A, present in the original situation, such
as B may fail to happen if A is not satisfied, is identified, by exploration performed through a transformation of the situation.”

The third model is the following (Boero et al., 1999, p. 141): “PGC3: A synthesis and generalization process is started with the exploration of a meaningful sample of conveniently generated examples.”

This is a typical example of production of conjectures in the context of natural numbers: pupils make explorations using a few concrete numbers and then formulate a general conjecture (for meaningful examples with young pupils, see Bartolini Bussi et al., 1999).

The fourth model is the following (Boero et al., 1999, pp. 141–142): “PGC4: The regularity found in a particular generated case puts into action expansive research of a general rule whose particular starting case was an example.”

In all these models, we see evidence of a process, that, started in time and with a well-controlled procedure, eventually becomes detimed. What makes the study of these processes meaningful from a didactic perspective is the fact that very often “the same mental exploration which leads to the conjecture is re-started by the student with entirely different functions during the proving process” (Boero et al., 1999, p. 142).

In each model, inner times, revealed by the language, play a relevant function (several examples of the narratives produced by students are in Boero article). In general, if we focus inner times as variables to be observed from clues of the protocol that remain, we see shifts from one to another that pave the way to the complete disappearance of time in the logical chain. A good problem solver (and a good mathematician who is able to accomplish the analysis-synthesis process) is able to master the freedom of these inner times and the continuous shift from one another. In the examples by Stefano and Yari, the students reconstruct this complex narrative structure in the written protocol. When this does not happen, it is the teacher’s job to nurture the process via systematic and intentional intervention.

The following pair of protocols will introduce the reader into the living world of “thinking aloud” with dialogues that occurred between partners working together to solve a complex problem.

Example 5. The Interaction Between Two Students: Different Inner Times

Dynamic software such as Cabri, The Geometer’s Sketchpad, GEX, or Cinderella among other things, fulfill of an ancient dream: the visible display of infinite mental experiments in the geometric environment. Dynamic software offers the possibility of dragging the points that, in the procedure of construction, have at least one degree of freedom and of seeing the transformation induced by dragging on the whole figure, provided that the geometric relationships given in the construction are maintained. Dragging introduces a new set of facilities in geometric activity that can be used in both phases: the exploration phase (when a conjecture is to be produced) and the validation phase (when a proof is to be constructed).

Two of the authors of this chapter are conducting a set of research studies about the mental processes involved with the use of dragging in solving open geometry problems in the Cabri environment (see Arzarello, 2000; Arzarello, Micheletti, Olivero, Paola, & Robutti, 1998a; Arzarello, Gallino, Micheletti, Olivero, Paola, & Robutti, 1998b; Olivero & Robutti, 2001). In this chapter, we report only preliminary results with the aim of showing different relationships between two or more inner times. In the articles cited here, we studied how students use dragging to explore a situation,
understand what it means, and make conjectures, with the final goal of justifying the conjectures with proof. During this process, students use different dragging modalities that reveal corresponding cognitive strategies in the same way as languages, gestures, or narratives show the students’ approach to a problem and their inner times. Dragging can be observed in the present, yet when it is an operation oriented toward a goal, it is realized according to a plan, and hence projected to the future. Moreover, when a known configuration is noticed, the past is in play. Particularly, dragging can show contemporaneity of time, for example, when students use it to explore a configuration without particular aims and in a casual way, such as a random motion (wandering dragging). It can also show the time of past experience, when it is used to control a construction, to see if it maintains the properties of the figure, during the movement (dragging test). In terms of past experience, dragging can, with different modalities, be used to check a conjecture, a property, a hypothesis, a thesis, or even a proof. In this case, the use of dragging is not random, but aimed at testing the statement. Another use of dragging is in the exploration of future time, when the pupils move a construction toward a particular configuration they want to see, and this is the case of a guided dragging.

In this example, the situation is more complex than in the previous ones because of two new aspects:

1. There are two subjects working together to solve the problem, so we must consider their individual approaches to the problem, together with their interaction.
2. There is an environment in which the students are working that is not simply static or passive but that offers a dynamic approach to the problem, together with continuous feedback to the students, consequent to their actions within it.

For the first point, we can analyze the individual contributions of each student’s argument, which reflects their respective inner times, then we can determine if there is an intersection of the two inner times, that is, do they proceed in the same direction (past or future) or not (empty set)? With regard to the second point, which is linked to the first, we have found (Arzarello et al., 1998b; Olivero & Robutti, 2001) that Cabri contributes more to the richness of students’ production in terms of conjectures, validations, and proofs than do other environments such as paper and pencil. Moreover, students’ use of dragging can provide information about their inner times, as described above. Here we have a 2-hour class session, in which the students (the 12th-grade of a scientific school with five classes of mathematics per week) have to solve a geometry open problem with Cabri. They are asked to explore the problem, make conjectures, and prove some of them. These students are able to use Cabri because they have used it some times before in similar problem-solving situations. The students work in pairs, and we observed two pairs. Here we have the protocol of one pair.

Two students (Anna and Matteo, 12th grade) are working together. They are creating the construction and exploring it by dragging.

PROBLEM. A quadrilateral $ABCD$ is given. On its sides, construct four squares externally. Find the centers of the squares and link them by line segments to obtain the quadrilateral $EFGH$.

1. After reading the text, explore the situation and produce your conjecture (according to the “if . . . then . . .” clause) about the configurations of $EFGH$ while varying $ABCD$.
2. Prove some of your conjectures.
At this point in the protocol, Matteo has the mouse, and Anna is speaking to him. The students are looking at the evolution of the figure caused by dragging.

The two students are working together with Cabri to solve a geometry open problem, and, even if their initial approach is similar (they noticed a particular configuration on the screen), their later approaches are quite different because Matteo moves at an empirical level, whereas Anna takes a theoretical approach.

In the Phase I, the two students use wandering dragging to investigate the situation as their first approach to the problem, then they use a guided dragging to move the figure until point C belongs to the side GF. Both the students concentrate on the moving figure and are projected toward the future: Their inner times are exploration times.

In Phase II, the students, by reasoning on a static figure, begin to go in different directions: While Anna tries to explain the situation at a theoretical level, Matteo moves at a more empirical one and wants to drag the figure to observe it once more. Instead, Anna needs a static figure to analyze it to construct a justification of the statement (four equal squares). Matteo is in the exploration time, whereas Anna’s inner time is the synchronous connection time.

In the Phase III, the divergence of the two inner times is total, because Anna wants to trace a circle for the vertices of the quadrilateral; Matteo doesn’t agree with her, and Anna needs to draw the circle on paper because she is in a contemporaneity time in which Cabri is not useful (*It takes too long*). The girl has to realize immediately her ideas to control them and the paper is the best environment to do this.

<table>
<thead>
<tr>
<th>An excerpt from the transcript of small-group discussion</th>
<th>Anna’s inner time</th>
<th>Matteo’s inner time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anna: <em>If</em> B = C <em>it becomes a triangle: against the hypotheses.</em></td>
<td>The <strong>contemporaneity time:</strong> Anna notices a particular case during the dragging.</td>
<td></td>
</tr>
</tbody>
</table>
Anna: *Try with C on GF.*

### The exploration (future) time:
Anna notices that a particular configuration occurs if the vertex C of ABCD belongs to the side GF of EFGH. A new exploration with dragging is begun.

Matteo: *We have an equal triangle.*

### The exploration time:
A particular case is noticed: when C belongs to GF, four equal triangles appear.

Matteo: *Wow, they are four squares!*

### The time of past experience:
Matteo sees four squares on the sides of ABCD.

The dragging is too fast and is therefore stopped. The students are reasoning on a static figure.

### Phase II

![Figure 21.6. Conjectures about quadrilaterals.](image)

**Anna:** Yes, it’s the midpoint.

### The synchronous connection time:
For the symmetry of the figure, the point C (and its similar) is the midpoint of GF.

**Matteo:** Why?

### The contemporaneity time:
Matteo is in the present time, and he wants to understand the reason for this particular configuration.
Anna: Let’s reason. We know that ABCD is a quadrilateral.

The **time of past experience**: now the students reason on the static figure (movement is not useful at this moment). Anna recalls the initial situation of a quadrilateral.

<table>
<thead>
<tr>
<th>Anna: FB is a half diagonal; BC and FC are congruent.</th>
<th>The <strong>synchronous connection time</strong>: Anna makes some connections between the elements of the figure.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Anna: If FC = CG, they are all congruent triangles.</td>
<td>The <strong>time of past experience</strong>: Anna recalls an observation she made before and formulates it in a conditional form.</td>
</tr>
<tr>
<td>Anna: These are diagonals; they form an angle of 90°—a right angle.</td>
<td>The <strong>synchronous connection time</strong>: Anna passes from an empirical level to a more theoretical one, because she uses the relationships as a logical consequence.</td>
</tr>
</tbody>
</table>

Matteo: It’s a square because the sides are equal.

The **time of past experience**: Matteo uses the relationships of the previous construction.

---

### Phase III

Matteo starts dragging.

The **exploration (future) time**: Matteo needs to move the figure, to investigate further.

Anna stops him.

The **contemporaneity time**: Anna needs to stop the figure because she is at a more theoretical level.

Anna: I need a perfect figure.

The **synchronous connection time**: She needs another figure to control an idea she has.

Anna: Let’s draw a circle: If it meets at all 4 points, it is a square.

The **synchronous connection time**: Anna explains the reasons for the construction of a circle.
Example 6: Similar Inner Times

Now consider a more complex example concerning the interaction of three students in a class situation in which students try to solve a geometry problem with a dynamic software (Cabri, version 1.7) by means of a suitable task. In this example, the situation is similar to the previous one: three subjects are trying to solve the problem, working together and in the Cabri environment.

In a 2-hour class session, the students (the same grade and school as in the previous example, but in a different classroom) must solve a geometry open problem with Cabri. They are asked to explore the situation, make conjectures, and prove some of them. These students are familiar with Cabri because they have used it many times before in similar problem solving situations. The students work in two or threes, and in this class session, we observe one group of three. Following is their protocol.

Then the process goes on (more details analysed in Arzarello, 2000). At the end the students succeed in finding (and stating) a correct conjecture about the degeneration of A’B’C’D’ if the initial quadrilateral ABCD is inscribed in a circle.

What is interesting for the purpose of this chapter is not to analyze the process until the very end, but to emphasize that to interpret the small-group process, some constructs related to the students’ inner times are interesting for our analysis, similar to the one of Example 5. Whereas in Example 5 we noticed a different approach to the problem by the two students who were moving in different inner times, in this example, the three inner times are often of the same kind, and the shifts from one to another are synchronous. This issue will be reconsidered in the following example.

In Examples 3 through 6, we have focused on students (i.e., the inner times of learners). When the teacher’s perspective comes into view, his or her set of inner times also comes important. Nonetheless, there is asymmetry in the roles of the teacher and of the students: The teacher, when interacting in the classroom, is not as engaged in his or her own solution of the problem (which should have been analyzed in advance) but is engaged in the interpretation of students’ processes to be able to interact with them appropriately. To reach this goal, the teacher tries to interfere with the inner times of students. In the following example (that reconsiders an example presented in detail in Mariotti’s chapter of this volume), we shall show which kinds of communicative strategies are available for this task.
Example 6. A group of three students is working with a computer, with paper and pencil at their disposal.

**PROBLEM.** You are given a quadrilateral $ABCD$. Construct the perpendicular bisectors of its sides: $a$ of $AB$, $b$ of $BC$, $c$ of $CD$, $d$ of $DA$. $A'$ is the intersection point of $a$ and $b$, $B'$ of $b$ and $c$, $C'$ of $c$ and $d$, $D'$ of $a$ and $d$. Investigate how $A'B'C'D'$ changes in relation to $ABCD$. Prove your conjectures. The figure produced by the students is at right.

![Figure 21.7. The perpendicular bisectors.](image)

An excerpt from the transcription of the small-group work of three students (E, M, and V)

**Phase I**

36. E: *Now what?* (E has the mouse.)
37. M: *One must see how it varies, as the external quadrilateral changes* [ABCD].

... 

41. V: *I think that not ... try moving ... the figure.* [E drags randomly point D] ... ’cause ... move this one [V indicates point B and E drags it randomly] ... it seemed to me that you had put the ... you know ... the function of the segment, that you can create without doing the points ... it seemed that you had not shot this one [A'] ... do you understand? ...

... 

48. V: *But if you already do it colored ... you get a small colored point.* [E colours the quadrilateral $A'B'C'D'$ and drags the point D]

49. E. *And let's try perhaps ... let's try to see what happens with regular external quadrilaterals.*

50 M: *I don't know ... let's start with a square, so that we see ...* [E drags B, C, D up to get a rectangle]

51. V: *Properties of the perpendicular bisectors?*

**Phase I**

Here we can notice that the student M is in the **exploration (future) time**, in which she tries to connect the shape of the first quadrilateral (ABCD) to the shape of the second one ($A'B'C'D'$) and this exploration is made randomly (by the student E), with a wandering dragging. The students’ inner times are here synchronised (the other students too are in the **exploration time**), because they are all looking for a configuration of the second quadrilateral related to a correspondent one of the first quadrilateral.

At #49 the student E gets a suggestion to the group: the idea of exploring the situation with regular quadrilateral, and the other students accept. In the meanwhile they try to remember the properties of the perpendicular bisectors (**time of past experience**). Using both measures and dragging, they explore the configuration of a square (#56), and stop the
52. E: The perpendicular bisector...how was it?
53. M: Hence...the perpendicular bisector passes through the midpoint...
54. E: It is perpendicular!...
55. M: ok!
56. E: Well in the square, in the...
[She measures the sides of ABCD, then drags randomly first point C then point B]
57. E: No, I was wondering...that...I was wondering!
[E stops the dragging with Fig. 21.9 aside]
58. E: No, that is...it degenerates into a point...is it?...if they are parallel...that is, if the sides are perpendicular...[she drags B]...

Phase II
59. M:...we are looking for...
60. E: I mean if the opposite sides are parallel [she continues dragging B], those [the perpendicular bisectors] are perpendicular. And up to this...Isn’t it?...and if they are equal the midpoint is on the same line. [She drags C till ABCD becomes a rectangle].
61. M: ok...so?
[E drags A randomly, then D; in the end she goes back to the original figure]
62. E: Please, tell me something!
[E drags B, D, C, A systematically]
63. V: What are you doing? Are you moving randomly?
64. E: No, I was wondering if I could construct a figure...

Phase III
65. V: Listen to me, please; let’s try thinking...just a moment...’cause of that we have done before...to finish the discourse, when it degenerates into a point, that...have I misunderstood or we have not explained it?
66. M: well, practically she is saying: since the properties of the perpendicular bisector are perpendicularity and the distance from a point...if...the different segments are parallel, then since they are perpendicular...Moreover if two of...like in a square for example, the midpoint must belong to the same straight line.
67. V: yes
[In the meanwhile E has dragged the points A,B,C,D in order to get a parallelogram]

movement when they observe a particularity: the quadrilateral $A'B'C'D'$ collapses into a point. Here (#58) the students are in a contemporaneity time: they were looking at a quadrilateral, and now instead of it there is a point.

Phase II
In this phase the students’ inner times are not so similar as the previous phase, in fact the student E is dragging the figure and in the meanwhile explaining why the perpendicular bisectors are (two by two) on the same lines, recalling some properties (time of past experience). She needs a feedback from her mates, but it doesn’t come, because they are in different inner times: V is in a contemporaneity time, at a perceptual level (she is looking at the movement of the figure), while M is in a exploration (future) time, waiting for the consequences of E’s reasoning.

Phase III
In this phase the students are coming back to a similarity in their inner times, because now the student V tries to reconstruct the reasoning, while M repeats the proof made by E, and E moves the figure to illustrate a more general case: the parallelogram. The students are in a synchronous...
68. E: I am doing a parallelogram... the sides are parallel, aren’t they? in the parallelogram. Hence also the perpendicular bisectors are parallel, isn’t it? They are parallel two by two.

69. V&M: yes

70. E: So also the segments A′B′ and C′D′ are parallel.

71. V: Hence it maintains... no, nothing!

[E drags the points B, C, D till she gets a rectangle]

72. E: hence the square has been proved... degenerate...

73. V: Hence if... when... and hmm, yes, that is natural, because when there are two... the two sides of the external one... the two sides parallel two by two, it is natural... that is it should always be that the perpendicular bisectors are...

74. M: it is so.

75. E: Because they are parallel... they are perpendicular to two parallel lines.

76. V... they are parallel...

77. E: let’s move the point very slowly to see what changes [she drags the point C for a while]. Now they are not any longer parallel, hence... these two [d, b] are not any longer parallel... sure, it is logic... and not for these two [a, c]... That is what we have said up now. [E drags slowly point C along line BC and back]

connection time, they are relating the particular case of the square, analysed above, to the general case of a parallelogram (#68: They are parallel two by two. #72: hence the square has been proved... degenerate...). In their very contracted expressive way, not so simple to understand by a person out of the group, the students communicate their ideas and understand the others’ ones.
Example 7. Time in Teacher-Student Interactions

The following example is discussed in detail by Mariotti (this volume). The teacher in this example intentionally uses the “interpretation game” and the “prediction game” to foster the shift from the procedure of making a Cabri construction to the justification of the procedure itself. Here we briefly describe the situation (for details, see the Mariotti chapter).

A 9th-grade class of 19 students starts creating a Cabri construction. The students, sitting in pairs in the computer room, have been given the following task: Construct a segment on the screen. Construct a square that has the segment as one of its sides.

Each pair has solved the task. The following day, the solutions are compared (i.e., each group presents its product). When an interesting product is presented on the master computer, the teacher describes what was done. The construction is reconsidered step by step, by means of the “history” command. At a certain point, the teacher interrupts the procedure and asks students to detect the reasons that brought their classmates to use those commands. The teacher’s intervention (which starts an interpretation game) aims to provoke the first shift from the procedure to a justification of the procedure.

If we look at this from the perspective of this chapter (i.e., the analysis of times involved), we see that the teacher is asking the students to reconstruct something that happened in the past; yet, to do so, the students must predict an answer, hence shifting themselves into the future. In this case, the “time of past experience” and the “exploration time” are strictly intermingled in the “contemporaneity time”: this does not happen in a spontaneous way but is forced by the teacher’s specific requests. Something similar happens in the prediction game, when students are asked to predict what might have been the next step in reaching the goal of the construction. Their hypotheses will be tested on a completed construction, yet are produced by looking at the future. This continuous wandering between the “time of past experience” and the “exploration time,” which is supposed to be typical of good problem solvers, is in this case forced by the teacher via specific linguistic activity during on-the-spot interaction, which draws on the availability of the history command. This command can reconstruct the steps taken to produce the construction. It is possible to go back and forth through the sequence of the construction steps and to repeat this process as many times as it needed.

This final example concerns the beginning of students’ processes as facilitated by the teacher during whole class discussion. We have seen that to understand what is happening, it is useful to introduce a time reference: The teacher is intentionally controlling the contemporaneity of time, as experienced by the students involved in the discussion, to force them to move between the “time of past experience” and the “exploration time.” Yet if we do not have this teacher’s general goal in mind, we might be disappointed by the discussion’s apparent confusion. The nonlinearity of classroom discussions, although orchestrated by the teacher, is evident to external observers but is, typical of this kind of classroom interaction, as Example 2 has clearly shown.

A Tentative Conclusion

We have collected several examples of theoretical constructs related to time taken from the literature on innovative teaching experiments in the classroom. Our aim in this paper was not to present a complete list (if there is one) of these theoretical constructs, but rather to discuss in which sense they are useful in innovative research paradigms and which kinds of methodological problems they produce in the development of research. We hope we have addressed the first issue in discussing the examples. Now we are addressing the second: contrasting the naive position of those who think
that (and act as if) the “universal” time of physics—as measured by the clock of the observer—would be enough to describe educational processes. While noting the problem is necessary, it is not enough to solve it. In the following section, we point at some methodological difficulties.

3. A SECOND-ORDER APPROACH

Preliminary Definitions

Starting from the examples described previously, different theoretical constructs related to time have been described, such as

- **Didactic memory** (Example 1): mobilization, use, or evocation of classroom facts, that are not objects of teaching, yet are important for learning
- **Stream of discussion** (Example 2): a time-ordered cluster of episodes within a discussion, a new episode being determined by a change in focus (new problem, new bit of information) or in the form of discourse (e.g., explaining one’s own reasoning; remembering; summarizing; generalizing, and so on)
- **Mathematical narrative** (Examples 3 and 4): individual recapitulation (usually in a linguistic form) of past experiences concerning mathematical activity by means of temporal clauses, which include at least two times—the time of the narration and the time of the events that are narrated
- **Inner times**: the time components of the mental processes involved in mathematical problem solving (Examples 3, 4, 5 and 6)

As the discussion of the examples has shown, at least in a basic way, to analyze and model the long-term processes of teaching and learning, it is useful to focus on these theoretical constructs and to study their relationships with each other and with other important theoretical constructs, which we have not considered here in detail (i.e., those related to mathematics and language). For instance, patterns of discourse (Example 2) are tools to analyze the structure of a complex interaction, organized into different streams, with reference to the intended goal. We have described the laminar and turbulent flow to understand the nonlinear process of reaching that goal. In another example (Example 7), we have considered language games, that is, language activities introduced by the teacher using a precise set of rules (we have described those related to the interpretation game and to the prediction game), which have the effect of inducing students to combine in a functional way the "time of past experience" and "exploration time" in the "contemporaneity time"; in other words, language games induce students to mold their own inner times.

Borrowing terminology from logic, we consider the occurrences of the above theoretical time constructs as cases of **first-order variables** and the relationships either between them or with other variables as **second-order variables**. We have argued elsewhere that a relevant research study for innovation in mathematics classroom deals with **second-order variables** (see also Arzarello & Bartolini Bussi, 1998). Surely the idea is not new. Vygotsky (1934) warned about the mistake made in classical psychology in which researchers attempted to gather knowledge by studying separate components of something. On this subject he suggested a useful metaphor: It makes no sense to study hydrogen and oxygen separately if one wishes to study water because they do not have the properties of water.

In Examples 1 through 7, we have addressed either the relationship between different theoretical constructs or between these constructs and the processes of mathematical problem solving or the construction of mathematical meaning. The
relevance of this second-order approach is confirmed by our analysis of time. Let us conduct a “mental experiment” on a research study concerning the relationships between the didactic memory and mathematical narratives produced by the pupils over a period of some months. The aim of this experiment is to model the processes of the teaching-learning activity (a case is presented in Bartolini Bussi et al., 1999). We can study didactic memory, taking note of the teacher’s ways of interacting in the classroom. The group’s comments are organized along a time line, governed by the observer’s clock. Nonetheless, the teacher’s strategies often revert to narrative forms in which parallel times are intertwined. Taking part in the activity that the teacher has structured (as Example 2 also shows), the pupils eventually follow a path according to their own inner times, through which they develop their experience and construct a meaning for the phenomenological work they perceive and act upon. There is a deep and complex interaction between these different times, which is crucial for learning.

This fact is confirmed by the comparison between what happens in Examples 5 and 6. If we compare the two protocols, we can make the following hypothesis: If the inner times of the students working in little groups are very similar or the same, then the interaction among them is more productive than if the operate at different inner times. This issue of synchronization is a major one in research about inner and external times. Synchronization reveals intriguing information when analyzing the individual performances of a single subject because of the different rhythms of inner time within the same person (see Varela, 1999b). The coordination of rhythms also becomes crucial when subjects interact with software (see the discussion in Arzarello, 2000). It is beyond the scope of this paper to enter into this subject, but much experimental research is needed in which different methods from different subjects must be used to analyze the rhythms of students engaged in teaching experiments.

**Methodological Problems**

The above discussion emphasizes phenomena related to time that could be studied in research for innovation in mathematics education, but it also draws attention to several unsolved methodological problems. The first problem we face is that it is impossible to have only one strain of analysis as far as time is concerned; the didactic memory develops over weeks or months, whereas some critical individual events may take place in microseconds (see the analysis of Dehane, 2000). The order of time is not even maintained if we look at students’ inner time: the time may be suddenly stopped, reversed, contracted, enlarged, cut, and displaced in the past or in the future according to the specific goal (e.g., remembering, planning, foreseeing) to accomplish dynamic mental experiments. In other words, a moment in time for the didactic memory has a very different size compared with a moment for mental processes. The difference in size makes it theoretically impossible to consider both in the same analysis.

Moreover, whereas the former is linear and allows a researcher to line up the events in a time line, the latter may be better described as a network in which different inner times are connected in different ways. Last but not least, the system of the variables must comprise mathematical and linguistic components. A meaningful example of a second-order analysis is given by Arzarello (2000): a short period of small group interaction (the same episode as our Example 6) is analyzed by considering the relationships between different sets of variables. Yet as far as we know, little attention (if any) has been given in the literature on mathematics education to the variables concerning time (no mention to such problems is given in Teppo’s 1998, monograph), perhaps because researchers believe that time is not important, because the final,
accepted product of mathematical activity usually has a detimed structure. We hope
to have shown, however, that this detimed structure is only a final form of a process
that not only develops over time but also needs, in order to be analyzed, the use of
different kinds of variables related to time. The difficulty of a teacher’s job is the man-
agement of different inner times, within a shared time of the classroom, with the goal
of constructing products that are detimed.

Open Questions

In innovative research paradigms, the fine-grain analysis of short-term processes is
coordinated with the analysis of long-term processes. How do researchers manage the
many problems raised in research for innovation? They use a variety of methods that
are not universal but that are functional to the particular research questions. For in-
stance, when the focus is on the teacher’s role and the aim is the modeling of the social
processes in the mathematics classroom, Leont’ev’s activity theory (1978) is one of the
possible solutions because, on the one hand, it comes from the Vygotskian tradition in
which the cultural (and asymmetrical) role of the teacher is emphasized and, on the
other hand, it offers a system of tools to relate the global level of activity developed
over time to the individual operations (i.e., communicative strategies) realized by the
teacher on the spot. Activity theory makes explicit (and functional) the distinction
among different levels of phenomena (Activity—Action—Operation), which occur in
different but interrelated periods of time. An example is the analysis of whole class
discussions from Bartolini Bussi (1998; see also the example given by Mariotti in this
volume).

Yet when the focus is on the individual processes in problem solving, activity the-
ory might not be the best solution. The analysis of pupils’ inner times given in our
Examples 4 through 6 is carried out without any reference to the teacher’s role and
to the instruments of Leont’ev. The “true life” observation of a long-term teaching
experiment is quite different from the observation of processes that take place in a
psychology laboratory (or in a surrogate psychology laboratory, that is, a classroom
in which only short-term processes are analyzed without any reference to the global
school activity), where the influence of the external events can be made as unobtrusive
as possible and where the different time variables can be separated. In a research study
for innovation in the mathematics classroom, the phenomena related to time must be
studied to produce scientific knowledge about them. This knowledge is important not
only for theoretical purposes, but also to synchronize the inner times of the students
and to push them in the same direction. Are we satisfied with the juxtaposition of
methods we have discussed here? Are other solutions possible? One might say that
it is too soon to seek coordination among methods of analysis because of the insuf-
ficient development of theoretical reflection. The classical methods of observation,
which have been carefully investigated in the past, might be meaningful only for the
first-order variables and not for second-order variables, which require different but
simultaneous observation methods.

We have a challenging situation: The study of variables related to time is supposed
to be relevant, but there are many methodological problems. Maybe the search for
methodological purity must be given up for a time, at least in innovative research
paradigms. Nonetheless, we feel compelled to continue as we witness developing
research trends that overcome the distinction between theoretical and pragmatic re-
evance (Bishop, 1998; Sierpinska, 1993), producing results with a sound theoretical
basis that have a profound impact on the practice of teaching. Only when the book was
in press, the authors read the paper by Lemke (2000), where similar methodological
issues are addressed from a broader perspective.
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The Problematic Relationship Between Theory and Practice

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In the field of education, different problems have arisen in different countries according to specific historical factors, social needs, and political choices. Such “local” conditions have led to the development of different approaches and styles to deal with these problems in the context of the cultural and teaching traditions of each country. Over time, research in mathematics education has turned the field into a discipline and generated different ideas about what it is or should be (Bieler, Scholz, Strässer, & Winkelmann, 1994; Malara, 1997; Sierpinska & Kilpatrick, 1998), because in each country, it possesses different features, as a reflection of the sociopolitical and cultural environment in which it has developed (Barra, Ferrari, Furinghetti, Malara, & Speranza, 1992; Blum et al., 1992; Douady & Mercier, 1992; Gelfman, Kholodnaya, & Cherkassov, 1997; Iwasaki, 1997; Mura, 1998; Rico & Sierra, 1994; Sowder, 1997). In particular, the following conceptions of mathematics education have arisen:

- Mathematics education as a theoretical and autonomous science, based on a conceptual system and on original methods of inquiry that are not borrowed from close disciplines and that are aimed at studying the phenomenon of mathematics teaching in its complexity and seen in its context (concepts mostly developed in France after Brousseau, 1986)
- Mathematics education as a scientific discipline including theory, development, and practice interacting with the social school system (teacher training, development of curriculum, mathematics classroom, textbooks and teaching aids, assessment) and with related fields (not only with mathematics and its history and epistemology but also with psychology, science education, sociology, etc.) and with a linking function between mathematics and society (Steiner, 1985).
- Mathematics education as applied or design science, or in general as science of practice that studies the concrete action of teaching by carrying out a mediation between pedagogy, mathematics (with its history and epistemology) and other disciplines (psychology, anthropology, sociology, etc.), from the integration of which it acquires its own uniqueness authenticity (Iwasaki, 1997; Pellerey, 1997; Speranza, 1997).
In the last case, mathematics education is not directed toward an abstract knowledge of some aspect of teaching but toward changing something in the teaching; it has an innovative character based on practical framework as a whole (Iwasaki, 1997). As stressed by Mason and Waywood (1996), one way to recognize alternative paradigms in mathematics education is to look at the products of research in relation to practitioners.¹

We should not understated this plurality of conceptions as if we were in a phase of prescience (Kuhn, 1962); on the contrary, as emphasized by Godino (1991), “the co-existence of competing schools of thought can be seen as a natural and rather ripe status in this field since it promotes the development of a variety of research strategies and the observation of the same problems from different points of view.”²

Even if different opinions exist about which conditions stand for its scientifi city, today mathematics education is progressively being acknowledged as a scientific discipline, but within the mathematics education community, it tends to be treated as a purely scientific discipline with no connection to social reality and to the most urgent needs of teachers. This is why the relationship between theory and practice is being discussed today rather than previously, when most studies concerned mathematical issues and curricula. Some earlier authoritative voices recognized this separation (Freudenthal, 1983; Kilpatrick, 1981). Many studies state that teachers and educational practitioners, who primarily aim at the improvement and renewal of teaching, are quite sceptical and disinterested in theorization (Margolinas, 1998; Silver, 1997; Verslappen, 1994). This means that new energy and resources are needed to ally these two worlds, and we must include the communication and dissemination of research results (Bishop, 1998; Lester, 1998).

In his plenary conference at ICME 7, Howson (1992) expressed something very interesting on this topic:

> I have written elsewhere of the danger that parts of “mathematics education” will detach themselves from mathematics teaching in much the same way that “philosophy of mathematics” has drifted well away from “mathematics” itself. . . . The importance of such studies is not be denied, but where does that leave the mathematics educator who wants to serve and help teachers, not just to study, count, or assess them? Perhaps it would be a useful check for all of us contributing to this congress to ask of our contribution: How will/could it help teachers, under what conditions and within what timescale? (p.)

This thought-provoking question emphasizes the fact that most studies are about teachers but not with and for teachers.

We agree entirely with these scholars and believe that research in mathematics education, especially theoretical research, finds its natural validation in practice, which means not only in the daily management of classroom activity, but also in teachers’ wholeness as living human beings. In fact, our idea of teaching acknowledges teachers as decision makers, influenced by important factors that the research should not neglect, such as knowledge, beliefs, and emotions. Within this framework, we articulate our discourse on the following points:

1. the relationship between theory and practice;
2. the perception of teaching as decision making;

¹Mason and Waywood wrote, “Some have practitioners in mind, some are oriented to the institutional and socio-political, some are oriented towards other researchers, and some are focused on the researcher’s personal development” (p. 1056).

²Godino moved from a statement by Shulman (1986a) with reference to social and human sciences and therefore to mathematics education.
3. the impact of teachers’ knowledge, beliefs, awareness, and emotions on their practice;
4. the consequences for theory (criteria of research quality—reproducibility, relevance, communicability);
5. the collaboration between teachers and researchers as the reconciliation of theory and practice; and
6. an example of reconciliation: the teacher–researcher in the Italian model for innovation research.

THE THEORY–PRACTICE RELATIONSHIP

Before analyzing this relationship, we offer definitions of the two words. Mason and Waywood (1996) undertook a deep analysis of the role of theory in mathematics education and in research. They identified various senses of the term *theory*: (a) an organized system of accepted knowledge that applies in a variety of circumstances to explain a specific set of phenomena, (b) a hypothesis or possibility, such as a concept that is not yet verified but that if is true would explain certain facts or phenomena, and (c) a belief that can guide behavior. They claimed that “what is in common in the use of the word ‘theory’ is the human enterprise of making sense, in providing answers to people’s questions about why, how, what. How that sense making arises is itself the subject of theorizing.” (p. 1056) Mason and Waywood emphasized that

Theories generated in mathematics education have various functions. They can have a descriptive function, providing a language to frame a way of seeing, and in this sense they affect an ideology. They may offer an explanation of how or why something happened, thus relating what has been observed to the past, whether through statistical correlation, cause-and-effect-analysis, influence, or co-evolving mutuality. They may attempt to predict what will happen in similar situations through stating necessary and appropriate conditions (and for this they need to specify what constitutes ‘similar’ and “situation”). They may serve to inform practice by sharpening or heightening sensitivity to notice and act in future. Which ever of these functions a theory contributes to, it comes from, belongs to, even constitutes, a *weltenschauung*, and communication between different world-views is at best problematic. (p. 1060)

In particular, they made a distinction between *foreground theory* in mathematics education, consisting of the studies aimed at locating, specifying, and refining theories about what does and can happen within and without educational institutions, and *background theory* of or about mathematics education, consisting in the unexpressed and often unconscious assumptions or beliefs underlying any act of teaching or research. This reference to background and foreground theories leads us to consider the distinction between teaching practice and research practice, and consequently to observe the theory–practice relationship in its many aspects. Many scholars (Brown & Cooney, 1991; Jaworski, 1994, 1998; Lerman, 1990, 1994; Mason 1990, 1998; Mousley, 1992) analyse this relationship with reference to teachers’ theory (their knowledge and particularly their beliefs); others consider it with reference to researchers, that is, the link between their knowledge and beliefs and the methods they use to conduct research (Burton, 1994); still others consider this relationship in mathematics and its influence on mathematics education (Vergnaud, 1998). The central problem, however, turns out to be the distance between theory—a corpus of knowledge on mathematics education in the hands of researchers—and practice—the actual teaching carried out by teachers.
A Way of Looking at Practice: Teaching as Decision Making

The multiplicity of variables involved in a pupil’s learning process, highlighted by the research in mathematics education, and the context constraints (the number of pupils in a class, the syllabuses to be followed, interaction with the colleagues and families, etc.) imply that the teaching process, too, is complex activity. In particular, we focus on an aspect of this complexity: Teachers repeatedly face situations that force them to make decisions (Carpenter, 1988; Cobb, 1988; Cooney, 1988; Crawford & Adler, 1996; Dalla Piazza, 1999; Mason, 1994a; Peterson, 1988; Shulman, 1985; Simon, 1995). These decisions involve not only finding solutions to problems that arise in the classroom but also the identification of the problems (Cooney & Krainer, 1996; Jaworski, 1998; Thompson, 1992). Therefore, teaching can be seen also as an activity of problem solving and problem posing.

Moreover, the constructivist approach to the learning of mathematics, particularly in its social context and within the context of Vygotskian principles (Lerman, 1992), has two important implications on the teaching. The first is that a teacher, as well as the pupil, is a person who has an individual interpretation of reality—and in particular of the texts, the syllabuses, and the teaching aims of his or her discipline (Arsac, Balacheff, & Mante, 1992; Carpenter, 1988; Cooney, 1994). Such an interpretation, as in the case of pupils, is not only influenced by the teachers’ knowledge but also by their beliefs and values.

The second implication is that in teaching mathematics, teachers must do much more than merely convey knowledge through the “right” words or actions (Cobb, 1988; Cobb, Yackel, & Wood, 1992; Jaworski, 1994, 1998; Pirie & Kieren, 1992; Simon & Schifter, 1991; Simon, 1995; Steffe & Kieren, 1994). They have the responsibility of creating an environment that allows pupils to build mathematical understanding, but teachers also must make hypotheses about pupils’ conceptual constructs and on possible didactic strategies to modify such constructs.

Some researchers (e.g., those who adopt a Vygotskian perspective) particularly emphasize the teacher’s role in the construction of mathematical understanding, stressing that they are a guide in the “zone of proximal development”; this role is then crucial in making decisions not only about the tasks but also in choosing the communicative strategies to be adopted in classroom interaction (Bartolini Bussi, 1998).4

These elements explain what happened in the mid-1970s: The research on teaching shifted from studies of observable phenomena such as teacher behavior to studies about the teacher’s decision processes (see the reviews by Shavelson & Stern, 1981, and Clark & Peterson, 1986). In these studies from that period, teachers are seen as

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3This is clear in the following excerpt (Carpenter, 1988, p. 190): “[Teachers] exhibit the same characteristics in solving problems of instruction that are employed by problem solvers in other contexts. Just as behaviorist analyses of problem solving proved to be inadequate to capture the complexity of the problem-solving process, viewing teachers simply as actors who exhibit certain behaviors is severely limiting. They do not blindly follow lesson plans in teachers’ manuals or prescriptions for effective teaching. They interpret them in terms of their own constructs and adapt them to fit the situation as they perceive it.”

4On teachers’ behavior, Bartolini Bussi (1998) wrote, “The teacher has to make a lot of ‘on-the-spot’ decisions in the flow of debate. The teachers’ position is similar to that of an actor in the ancient commedia dell’arte, in which improvisation on the plot played a major role: Yet improvisation was also not governed by chance but by very refined actor education; it was rather a science, based on a personal repertory of variant (which kind of jokes, which kind of provocation for the audience, which kind of reaction to some words from the audience, etc.)” (p. 23). She emphasized that one should analyze classroom discussions to give teachers a repertoire of communication strategies to be used.
“thoughtful professionals” (Shulman & Elstein, 1975) who make judgments and carry out decisions in a complex environment. Therefore, determining the decisions teachers make that influence pupils’ learning and the nature of this influence and finding out which factors influence these decisions became important to research on teaching.

Teachers’ decisions can be classified according to various criteria, the most significant of which is related to time and to the crucial moment of the interaction with the students. Such a criterion distinguishes the decisions made in the preactive phase from those made in the interactive phase and in the postactive phase (Brown & Borko, 1992; Jackson, 1968). Decisions can also be classified by typology: however, Cooney (1988) mentioned cognitive decisions (related to the content), affective decisions (related to the more interpersonal aspects of teaching), and managerial decisions (including the allocation of time). Of course, these two criteria can be combined; for example, a content decision that occurs in the preactive phase consists in deciding which issues to present and which to exclude from the instructional programme.

As Carpenter (1988) emphasized, the model of the teacher as problem solver or decision maker, shared today by most of researchers in mathematics education, suggests the need to integrate research on teaching and research on problem solving. In fact, the factors recently taken into account by the research on teaching are also relevant to research on problem solving. Researchers formerly stressed the teacher’s knowledge, whereas the evolution of research on problem solving and on learning processes—and particularly the importance given to metacognition—encouraged investigation of other aspects, such as the teachers’ beliefs, their awareness, and their emotions. The next section concentrates on these issues.

THE IMPACT OF TEACHERS’ KNOWLEDGE, BELIEFS, AWARENESS, AND EMOTIONS ON THEIR PRACTICE

As acknowledged by many research findings, teachers’ knowledge, beliefs, awareness, and emotions have a significant impact on practice. We analyze these factors individually, but this distinction is purely theoretical and artificial because in reality they always interact and intermingle.

The Teachers’ Knowledge

Although the strong influence of a teacher’s knowledge on his or her decisions is generally acknowledged, there is no agreement on what the fundamental knowledge is. Initially, the research on the teachers’ knowledge mainly concentrated on the specific knowledge of the subject matter of mathematics. The first studies that tried to demonstrate a cause–effect relationship between teachers’ subject knowledge and pupils’ learning (Eisenberg, 1977; School Mathematics Study Group, 1972) were not very successful. Such failure, however, must be analyzed because the criteria used to assess the teachers’ knowledge (number of courses completed or performance on a standardized test), as well as those to assess the pupils’ achievements (performance on standard exams), were not subtle.

Some years later, Shulman (1986b), reconsidering some of Dewey’s (1902) ideas, eventually identified seven domains of knowledge: (a) knowledge of subject matter (or content knowledge), (b) pedagogical content knowledge, (c) knowledge of other content, (c) knowledge of the curriculum, (d) knowledge of learners, (e) knowledge...
of educational aims, and (f) general pedagogical knowledge. In the context of mathematics education, particular attention was paid to content knowledge, curricular knowledge, and pedagogical content knowledge. The most innovative idea is that of pedagogical content knowledge, which integrates knowledge of content and knowledge of pedagogy.

In addition to Shulman, other researchers have stressed that the teacher’s knowledge has manifold components. Different frameworks for analyzing teachers’ knowledge were proposed (e.g., Elbaz, 1983; Peterson, 1988; Leinhardt & Greeno, 1986; Leinhardt, Putnam, Stein, & Baxter, 1991; for an analysis and a synthesis of these models, as well as for a model for research on teachers’ knowledge, see Fennema & Franke, 1992).

Within research on the influence of the teachers’ knowledge over their decisions, there are some interesting studies belonging to the project called Cognitively Guided Instruction (CGI). On planning this project Carpenter, Fennema, and Peterson intended to integrate the perspectives of cognitive and instructional science to study teachers’ pedagogical knowledge in the area of elementary arithmetic and to analyze how that knowledge influences classroom instruction and students’ learning. More precisely, as part of CGI, several studies have been conducted to determine whether knowledge about research on addition and subtraction would influence teachers’ decisions (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter, Fennema, Peterson, Chiang, & Loaf, 1989). The results of these studies, based on the research investigating young children’s learning of addition and subtraction (Carpenter & Moser, 1983), suggest that teachers’ knowledge of children’s thinking can have an important influence on teachers’ decisions and therefore on classroom learning: Teachers prepared in CGI listened to their pupils, were able to attend to individual children, and spent more time in activities involving problem solving.

By demonstrating the importance of pedagogical content knowledge, these studies suggest that teachers had access to research results. Of course, these results should not become models to imitate but rather should enable teachers to make the right decisions; as highlighted by Balacheff (1990, p. 269):

> the aim is to construct a fundamental body of knowledge about phenomena and processes related to mathematics teaching and learning. The social purpose of such an enterprise is to enable teachers themselves to design and to control the teaching-learning situation, not to reproduce ready-made processes. This knowledge should allow teachers to solve the practical problems they meet, to adapt their practice to their actual classroom.

### Teachers’ Beliefs

No matter which framework one chooses to analyze teachers’ knowledge, it is always necessary to consider teacher knowledge as a large, integrated, functioning system.

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6Content knowledge “refers to the amount and organization of knowledge per se in the mind of the teacher. ... To think properly about content knowledge requires going beyond knowledge of the facts or concepts of a domain. It requires understanding the structures of the subject matter” (Shulman, 1986b, p. 9).

7Curricular knowledge is the knowledge about instructional materials, which includes knowledge of “the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances” (Shulman, 1986b, p. 10).

8Pedagogical content knowledge includes “for the most regularly taught topics in one’s subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing the subject that make it comprehensible to others. ... [It] also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to learning” (Shulman, 1986b, p. 9).
in which each part is difficult to isolate (Fennema & Franke, 1992). In particular, it is impossible to separate teachers’ knowledge and beliefs.

Research on beliefs mainly developed in the late 1970s, simultaneously to the shift in paradigms for research on teaching, from teachers’ behavior to teachers’ thoughts and decisions (see the surveys by Thompson, 1992, and Hoyles, 1992). Today the research on teachers’ beliefs is an important field in the more general research on teaching (in addition to Thompson and Hoyles, see Pehkonen & Törner, 1996, and Krainer, Goffree, & Berger, 1999).

Teachers’ beliefs that are usually investigated in mathematics education concern two points (Thompson, 1992): (a) beliefs about mathematics and (b) beliefs about mathematics teaching and learning. This categorization includes a wide variety of inquiry fields, but other issues to be taken into account do not belong to mathematics specifically. Quite relevant are the beliefs that teachers develop about their pupils (Höfer, 1981). The study by Rosenthal and Jacobson (1966) on the so-called Pygmalion effect is a pioneer in this field, showing that teachers make themselves of pupils has a very strong influence on the pupils’ performance.

The importance of teachers’ beliefs is evident when it comes to creating new syllabuses or experimental projects. Although teachers apparently agree with the aims of a project and its features, and despite the fact that many projects are very specific as to the didactic choices to be made, it often happens that a sudden choice made by the teacher goes against the spirit of a project; in other words, the relationship between teachers’ stated conceptions (i.e., teachers’ theory) and practice turns out to be problematic. More generally, it happens that the beliefs espoused are inconsistent with practice (Furinghetti, 1997; Raymond, 1997; Nesbitt Vacc & Bright, 1999).

This mismatch between espoused beliefs and beliefs in practice, demonstrated by many studies on teachers’ beliefs (Hoyles, 1992), confirms the results of research on problem solving (Schoenfeld, 1989): The beliefs that teachers declare are in the end different from those that guide their problem solving processes and their behavior in general.

This compels us to discover which teachers’ beliefs most influence their decisions, and traditional tools such as questionnaires, interviews, and Likert scales are appropriate to detect these beliefs. Lerman (1994) maintained that context strongly influences teachers’ beliefs. The shift from one setting to another allows the appearance of factors that significantly change teachers’ actions from those they would profess or wish to apply. So it appears of little significance to examine teachers’ beliefs with an instrument, such as interviews or questionnaire completion in a laboratory, in one setting and their impact in another setting, typically the classroom.

Generally speaking, there is a strong need for social and anthropological approaches (Arsac et al., 1992; Bishop, 1998), that is, to study teachers’ beliefs in their natural context. Therefore, many studies suggest nontraditional methods, such as narratives (Brown & Cooney, 1991; Cooney, 1996; Chapman, 1997; Krainer, Goffree, & Berger, 1999). Indeed, narrative includes “tacit knowledge” underlying practice, which cannot be expressed in propositional or denotative form (Polanyi, 1958); tacit knowledge embeds teachers’ deep beliefs that influence practice. In particular, some researchers

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9The beliefs about mathematics teaching and learning include, for example, the causal attributions of failure and success (Fennema, Peterson, Carpenter, & Lubinski, 1990) and the theories of success (Zan, 1999a).

10Thompson (1984) stressed that “teachers’ conceptions are not related in a simple way to their instructional decisions and behavior. Instead, the relationship is a complex one. Many factors appear to interact with the teacher’s conceptions of mathematics and its teaching in affecting their decisions and behavior, including beliefs about teaching that are not specific to mathematics” (p. 124).
use metaphors to represent teachers’ knowledge grounded in experience and to provide coherence for teachers’ practice.

What seems necessary to measure changes in teachers’ beliefs is to study individual teachers in depth and to provide detailed analyses of their cognitive processes (Thompson, 1992). In particular, it is possible to get teachers to research their own practice “from the inside” rather than as objects to be studied (Mason, 1994b). Such studies, which are gradually becoming more common (Borasi, Fonzi, Smith, & Rose, 1999; Brown & Cooney, 1991; Cobb et al., 1992; Cooney & Krainer, 1996; Llinares & Sanchez, 1990), highlight the importance of reflection and awareness for effective changes.

**Teachers’ Awareness**

Recent research in mathematics teaching points out the need for teachers’ reflection about their own practice (Jaworski, 1994, 1998; Lerman, 1990; Mason, 1990, 1998). Jaworski (1998) used the following words to define the kind of practice that results from such reflection, that is, reflective practice: “The essence of reflective practice in teaching might be seen as the making explicit of teaching approaches and processes so that they can become the objects of critical scrutiny” (p. 7).

This notion appears to be in accord with Schön’s idea of the “reflective practitioner” (1983, 1987): Knowledge about practice grows from knowing-in-action, through reflecting-on-action to reflecting-in-action. Reflective practice mends, according to Schön, the rift between theory and practice and between practitioners and “experts.” Through reflective practice, teachers become aware of what they are doing and why; awareness is therefore the product of the process of reflection.

Mason (1998) emphasized the role of awareness in teaching. More precisely, he argued that being a real teacher involves the refinement and development of complex awareness on three levels: (a) awareness-in-action; (b) awareness of awareness-in-action, or awareness-in-discipline; (c) awareness of awareness-in-discipline, or awareness in counsel. Mason suggested that awareness-in-discipline is what constitutes the practice of an expert, but what supports effective teaching in that discipline is awareness in counsel.

The strategies Jaworski (1994) used to increase teachers’ awareness are similar to those suggested by other researchers (Garofalo, Kroll, & Lester, 1987; Schoenfeld, 1987) to develop pupils’ metacognitive skills and improve problem-solving abilities. In both cases, the subjects are continuously asked “difficult” questions about their thinking processes. Over time, the subjects begin to anticipate questions and to ask their own questions.

Awareness deals with metacognitive skills. More precisely, it deals with the first aspect of metacognition (Schoenfeld, which is the awareness of one’s abilities; 1987), the second is self-regulation or control of those abilities. The two aspects of metacognition are strictly linked in the sense that awareness of one’s resources can lead one to activate regulating processes. In fact, the influence of teachers’ increasing awareness on their control processes are clearly emphasized in most research (Borasi et al., 1999; Jaworski, 1994; Lerman, 1990; Thompson, 1984).

**Teachers’ Emotions**

Still, the shift from being aware to enacting control processes is not automatic, and it is influenced by many factors. Many studies have focused on the role of emotional factors in this context. In particular, a sense of self-efficacy and an enjoyment of learning flow from individual strategic events but eventually return to energize strategy selection and monitoring decisions, that is, executive processes (Borkowski, 1992).
In mathematics education, the importance of affect was initially emphasized in the context of problem solving (McLeod & Adams, 1989), and then, more generally, in the field of mathematics learning. However, the research on affect in mathematics education has mainly focused on beliefs and attitudes rather than on emotions (McLeod, 1992).

The most frequent approach to emotions in mathematics education, and in particular in problem solving, is borrowed from cognitivist psychologists (Mandler, 1984). This approach can help us understand how teachers’ decisions are influenced by their emotions, which the research on teaching often tends to forget. Moreover, this approach describes the model of practice that we use. According to Mandler (1984), the emotional experience is the result of a combination of cognitive analyses and physiological responses. If a sequence of actions is interrupted, or if a cognitive or perceptive discrepancy occurs between facts and expectations, the consequence is visceral arousal. The subjective experience of emotion is a combination of visceral arousal and a cognitive assessment of the experience.

Therefore, it is not the experience itself that causes emotion, but rather the interpretation that one gives to the experience. This interpretation is influenced by an individual’s beliefs. Beliefs also play an important role in perceptive or cognitive discrepancies. As McLeod (1992) suggested, Mandler’s theory is particularly interesting because among the cognitive theorists, Mandler has done the most to apply his ideas to problems in mathematics education (Mandler, 1989). Nonetheless the efforts of Ortony, Clore, and Collins (1988), although not regarding mathematics per se, are interesting too, since they attempt to categorize the various emotional responses.11

The acknowledged importance of emotional aspects in problem solving12 and decision processes suggests that we consider emotions to be relevant in the teaching process, too. Many researchers state that teachers must take into account emotional aspects in their teaching (Adams, 1989; Cobb et al., 1989; Grows & Cramer, 1989; Middleton & Spanias, 1999; Simon & Schifter, 1991; Sowder, 1989).

It is not enough to consider pupils’ emotions however; teachers’ emotions, too, play a fundamental role in the teaching–learning interaction because they influence teachers’ decisions, exactly as happens with pupils (Shulman, 1985). As a consequence to the approach described, context constraints have a double influence on teachers’ decisions. They have a direct influence on them because they are objective bonds (consider for example, factors such as the time needed to explain a topic, syllabus prescriptions, and number of pupils). These bonds, however, are also perceived and interpreted by teachers according to their aims, values, and beliefs, and this interpretation elicits emotions, which influence teachers’ decision processes. Time is a typical

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11Ortony et al. (1988) distinguished three main types of emotions, which they classify as reactions to objects, events, and agents. Emotions resulting from reactions to objects (“attraction” emotions) are all variations of the affective reactions of liking or disliking. They are influenced by subject’s attitudes and tastes (typical examples are love and hate). Affective reactions to events are variations of being pleased and displeased. These reactions arise when a person construes the consequences of an event as being desirable or undesirable and are influenced by the subject’s goals (typical emotions are joy, hope, fear). Affective reactions to agents involve approval and disapproval. They are influenced by the subject’s beliefs and values (typical emotions are pride, shame, admiration, reproach). From these three classes derive more complex emotions such as anger, in which the reaction to an unpleasant event is attached with a factor considered to be responsible for this event. In this sense, anger is more complex that other emotions (disgust, for example) because the interpretation process that gives life to it is more complex.

12In this context, the opinion expressed by Goldin (2000) is original. He suggested that affect, like language, should be seen as fundamentally representational as well as communicative. The affective states that he describes are not global attitudes or traits, but “local changing states of feeling that the solver experiences and can utilize during problem solving, to store and provide useful information, to facilitate monitoring, and evoke heuristic processes” (p. 209).
example in this respect because it influences the teachers’ decisions not only by imposing an objective constraint, but also because it arouses anxiety, which, also has a strong influence on decisions.

Even if emotional aspects are seldom the direct object of research on teaching, many studies demonstrate their importance. Arsac et al. (1992), for example, facing the issue of reproducibility of didactical situations, considered the problem of the teachers’ role in the class, when they have to follow a predefined scenario. Through two case studies, the researchers discovered two factors that hamper fidelity in reproducing the given scenario: constraints resulting from the didactical system, such as time constraints, and teacher’s conceptions about mathematics and learning. They observed that a teacher’s decisions for coping with these constraints tend to oppose the devolution of the problem situation to students. In our opinion, teachers’ behavior that diverges from the planned scenario (such as making up questions that induce answers, not writing false statements on the blackboard, bypassing the processes they considered to be too uncertain), can be derived but not directly from the context, from context-provoked emotions, in particular, from the anxiety elicited by time constraints and by the difficulty of managing uncertainty.

If we observe decisions made in the preactive, interactive, and postactive phases, we can see that most context bonds belong to the interactive phase, in which time to decide is short, and there is no possibility of pondering before making a decision. This is why teachers’ decisions are strongly influenced by emotions in this phase. In particular, in this phase the emotions connected with the interaction between teachers and pupils are very important. Salzberger-Wittenberg, Polacco, and Osborne (1983) faced this problem from a psychoanalytic point of view. They stated that in front of pupils, teachers can feel fear of factors such as criticism, hostility, or loss of control. Moreover, the authors stressed that the attitude and expectations teachers have toward pupils can also influence their perception and interpretation of pupils’ behavior, as well as their reactions to such behavior.

Here, too, awareness appears to be crucial to minimizing the consequences of this influence (Salzberger-Wittenberg et al., 1983). If we assume this point of view, the studies that consider teachers’ emotions are of particular important, especially, those that study their influence on decision processes. The aspects analyzed here (knowledge, beliefs, awareness, emotions) must be seen in their interdependent state, but also in the more general framework of values (mental, moral, and aesthetical), on which research in mathematics education still has not concentrated sufficiently (Vinner, 1997).

**CONSEQUENCES FOR THEORY**

Seeing the teacher as decision maker or problem solver, rather than as executor of procedures, has a strong influence on the theory–practice relationship, as well as on theory itself. In mathematics education, theory was born with studies that were strongly characterized by the positivist paradigm, the latter being considered a synonym of scientficity, especially in the science of education. Therefore, up to the 1970s, the predominant methodology was statistical in nature.

This approach turned out to be unsatisfactory as soon as the complexity of learning processes was acknowledged, however. Because learning is a complex activity, we have an “uncertainty principle of didactic variables” (Arzarello, 1999); being able to have all variables under control is an illusion, as Mason (1994b) noted:

> education may not be best served by continuing to employ a solely cause-and-effect perspective. ... In scientific enquiry, all factors are held as constant as is possible; in education, no factor remains stable when another is perturbed. (p. 194)
This in the end raises questions about the positivist paradigm as synonym for scientific method under discussion\(^\text{13}\) (Kilpatrick, 1993; Schoenfeld, 1994; Sierpinska & Kilpatrick, 1998).

In particular, it becomes increasingly important to acknowledge the influence of teachers’ decisions on pupils’ learning processes, as well as the complexity of this influence and of these decisions. For a long time, teachers were treated as a “constant” in classroom studies (Chapman, 1997) or when curricula were developed (Fennema & Franke, 1992). However, as we have seen in the previous paragraph, the failure of many innovative programs, even if their developers were extremely careful in foreseeing most of the important decisions for the teacher (regarding, for example, content, activities, and assessment), and the difficulties in reproducing experimental situations (Balacheff, 1990; Artigue & Perrin-Glorian, 1991; Arsac et al., 1992) underscore the dramatic importance of the teacher variable.

Although researchers have adopted the language of treatments and variables, the objects they so named often failed to have the requisite properties, as underlined by Shoenfeld (1994):

> oftentimes, for example, an instructional “treatment” was not a univalent entity but was very different in the hands of two different experimenters or teachers. Similarly, if an instructional experiment used different teachers for the treatment and control groups, the teacher variation (rather than the instructional treatments) might account for observed differences; if the same teacher taught both groups, there still might be a difference in enthusiasm, or in student selection. In short, many factors other than ones in the statistical model—the variables of record—could and often did account for important aspects of the situation being modelled. (p. 701)

In other words, in mathematics education a typical phenomenon for complexity takes place, known as “butterfly effect”\(^\text{14}\). “Microscopic” teacher’s decisions can have “macroscopic” effects in the dynamics of situations.\(^\text{15}\)

### Criteria for Quality of Research

Once complexity is acknowledged, many different approaches and methods are needed because only the consideration of many different points of view can help describe a complex situation (Arzarello, 1999; Barolini Bussi, 1994; Kilpatrick, 1993; Lester, 1998; Mason, 1994b; Pellerey, 1997; Schoenfeld, 1994; Steiner, 1985).

Being open to a multiplicity of methods borrowed from other disciplines (psychology, sociology, linguistics) means, of course, that the quality of research must be always kept under strict control (see Zan, 1999b, and references therein). In recent

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\(^{13}\) Other disciplines, too, state the need for new approaches. In particular, in the science of education Cohen and Manion (1994) referred to two different and complementary perspectives in research on education: “The first, based on the scientific paradigm, rests upon the creation of theoretical frameworks that can be tested by experimentation, replication and refinement. . . Against this scientific, experimental paradigm, we posit an alternative perspective that we describe as interpretative and subjective, a focus we hasten to add that should be seen as complementing rather than competing with the experimental stance” (p. 106).

\(^{14}\) The so-called butterfly effect in meteorology says that “If a butterfly flaps its wings on the Caribbean Sea, the weather in North America could change.”

\(^{15}\) Artigue and Perrin-Glorian (1991) wrote that: “Various recent researches, for instance (Arsac, 1989), have highlighted the macroscopic effect of decisions which can be qualified as microscopic if one refers to the level of observation, and the bifurcation in the dynamics of a classroom which can be caused by an apparently innocent remark, or even a simple movement or expression by the teacher. They clearly prove that the teacher can exert a close control over the dynamics of situations, at this microscopic, nearly invisible level, in order to reproduce what he perceives as necessary, or at least important, to reproduce through the description given to him” (p. 14).
years, this issue has been explicitly addressed by several researchers, even at international conferences (the symposium on Criteria for Scientific Quality and Relevance in the Didactics of Mathematics, held in Gilleleje, Denmark, in 1992; the ICMI study conference What Is Research in Mathematics Education and What Are Its Results? held at the University of Maryland in 1994; the Working Group 25 Didactics of Mathematics as a Scientific Discipline at ICME 8, held in Seville in 1996). In particular, the meaning of constructs such as relevance, validity, objectivity, originality, rigor and precision, predictability, reproducibility, and relatedness in the context of different kinds of research methodologies is discussed; moreover, it is discussed whether, and in which form, these constructs should continue to be regarded as fundamental criteria for assessing mathematics education research.

The discussion of some of these criteria is strongly influenced by the model of teacher as decision maker. The fundamental role of teachers’ decisions, and the butterfly effect observed when one tries to reproduce a teaching experiment, generates a long series of problems on the reproducibility of these studies.

Reproducibility

The problem of reproducibility has often been studied by the French school within research on didactic engineering (Arsac et al., 1992; Artigue & Perrin-Glorian, 1991; Balacheff, 1990). Even if Artigue and Perrin-Glorian spoke of internal and external reproducibility, and emphasize the need for “rejecting an over-simple assimilation between internal and external reproducibility,” they acknowledged the existence of obstacles created by teachers’ unforeseen decisions.16

In the study by Arsac et al., the authors analyzed very deeply and explicitly the role of teachers as to such reproducibility. With two case studies, they showed that two factors hamper fidelity in reproducing a given scenario: (a) constraints on the teacher resulting from the didactic system and (b) teachers’ conceptions about mathematics and learning. In our opinion, these two factors are not separate: As we have indicated, teachers’ beliefs and the constraints resulting from the didactic system interact, and their interaction elicits negative emotions such as anxiety, which in turn influence teachers’ decisions.

Teachers therefore play a fundamental role in facing the problem of the reproducibility of a teaching experiment. The teacher variable must be considered with and among all other variables. Furthermore, to allow research to be reproduced, it is extremely important that teachers undergo preliminary training about all the aspects that, as we noted, influence decision processes in doing research (knowledge, but also metacognitive skills, beliefs, emotions). From this perspective, the singling out of teachers’ decisions and actions is no longer the last link of a chain, unessential to the previous ones, that researchers can neglect or delegate to others. On the contrary, it is related to the quality of research itself. This way, the question of the theory–practice relationship becomes unavoidable: Researchers’ theory cannot exist without teachers’ practice.

This issue is analyzed in Italian research for innovation, aimed at producing paradigmatic examples of improvement in mathematics teaching, and at studying

16Artigue and Perrin-Glorian (1991) wrote that “it is well known that the effectiveness of the transmission of the products of didactic engineering is not self-evident. Several researchers have encountered this problem within the research process itself despite the privileged conditions in which such a process takes place: they do not usually do the teaching themselves, so in some way their engineering partly escapes them at the time of the experimentation. Even when the teachers carrying out the experimentation have closely participated in its development, when experimenting they frequently take unforeseen initiatives which disturb the functioning of the research process” (p. 14).
the conditions for their realization, as well as the possible factors underlying their ineffectiveness. Arzarello and Bartolini Bussi (1998) stated that the very moment of this research is the phase (before but also during the implementation of the experiment) in which teachers receive specific training:

Besides, the critical role played by the teachers in all the phases of the research study requires long-term training of the teachers before, and during the implementation of the experiment. Hence, even if specific issues of teacher training are not theoretically addressed in the research study, they are always in the background as a strong pragmatic component. These two elements are present in the development of every teaching experiment, even if they may not always be addressed in the reports- in order to meet the space and time constraints of either international journals or international conferences. (p. 225)

Relevance

Nonetheless, reproducibility is not the only quality standard influenced by the model of teacher as decision maker. This model implies also a revision of the standard of relevance. This criterion, regarded as fundamental for research in any discipline (Polanyi, 1958), is linked to the ultimate goal of research, and therefore it is developed in different ways according to the typical values of each discipline. Because mathematics education is a relatively young discipline, it is difficult to identify any typical values, that is, shared by all researchers; still, the improvement of the practice of teaching seems to be quite unanimously considered as the ultimate goal (Vinner, 2000).

Even assuming that this is the ultimate goal, relevance is an ambiguous term. One reason for this is that the criterion of relevance— unlike other criteria, such as validity— can be referred to the various components of research; so we can speak of relevant research problem, but also of relevant method or relevant results. Sierpinska (1993) suggested that we make a distinction between pragmatic relevance and theoretical relevance:

something is pragmatically relevant in the domain of mathematics education if it has some positive impact on the practice of teaching; it is cognitively relevant if it broadens and deepens our understanding of the teaching and learning phenomena. (p. 38)

She observed that if we accept the idea that the ultimate goal of research is the improvement of the practice of teaching, each theoretically relevant research must be pragmatically relevant, too; the only distinction, in this case, is between more direct and less direct pragmatic relevance.

Kilpatrick (1993) also mentioned direct relevance. He observed that “a research study may be of direct relevance to teachers, but more commonly its direct relevance is to other researchers.” The meaning of the term direct relevance is not evident. We suggest a distinction between direct relevance for teaching and direct relevance for teachers. If direct relevance for teaching can mean a direct useability of some parts of research in practice (but then, in this sense, there is very little relevant research), direct relevance for teachers is a subtler question. Teachers’ role in mediating between theory and practice does not necessarily consist of properly modifying experience deriving from research to adapt it to the classroom. The teachers’ role is different: Because theory modifies teachers’ knowledge, metacognitive skills, beliefs, and emotions, it

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17As underlined by Balacheff (1990) about teaching experiments, however, “the result is not the teaching setting itself but the answer to the initial research question or a new formulation of it, or the evidence of intrinsic links between pupils’ behavior and some set of variables whose control conditions the teaching process, or even the principles of the teaching design” (p. 270).
modifies teachers directly. In particular, theory modifies teachers’ decision processes and consequently their practice. This change does not take place through external intervention (where the teacher is told to “do this, not that” or “think differently”); this change occurs as a progressive increase in teachers’ awareness induced by a theory and by the reflection on it.

From this point of view, the model of teacher as decision maker knits together the break between pragmatic and theoretically relevant research. As a matter of fact, teachers’ decisions are influenced—over a rather long time—by changing teachers’ knowledge, metacognitive skills, beliefs, and emotions, as shown in Fig. 22.1. Nonetheless, to make sure that theoretically relevant research has a direct influence on teachers, two conditions are needed: Teachers must be able to “absorb” such research, in particular, they must be aware of their role as decision makers. In addition, the research itself must be conveyed in forms that are accessible also to practitioners. The first point is particularly important. Without appropriate training, teachers (but more generally those who do not do research in mathematics education) tend to prefer research results that seem immediately applicable in the didactic practice.

To clarify the problems regarding the tension between results needed by the school system (teachers, administrators, etc.) and results discussed and offered by researchers, Boero and Szendrei (1998), proposed a specific classification of results in mathematics education. The categories suggested are the following:

- **Innovative patterns** to teach a specific subject or to develop some mathematical skills, or, more generally, innovative methodologies, curricula, projects, etc.
- **Quantitative information** about the consequences of educational choices concerning the teaching of a specific mathematical subject, of general methodologies, and of curricular choices (including comparative and quantitative studies).
- **Qualitative information** about the consequences of some methodological or content innovation, or some general or specific difficulties concerning mathematics, etc.
- **Theoretical perspectives** regarding the relationship in the classroom between teacher, pupils, and mathematical knowledge; the role of the mathematics teacher in the classroom, the nature of the relationship between school mathematics and mathematicians’ mathematics, topics to be taught, the relationship between research results and classroom practice in mathematics education, and so on.

Boero and Szendrei (1998) observed that for many teachers, and for many mathematicians as well, the most useful research results are those offering innovative patterns or quantitative information. Still, they emphasized that results offering qualitative information and theoretical perspectives are important not only as such, but also because they allow teachers (and researchers) to keep the other kinds of results under control. And indeed, without proper warning, there is a risk of offering naïve interpretations. Artigue and Perrin Glorian (1991, p. 14) stated that this risk exists with regard to internal and external reproducibility:

it is then, for obvious reasons of communicability, accompanied by a flattening-out of scientific didactic language into the common language of teaching. It is not at all certain that, by doing this, we really reduce the problem of transmission. We give an illusion of communicability—but only an illusion. In fact we encourage naïve interpretations and therefore possibly make internal reproducibility more difficult to obtain. (p. 14)
Communicability

These considerations lead us to the problem of communicating research and to a revision of the quality standards for research reports. In this perspective, standards such as clarity, organization, and synthesis, which belong specifically to the phase of communication, become important for the quality of the research as a whole.

One thing that should not be forgotten when it comes to communication is the wide variety of methods used in mathematics education research—and above all the variety of disciplines to which mathematics education research refers (mathematics, epistemology, psychology, linguistics, sociology, anthropology, etc.). The coexistence of different, sometimes contradictory, paradigms is complicated by the fact that sometimes researchers do not declare their choices explicitly (Dörfler, 1993; Mason & Waywood, 1996). Moreover, these choices often derive from very personal beliefs, which should also be made explicit (Burton, 1994; Mason, 1994b; Schoenfeld, 1994).

Of course, it is not enough simply to offer such information, because the way this information is conveyed is extremely important. To be appreciated and have any feedback, research must be communicated. Sometimes the quality of reports makes communication difficult even among researchers, but this problem is most significant between researchers and practitioners. Many fundamental details are often taken for granted, and language is understandable only by “initiates” (Bishop, 1998; Hanna, 1998; Lester, 1998; Lester & Lambdin, 1998). According to Mason (1998):

> the more familiar and overt products of research, namely reports, articles, books, professional development materials, and classroom materials all suffer from what might be called a “research transposition,” following Chevallard’s (1985) transposition didactique.

(p. 370)

Highlighting the failure of research in mathematics education to resonate with teachers, Lester (1998) recognized one of the many explanations in the fact that:

> researchers and teachers have accepted different ways to frame their discourse about what they know and believe about mathematics teaching and learning. By and large, teachers communicate their ideas through, what Schwandt (1995) calls “the lens of dialogic, communicative rationalism.” By contrast, researchers typically communicate their ideas in terms of (monologic) scientific rationalism…. To accept dialogic rationalism involves accepting that reason is communicative: “It is concerned with the construction and maintenance of conversational reality in terms of which people influence each other not just in their ideas but in their being” (Schwandt, 1995, p. 7). It aims to actually move people to action, in addition to giving them good ideas. Dialogic rationalism, then, has something to say to mathematics educators about how we make and justify claims in our research. In particular, dialogic rationalism attempts to avoid treating students and teachers as objects of thought in order to make claims about them that will guide future deliberative actions. Instead, it aims to include teachers (and students?) in dialogic conversations in order to generate practical knowledge in specific situations.

(pp. 203–205)

To move people to action, however, it is necessary to create a social practice and speak directly to people’s experience (Mason, 1998). It is therefore important to present research in forms that promote personal construal, in which readers find themselves seeing their past experiences in a fresh light and sensitizing them to potential incidents to notice in the future (Mason, 1994b).

This kind of communication does not consist only in research reports. Communication can also go other ways. Conferences where researchers and practitioners can meet (such as CIEAEM) allow a real two-way exchange, and not only from researchers to practitioners, as it usually happens with reviews.
COLLABORATION BETWEEN TEACHERS AND RESEARCHERS AS RECONCILIATION BETWEEN THEORY AND PRACTICE

Examples of collaboration between researchers and teachers in mathematics education exist at various levels, with different partner roles (Burton, 1994; Mason, 1999). One of these roles, witnessed in literature, is the teacher-researcher. Scholars have different opinions about its validity, according to their idea of mathematics education as a discipline. Some scholars refuse to believe that the two roles can be played by the same person, because such roles are seen to belong to separate cultures. Vestappen (1994), for instance, emphasized that there is a separation between pragmatism, typical of the teaching profession, and theoretical speculation, which is typical of research; he nevertheless admitted that “there are exceptions, who actually come off their practice, who are inspired by something, who show a different teacher/theoretician, operating clearly diachronically and fundamentally upon ideas of Clairaut, Piaget, Freudenthal and others” but states that “only in a dream we can expect such vision and ability from all the teachers” (p. 60).

This idea, moving from the notion that teachers are strangers to the research community, is the extreme result of an overgeneralization, by which all teachers should then be considered researchers. Balacheff’s position (as expressed in Sierpinska & Kilpatrick, 1998) is even more radical: he stated that a teacher simultaneously cannot be a researcher, just as one cannot be one’s own psychoanalyst. Even if France does not lack collaboration between teachers and researcher, this position can be justified if we think about the nature of French research, which is not specifically aimed at innovation but is mainly theoretical.18

In England, thanks to the action research movement, the teacher-as-researcher is acknowledged and appreciated.

According to Jaworski (1998), the origin of the movement dates back to the 1960s, when Stenhouse did his first explorations; in the 1970s, it spread internationally, with the realization of large-scale projects and the creation of a review (International Journal of Educational Action Research) that documents a wide variety of theoretical perspectives and studies undertaken by teachers. Still, Jaworski stated that many teacher researchers are motivated by gaining an academic degree; they most often seem to be linked to graduate-level programmes, to externally directed projects led by university researchers, or both. Teachers’ research has its limit in this framework and is written according to the academic standards corresponding to the degree they pursue rather than aimed at developing their teaching. She stated that it is unlikely teachers carry out research outside the academic environment, even if she quoted examples of research they have carried out within associations or working groups.

18Margolinas (1998) wrote, “In French research, the teachers play various roles but, most of the time, they are integrated into research teams. They often become very active in these teams thereby producing their own autonomous research activity. Within certain structures (notably in the IREM) these teachers can get teaching time credits (halftime being the maximum), but they have to apply every year and the credits are not easy to obtain. The absence of a real part-time ‘researcher status’ in primary or secondary schools is a deterrent to the development of more interactive relations between university researchers and teachers of different levels. . . . If we turn now to experimental research in the classroom, what awaits a researcher is the testing of statements originating from theory, which means the production of phenomena. . . . But the motivation of the teacher who participates in this type of research derives frequently from the need to fight against ageing of teaching situations. The teacher not only expects changes but also improvement: he or she is oriented towards ‘innovations’. There are therefore frictions between motivations of the researcher and those of the teacher or frictions between innovation and research” (p. 354).
Jaworski began an interesting study in which she observed a group of six teachers for a year, while they tried to develop their own projects, which each of them created autonomously. University researchers did almost nothing to develop classroom activities; their role was supportive, enabling teachers to undertake and sustain research. The result of this study is a model of teacher-researcher that Jaworski called **evolutionary**: Teachers do not rigidly decide their interventions in advance, which would be typical for action research (planning, acting, observing, reflecting, replanning, further action, further observation, and further reflection), but observe and reflect systematically on what happens in class, gradually developing an increasing metacognitive attitude of control on, in, and for action. On this, she wrote the following:

The value, and to some extent inevitability, of this evolutionary process in research is in its recognition of the complexities of teaching. Compounding these complexities is the interrelatedness of substance and methodology, which are rarely distinct. The cognitive development of the researcher parallels closely the development of the research process and analysis of data. Knowledge grows through experience and cognitive challenge (made overt in the research process) within a social situation (Piaget 1950; Von Glasersfeld, 1995). Such is the metacognitive position to which these teachers were developing in their research activity. The strength of the process can be seen in terms of the teachers’ developing knowledge and practice. (p. 21)

In this context, however, teachers are alone in front of research with their own knowledge, culture, and sensitivity.

In the Italian case, quite different from the last example, the teacher-researcher plays both roles. We shall briefly present the origin and features of this model in the next section (refer to Arzarello and Bartolini Bussi (1998) or Malara (1999) for further details on the Italian situation).

**AN EXAMPLE OF RECONCILIATION: THE TEACHER-RESEARCHER IN THE ITALIAN MODEL FOR INNOVATION RESEARCH**

In Italy, from the institutional point of view, mathematics education research started in 1975 for social reasons, and it was aimed at renewing and improving mathematics teaching at every school level. From the beginning, the intention was to help teachers, because no renewal is thinkable without their being involved with awareness and motivation. **Nuclei di ricerca didattica** (Nuclei of didactic research) arose, where university researchers and school teachers of all levels could meet, so that by joining together their different competencies is and experiences, they could collectively construct a more adequate answer to the needs of society. The chief organizers of this policy, who are affiliated with universities, are pioneers the movement to involve practitioners in research as Bishop (1998) recently expressed,

Researchers clearly need to take far more seriously than they have done the fact that reforming practice lies in the practitioners’ domain of knowledge. One consequence is that researchers need to engage more with practitioners’ knowledge, perspectives, work and activity situation, with actual materials and actual constraints and within actual social and institutional contexts. (p. 36)

Teachers who participate in the **nuclei** are volunteers, they receive neither money nor help from their own institutions. Their motivations are primarily idealistic and cultural reasons, and this is probably what has made them successful.

Collaboration between researchers and teacher-researchers develops in a dialectic process between theory and practice. Researchers, especially at the beginning,
offer access to theory. They suggest readings, highlight problems, propose research hypotheses, and, in the end, they act as models in conducting research. Through seminars, teachers are introduced to the study scientific literature and are involved in epistemological analysis of concepts or theories that are at the basis of specific teaching contents until they achieve a common theoretical background concerning the questions related to such teaching and on the guidelines of experimental researches. Borrowing Even’s words (1999), the aim is that “the participants build upon and interpret their experience-based knowledge using research-based knowledge and vice versa they examine theoretical knowledge acquired from reading and discuss research in the light of their practical knowledge” (p. 12).

The methodology adopted in these research practices is complex but can be summarized in the following activities:

- Explicitations and comparisons among the various positions, collective formulation of research hypotheses.
- Common planning of classroom interventions, realization of discussion drafts, selection construction of questions for verifying research hypotheses, a priori analysis of difficulties.
- Joint qualitative analyses of the pupils (collective or individual) productions; analysis of the difficulties arisen.
- Reflection on the results obtained, classroom feedback as to innovation and as to teachers’ beliefs.

The movement between practice and theory is fundamental in this model: teacher-researchers have a crucial role in the whole research process. They participate in determining the innovation experiments because they best know the children’s level of development and how much time is available in school, which is often different from what researchers would think. They also prepare the tests, sometimes on their own and sometimes correcting the language in which the tests were written so that it becomes less rigorous and more effective for pupils. They make hypotheses on the pupils’ possible productions with reference to specific activities conceived and help formulate criteria for the assessment of such productions.

As teachers, they develop their didactic interventions in the classroom starting from problem situations that lead to the construction of mathematical knowledge through a network of interactions among the members of the class. Each teacher is therefore a question provoker, a listener, and an orchestretor of the activities rather than a mere vehicle of knowledge so that the acquisition of such knowledge does not proceed by simple transmission but derives from the conscious and reflected participation of the pupils in the activities suggested.

While leading the classroom activities, teachers can play the double role of participant and observer with remarkable control (Davis, 1992; Eisenhart, 1988); they are able to separate the observing subject from the observed subjects in their dialogue relationship (Arzarello, 1997). Moreover, teachers claim the role of observers to be peculiar to the teacher-researcher because they believe that any external observer could give misleading information, if not integrated or triangulated by the teacher’s observation process.19 They always give the first interpretation and assessment of the pupils’ answers, even if there is a joint analysis (teacher-researcher–university-researcher) of

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19 An interesting example of different views in the assessment of a protocol by a “weak” pupil (on the development of a demonstration of a simple arithmetical property) can be found in Malara and Iaderosa (1999).
the pupils’ protocols with a selection of documents considered particularly meaningful (not only for witnessing the fact that the proposal was good but also to document ways of facing the problem or difficulties encountered).

Moreover, the collaboration between researcher and teacher-researcher also offers a model of training process (for researchers as well as for teachers). Through the interaction with theory and thanks to the model researcher with whom they collaborate, teacher-researchers gradually achieve the professionalism of a researcher, and with time they start belonging to the research community, sharing knowledge, beliefs, and values. Many of these teachers are then acknowledged as members of the community (Malara, Ferrari, Bazzini, & Chiappini, 2000) in that they publish their articles on reviews and proceedings of international conferences autonomously (Ferri, 1992, Garuti, 1997; Iaderosa, 1999, Navarra, 1998; Paola, 1999; Scali, 1999). In particular, teacher-researchers get new awareness of the complexity of pupils’ learning processes. As reported in Malara and Iaderosa (1999),

although the aim of choosing this way of working was to create prototypes of didactic innovation, the most important achievement was the constitution—through almost continuous dialogue and exchange—of a deep and mature awareness in the teachers, not only towards the mathematical contents to be taught, but also towards the attitudes that should be promoted in the pupils as well as the dynamics to be developed in class.

(p. 39)

This awareness gradually modifies their “practice”: The role of researcher creates a new model of teacher that slowly replaces the previous one. This evolution is the result of a training process enacted along the relationship with theory, which influences teacher-researchers’ choices and the decisions by modifying their knowledge, beliefs, awareness, and emotions. The Italian tradition of teacher-researcher has many examples of such an experience.

As to the growth of knowledge (both content and pedagogical content knowledge), for example, as reported in Malara (1999) and in Malara and Iaderosa (1999), the need for starting an innovative project on the approach to algebraic thinking, centered on a relational teaching of arithmetic with an early use of letters. This highlighted the need to create the right cultural background and therefore led to studying literature on the teaching–learning problems of algebra, and in particular to literature on the difficulties caused by a procedural teaching of arithmetic (Kieran, 1989, 1990, 1992) and on questions linked to the translation of verbal statements into algebraic ones and vice versa (Clement Battista, 1981; MacGregor, 1991). Similarly, the study of literature on classroom discussion (Bartolini Bussi, 1991, 1994) led them to a ripening of methodological aspects.

This study had a strong influence on changing of beliefs. At the beginning, for example, some teachers believed that dealing with proofs is too difficult for students of middle-school level. It was found that pupils’ work in elementary number theory revealed unexpected skills in the context of logical reasoning, and consequently teachers radically changed their beliefs about what the pupils could do. In the subsequent investigations, they devoted more time to this subject, focusing their attention to the passage from argumentation to proof. As a result of their discoveries, the teachers began to disseminate the results of their research among other colleagues, highlighting its didactical value (Malara & Gherpelli, 1997).

The teachers also developed a fresh attitude toward mistakes, which are now sought after to be discussed and mastered by considering with the pupils their possible origin. This even produces a change in emotions. There is no longer fear and frustration in the face of pupils’ mistakes, but curiosity and a stimulus to probe more deeply to do something.
In sum, teachers develop a new way of perceiving their profession; study and research become an essential part of it. The teacher-researcher R. Iaderosa, in Garuti and Iaderosa (1999), declared:

Meeting the world of research puts a teacher in a condition of tension towards a study that, beyond every deadline, never ends, because one sees that knowledge must be built day by day, it is not a ready-made stock to be conveyed: this is very important and it belongs to the teaching profession as soon as it becomes an attitude to be conveyed, with one’s experience, to other teachers too. (p. 315)

These examples show what we have already observed, that is, that any separation between knowledge, beliefs, and emotions is abstract and artificial. In fact, all these factors interact profoundly. In particular, a change in knowledge can gradually influence beliefs as well as emotions.

On the basis of these considerations, we can make a few observations about the conflict, described by some researchers (see, for instance, Ainley, 1999; Mason, 1999; Wong, 1995), between the two roles of teacher and researcher in the same person. It is not really a conflict between teacher and researcher, but rather a conflict between two models of teacher (often implicit) that coexist in the same person: the traditional teacher, who preexists the researcher, and the new one introduced by coming a researcher (see also Wilson, 1995).

The new model interacts with the preexisting one and can generate conflict. It is a cognitive conflict, temporary and fruitful, connected to one’s process of transformation as a teacher, which is forced by one’s role of researcher. The solution of this conflict, which can last a long time, leads the teacher to a growth in awareness.²⁰ Sensing that there is a conflict between two ways of assessing (the typical one for school, and the one related to research) and that the context bonds are very rigid is an unavoidable consequence of this transformation. As it always happens, awareness, if it is not accompanied by the right control processes, can give vent to negative emotions such as anxiety and frustration. In Garuti and Iaderosa (1999), the teacher-researcher R. Garuti tells about one of her teaching experiments:

This started as a series of questions about my role: Isn’t there a risk of ‘killing’ principles, if one introduces mathematical models to soon? Which are the consequences? Sometimes the mathematical model co-exists coherently with the pupils’ conceptions, but in some situations the mathematical model gets abandoned, and the conceptions, which cause mistakes, arise again. In these cases a teacher feels [as if she] built something on the sand. (p. 319)

In the end, the real conflict between the teacher role and the researcher role consists in having different goals, if we assume that the primary goal of research is to understand, whereas the primary goal of teaching is to help students learn (Ainley, 1999; Mason, 1999; Wong, 1995). This conflict becomes smaller in the long term if we see as the ultimate goal of research the improvement of mathematics teaching and learning and as the ultimate goal of teaching the improvement of students’ learning. As Arzarello (1999) noted, the difference then becomes primarily one of time and context—short

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²⁰ This is exemplified in the following excerpt by Ferri (1992): “a teacher-researcher faces three major and contemporaneous factors of the teaching activity:— the object of teaching;— the teaching process;— the learning process. These three factors are intertwined in the teaching activity. . . . Yet, it is only by assuming, at least in part, the researcher’s perspective, that she becomes aware of the dynamic between the three aspects and directs consciously her actions in the teaching activity.”
periods of time for the teacher (in a particular classroom with students), long periods of time for the researcher (any time, anywhere, with any student).21

CONCLUSIONS

The relationship between theory and practice has been studied and discussed in every discipline. All the various ways of perceiving it within mathematics education are relevant. One way concerns the relationship between mathematics as a practical activity and mathematics as a theoretical body of knowledge (Vergnaud, 1998). Another dichotomy exists between teachers’ theory and teachers’ practice: Research on teachers’ awareness states that such separation can be mended through reflection (Jaworski, 1994, 1998; Mason, 1998). Finally, there is the gap between theory and practice in research, which depends on assuming that research practices, that is, the methods chosen, are independent from any methodological base. To mend this separation, researchers must make explicit (first of all to themselves) the choices concerning perspectives and methods. These choices, which represent a choice of values (Schoenfeld, 1994), indeed play an important role in influencing “what is observed” (Mason, 1994b).

In this chapter we have chosen to analyze the theory–practice relationship in the most traditional way, that is, by considering researchers’ theory and practitioners’ practice, even if, as observed by Brown and Cooney (1991); “intelligent reflection on the actual and potential relationships between researchers and practitioners may be better achieved by locating the place of both theory and practice in each of these communities rather than by dichotomizing them” (p. 112). Of course the analysis of this relationship is strongly influenced by the model of practice that one chooses. The one we have suggested is teaching as decision making. Not only does this model influence the theory–practice relationship, but it even has consequences on theory itself, on influencing its standards of quality such as reproducibility and relevance. In particular, teacher training seen as an action taken on knowledge, but also on beliefs, emotions, and awareness, becomes an important aspect of certain kinds of research to inform practice.

Italian research for innovation faces the problem of teacher training within the research project. A fundamental role in this research is played by the teacher-researcher, who represents an element of reconciliation between theory and practice in two different moments and at two different levels. On one hand, theory and practice move closer together by helping researchers get in touch with practice and practitioners with theory. On the other hand, it is an example of a gradual training process, in a way provoked by the researcher role existing in teachers.

The commitment required (to researchers and teachers) to this training process, and the small number of teachers that can be involved in this experience, make it rather difficult to see the teacher-researcher product as immediately generalizable. Its exceptional nature must be acknowledged because, as stated in Bishop (1998),

Exceptional situations should be recognized as such, and not treated as “normal” or generalizable. Indeed, it is better to assume that every situation is exceptional, rather than assume it is typical. Typicality needs to be established before its outcomes can be generalized. (p. 43)

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21 Garuti ended her reflection in Garuti and Iaderosa (1999) as follows: “From the point of view of research, the pupils’ conceptions and principles represent a rich and important chapter, but research can wait, can study, can go deep into things, create instruments for interpretation. Teachers, on the contrary, must make choices in a moment. Knowing all this doesn’t limit anxiety, but rather makes responsibility towards pupils even heavier. In this case teaching and research have different timings.
Still, even if the products do not seem immediately generalizable, the processes that allowed us to create these products can achieve more general feedback. In this perspective, if researchers continuously observe the training process of teacher-researchers, they can then provide important information to develop training projects and make theory and practice even closer.

The importance of teacher training is also linked to theory through another quality standard of research: relevance. The model we have chosen (teaching as decision making) suggests that research influences teaching and teachers’ decisions. Teachers must increase their awareness to broaden their sensitivity to the possibility of making decisions moment by moment. Theory has a crucial role in this process because it changes teachers’ knowledge and therefore beliefs and emotions. On one hand, theory must be communicated so that even nonresearchers can appreciate it; on the other hand, teachers must have a basis preparation that allows them to appreciate research results.

We have characterized teachers’ practice as decision making, but research activity, too, implies making continuous decisions about aims and users and consequently about the choice of research problems, the theoretical frame of reference, the methodology and the modes of communication. But in addition, the interaction of the researcher with the teacher influences not only the choice of the research problems, but also the strategies chosen to face them. If the contact with theory (slowly) changes the teachers’ decision processes, and therefore the practice, the contact with practice (slowly) changes the researchers’ decision processes, and therefore the theory (see Fig. 22.2).

The two processes that we have considered separately starting either from practice or from theory, related to the changes of teachers and researchers, have to be seen as connected components of a same “object,” as in a Möbius strip.

For researchers, a change also happens through a renewal of knowledge, beliefs, awareness, and emotions, because these factors influence their decision processes. Emotional aspects, in particular, often neglected by research as if they only polluted thinking processes, play a crucial role in choosing what to observe, in defining goals, in directing actions. The process of knowledge is always extremely personal. As Polanyi (1958) noted, “I have shown that in every act of knowledge there is a passionate contribution of the person who knows what gets known, and that this component is not an imperfection but a vital factor of knowledge” (p. 8).

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Linking Researching With Teaching: Towards Synergy of Scholarly and Craft Knowledge

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This chapter focuses on relationships between researching and teaching and between researchers and teachers working to develop a knowledge base for mathematics teaching. In so doing, it treats researching and teaching, researcher and teacher simply as convenient typifications, recognizing the possibility that institutions and individuals may participate in both practices and take on both roles. Moreover, this chapter will reserve the term practice for use in the sense of social practice; it will employ the more direct terms of teacher and teaching to refer to what some sources describe as practitioner and practice. The knowledge base for teaching is seen here as drawing both on scholarly knowledge created within the practice of researching and on craft knowledge created within the practice of teaching. A particular concern will be with how greater synergy can be fostered between these distinctive practices, their characteristic forms of knowledge, and the associated processes of knowledge creation.

MATHEMATICS EDUCATION AS A RESEARCH DOMAIN: BETWEEN ACADEMIC ACCEPTANCE AND PEDAGOGICAL PERTINENCE

A recent study, under the auspices of the International Commission on Mathematical Instruction (ICMI), suggests that “mathematics education as a research domain” is still engaged in “a search for identity” (Sierpinska & Kilpatrick, 1998). The study reveals a community held together not by a common idea of research, but by research as a common ideal. Although most of the contributors identified the development of knowledge and resources capable of supporting the teaching and learning of
Invited to reflect on the deliberations of the study within the ICMI publication itself, Bishop (1998) expressed concern over “researchers’ difficulties relating ideas from research with the practice of teaching and learning mathematics” (p. 33). While noting “some signs that research and researchers are relating more closely to the ideas of reform in mathematics teaching” (p. 35), he argued that “researchers need to engage more with practitioners’ knowledge, perspectives, work and activity situation, with actual materials and actual constraints, and within actual social and institutional contexts” (p. 36). Reviewing the ICMI publication, King and McLeod (1999) found “surprisingly little ... about the implications of well-developed research areas for classroom practice” (p. 232) and suggested that “At the same time that researchers in mathematics education have been shifting paradigms, other researchers with more traditional views have been busy ... arguing against educational reform” (p. 231). More trenchantly, Steen (1999) concluded, “From this thicket of meaning-challenged words and related disputations about goals, aims, methods, criteria, and results emerges a single irony that seems to enjoy widespread assent: Research has had essentially no impact on the practice of mathematics education” (p. 240).

The ICMI study expresses a trend that can be seen readily by examining how, over the last 30 years, the concerns of international congresses in mathematics education have shifted and the contents of leading international journals have changed (Bishop, 1992; Boero & Szendrei, 1998). There has been a move from international exchanges about the teaching of mathematics toward supranational dialogues about research in mathematics education. Hence, although there remains a very real sense in which, as Sekiguchi (1998) put it, “educational research is essentially local practice, the major part of which consists of practical studies in socially and culturally bounded places and communities” (p. 395), the emerging international research community has become an important audience for—and potential influence on—such studies. It has created its own distinctive culture of research, detached from the more pragmatic and locally contextualized concerns of national teaching communities.

As Silver and Kilpatrick (1994) noted, this international community has “tended to take a path of least resistance, focusing on topics that are relatively easy to discuss internationally” (p. 749) so that “conversations about the learning of specific mathematics content and processes or about theoretical issues have tended to occur at a level that allows members of the community to bypass important aspects of the conditions and traditions of educational practice within countries” (p. 750). Consequently, “important research questions may be largely ignored within the international community because they do not relate readily to abstractions or universals, requiring instead attention to the nuances of local educational settings” (p. 750).

Yet such nuances—and their significance—may be too readily taken for granted. Gouldner (quoted by Hargreaves, 1999) contrasted conceptions of the applied social scientist as technician and as clinician, suggesting that the technician tends to take problems at face value, as formulated by the client, whereas the clinician makes his or her own independent assessment of the client’s problem: “Not only does the clinician assume that the client may have some difficulty in formulating his own problems, but he assumes, further, that such an inability may in some sense be motivated and that the client is not entirely willing to have these problems explored and remedied.” Therefore, the clinician “does not take his client’s formulations at their face value,” argued Gouldner, “but he does use them as points of departure in locating the client’s latent problems” (pp. 241–242). The formation of an international research community has created new possibilities of recognizing, examining, challenging, and suspending local assumptions.

The development of this international community has been encouraged by the widespread trend for universities to play a greater part in teacher education and
the corresponding drive to gain recognition for education as a research field. This has created a new generation of career mathematics educators with a much stronger identity as academics, influenced by dominant representations of research and valorizations of it. Silver and Kilpatrick (1994) pointed to a “prevalent tendency to emphasize the connection of one’s scholarly work to the academic disciplines rather than to educational practice.” This they attributed to the “important pragmatic concern [of] many mathematics educators . . . to establish . . . the academic quality and rigor of their research” (p. 739).

The often precarious status of mathematics education as a research subspecialty—sometimes located within mathematics, more commonly within education, but consistently on the margins of the host field—has heightened aspirations for academic acceptance. Institutional location also has shaped the terms of such acceptance, influenced by established models of knowledge and enquiry in the host field. Against this background, Valero and Vithal (1998) suggested that researchers may be prone to choose methodological integrity over educational relevance when these appear to conflict. Specifically, researchers may prefer to work in less problematic research environments and to address topics of only marginal relevance to the mainstream of mathematics teaching, rather than risk having their research considered methodologically poor.

Here, there are echoes of Schön’s (1987) wider critique of technical rationality as a basis for professional activity:

On the high ground [of professional practice], manageable problems lend themselves to solution through the application of research-based theory and technique. In the swampy lowland, messy, confusing problems defy technical solution. The irony of the situation is that the problems of the high ground tend to be relatively unimportant to individuals or society at large, however great their technical interest may be, while in the swamp lie the problems of greatest human concern. The practitioner must choose. Shall he remain on the high ground where he can solve relatively unimportant problems according to prevailing standards of rigor, or shall he descend to the swamp of important problems and nonrigorous enquiry? (p. 3)

Nevertheless, pedagogical pertinence remains a prominent concern of mathematics education researchers, not least because most also work concurrently as mathematics teacher educators, as teachers of mathematics, or both. This provides motivation to establish a persuasive and productive relationship between their research and teaching activities, a motivation likely to be strengthened by the expectations they encounter, as researchers, in forming working relationships with teachers.

RELATIONSHIPS BETWEEN RESEARCHERS AND TEACHERS: PURPOSES, PERSPECTIVES, AND POWER

Relationships between researchers and teachers can be characterized in terms of three ideal types, starting with traditional data-extraction agreements and shifting toward more intensive and reciprocal collaboration in the form of clinical partnerships or colearning agreements (Wagner, 1997). In data-extraction agreements, collaboration extends only as far as negotiation between researchers, as seekers of data, and teachers, as sources of data or gatekeepers to it, regarding reasonable terms of access. Although collaboration on this basis may offer useful insights into the practice of teaching, it provides little opportunity for interaction between the thinking of researchers and teachers. In a clinical partnership, collaboration extends to give teachers a part in formulating and conducting investigations; however, researchers retain responsibility for the process of enquiry, and it is the practice of teaching that is the subject of analysis...
and reform. In a colearning agreement, teachers become more active counterparts in the process of enquiry, and the practice of researching also becomes subject to analysis and reform.

Closer collaboration and deeper interaction between researchers and teachers is liable to lay bare important differences of perspective, calling for sensitive management and constructive dialogue. In exemplifying and examining such differences, this section will draw on two sources: a refreshingly candid account by Wiske (1995) of a collaborative program in which researchers worked with teachers to find ways of using information technologies to teach mathematics, science, and computing more effectively at the high school level, and the reflections of Newman, Griffin, and Cole (1989) and their collaborating teachers as they worked together to develop curriculum modules aimed at supporting cognitive change. Episodes have been chosen as illuminating breakdowns in collaboration; they should not be taken as typifying the projects concerned, both of which were largely successful in building cordial and productive working partnerships.

In one vignette, Wiske described the tensions emerging within one project group in which teachers were concerned that students often had no idea of what operation to use in solving word problems, tending to rely on ritual maneuvers rather than problem analysis. The researcher in the group construed the situation differently, as symptomatic of a more fundamental lack of understanding of “intensive quantities” on the part of students.

The teachers defined the [issue] in terms of the types of problems, taken from their texts, tests, and workbooks, that students frequently failed. The professor… defined the [issue] in terms of an underlying mathematical concept, described with language that was unfamiliar to most of the teachers… [The professor] recalls the early conversations with teachers… as full of conflict. He and the teachers became “polarized” over the way they defined the important questions worth investigating… [His analysis] “was largely construed as abstruse and theoretical and without purpose.” (Wiske, 1995, p. 193)

Here, teachers are intent on addressing students’ needs and exploiting the resources available to them, resources that help to define these needs and the means by which they can be addressed. Prime among these resources are texts and tests, as well as pedagogical approaches developed in the course of, and in relation to, working with these resources. This is characteristic of the way in which skilled practical thinking incorporates the task environment and exploits setting-specific knowledge (Scribner, 1986). Equally, the researcher is intent on analyzing students’ difficulties, as defined within his task environment, exploiting the resources available.

Probing deeper, we see that, far from collaborating within a common task environment, researchers and teachers are seeking to coordinate their distinctive practices and to cooperate within them. Another vignette further draws this out.

[A teacher] participated in pilot testing some lessons with small groups of students… and recommended several significant alterations to make the lessons more practical with whole classes… Her recommendations were based on two concerns. First, the “flow of ideas” was not sufficiently clear and, second, the lessons involved too much telling by the teacher… “You don’t just tell the students to do this… you have a discussion so they understand what they’re doing.” When she was asked a few months later to teach the revised experimental unit in her class, she found that “some of the things we had said absolutely would not work had been put back into the lessons.” (Wiske, 1995, p. 197)

Here, the teacher was drawing on her craft knowledge in adapting the lesson designs of the researchers, grounded in cognitive theories, to the conditions of class teaching and the characteristics of students, but because such issues were not salient
in the researchers’ theoretical frame, the teacher’s advice did not register. Eventually, the researchers came to recognize how this teacher could help them to revise lessons to make them more feasible for classroom use, but it appears that this happened only when they themselves began to engage with the teacher’s task environment through coteaching. As one research assistant put it, “When the teachers say it’s not possible you think, well, they just don’t understand. . . . But the fact is you don’t know until you do it. . . . Having to think what you want to do and do it at the same time is not easy” (p. 198). In the words of another research assistant, “You have an idea on paper, but when you try to chunk it up into classes, the connections get lost. . . . Watching one kid [in a clinical study], you forget how much of the problem comes from the constant distractions in a class” (p. 198). Even here, however, classroom experience seems to be being construed unfavorably against research norms.

Another vignette brings out the different purposes and perspectives of teachers and researchers. Wiske described the reaction of one teacher to conducting a series of clinical interviews jointly with a researcher, in which students were expected to puzzle over a complex system, even to the point of frustration. The teacher regarded the clinical interview as an educational experience for the student. She wanted the child to be treated as the teacher would have treated her in class, not allowed to feel stupid or discouraged by a prolonged period of ignorance unlike anything the teacher would willingly sustain in class. (Wiske, 1995, p. 203)

Not only is this episode illustrative of ‘struggles between researchers eager to understand how children’s minds work and teachers who felt pressed to educate these minds’ (p. 195); one also senses a teacher expressing a concern to respect and nurture students, to set a moral example as well as an intellectual one.

Newman et al. (1989) encountered similar issues of differing perspectives and purposes.

One of the basic conflicts between teachers and researchers is in the fact that, for the teacher, it is important to find ways in which children can succeed as well as possible in their academic work. Yet . . . the researchers . . . were also interested in the ways and situations in which children were having difficulties with cognitive tasks . . . [One] teacher took it as her responsibility to make sure that lessons went as well as possible once the planning phase was over, no matter what the logic of the research demanded. Sometimes she would modify the lesson, using her intuitions about the needs of individual students. This complicated life for the researchers. It would have been convenient, from our viewpoint, for her lessons to be uniformly structured. . . . But the changes eventually became part of the data. . . . [R]esearch, as well as teaching, often needs to be modified as the process under observation unfolds. (pp. 145–146)

Quinsaat, another of the collaborating teachers, explained her “advocacy” for students during the course of the research, “Research is intended to be a benefit for the children in the long run. But in the immediate circumstances, it is up to the teacher to protect the child from research situations that might violate their rights” (p. 143). She identified similar issues affecting teachers.

Many teachers I know assume that educational researchers end up exposing and criticizing the practitioner. . . . It is easy to see how teachers might get this impression from the kind of research that is published about teachers and schools. . . . Why, one might ask naively, should a competent teacher worry? If everything was going alright there would be nothing to hide. This point of view really is naive. I am willing to admit that things go wrong in my classroom more often than I would like, as would any honest professional. . . . It would be extremely easy to find cases which could be embarrassing. (Quinsaat, quoted by Newman et al., 1989, p. 144)
Wiske, too, acknowledged issues of status, influence, and power, noting the way in which university-based researchers tend to be "more equal" than school-based teachers, so that "[w]hile most . . . participants recognized that the academics made a good faith effort to collaborate, school people found that the university people's world view tended to predominate in the design, conduct and interpretation of the research" (p. 206). Nonetheless, through what Wiske (1995) characterized as sustained commitment, reciprocal exchange, and mutual education, the collaborative program was able to progress beyond conflicting perspectives to arrive at cooperative purposes.

When the [professor] shared readings that informed his conceptual framework [for understanding word problems], other members of the group were able to join him in further refining and applying this framework to their shared work. When a researcher and teacher sat down long enough to explain to each other their expectations about the appropriate way to conduct a clinical interview, they were able to invent a way of proceeding that made sense to both of them. When researchers traded roles of teacher and observer with collaborating teachers, their eyes were opened to insights previously invisible to them. (Wiske, 1995, pp. 208–209)

In terms, then, of the ideal types sketched at the start of this section, both of these projects seem to have been conceived originally as clinical partnerships but to have moved to some degree toward colearning agreements, as the assumptions and methods—not just of teacher but of researcher as well—came under scrutiny and became the subject of negotiation. These projects illustrate how the practices of teaching and researching each involve distinctive types of purpose and perspective. These differences were accommodated not so much by establishing common purposes and perspectives as through finding ways in which the purposes and perspectives of the two practices could be coordinated. This made possible the cooperation of the practices of teaching and research. Such accommodation, coordination, and cooperation was facilitated by some degree of engagement of each group in the practice of the other. Equally, there seems to have been a process of tacit negotiation through which each group regulated its degree of accommodation to the other, thus shaping the character of the cooperation. These issues of purpose and perspective, and of the exercise of power, emerge as central to understanding collaboration between researchers and teachers.

ROLE DIFFERENTIATION AND ROLE (DIF)FUSION: THE EXAMPLES OF DIDACTICAL ENGINEERING AND RESEARCH FOR INNOVATION

An illuminating contrast in the conceptualization of the respective roles of researchers and teachers in collaborative research has arisen between the French school of "didactical engineering" and the Italian school of "research for innovation." The two approaches share a concern for developing teaching designs and identifying and analyzing the didactical variables coming into play and the related didactical strategies available. However, didactical engineering aims to develop highly precise designs that will be reproducible under suitably controlled classroom conditions, and to do so through systematic and exhaustive analysis of variables and strategies, framed in terms of an overarching didactical theory. By contrast, research for innovation aims to develop prototypical examples of designs in the expectation that they will be adapted to differing classroom circumstances, and to study these under the resulting conditions of natural variation, guided by a serendipitous theoretical eclecticism.

Bartolini Bussi (1994) suggested that, in comparing the two approaches, the role accorded to teachers "acts as a litmus paper" (p. 123). She contrasted the sharp
differentiation of teacher and researcher roles in didactical engineering, against their fusion in the role of teacher-researcher within research for innovation. This relates, in turn, to two further contrasts: one between the responsibility of teachers, in the former, to implement precisely defined designs engineered by researchers, as against the expectation, in the latter, that teacher-researchers will not just contribute to the development of prototypical designs, but adapt them to their circumstances; the other, between the emphasis, in the former, on systematic observation of classroom activity by detached (albeit often teacher) observers, as against, in the latter, participant observation of their own classrooms by teacher-researchers in action. However different in character, each of these approaches has proved effective in sustaining longstanding research collaborations within its particular cultural setting.

Brousseau (in a text dating from 1975) outlined a basic experimental procedure for didactical engineering:

[The research team] wishes to create a phenomenon in a precise, reproducible way, and to observe it. . . . The development of the lesson is provided down to the smallest detail on the didactical proforma. This sheet is given to all observers before the lesson so that they can peruse it. . . . At the end of the lesson, the children’s work is collected; the written records and the observation grids are brought together. These documents are then examined during the working session which takes place after the observation. Everyone offers an opinion. It is necessary to determine in this way whether the predetermined objectives have been attained. (Brousseau, 1997, pp. 277–278)

For the teachers involved in this process, Greslard and Salin (1999) suggested that establishing agreed-on written rules governing the collaboration and institutional channels through which conflicts can be signaled and regulated, is important. It allows teachers to be sure that, in the last resort, their professional prerogative will take precedence. Nevertheless, they added that “this is true only if the reasons which make them refuse to do what the researcher proposes are based on the ordinary constraints of a teacher” (p. 31). This approach to didactical engineering, then, can be seen as a highly codified form of clinical partnership in which teachers agree to give researchers’ purposes and perspectives an unusual degree of influence over the way in which they work.

Explaining the crucial points of contention that emerged between didactical engineering and research for innovation, Bartolini Bussi (1998) cited not only “the limited role (if any) that the teacher had in the early elaboration of the theory,” but the emphasis on detached observation that “put the teachers under a lens directed by the university researchers [and] clashed against the tradition of a peer cooperation.” Yet there are further nuances to this issue of observation. The approach employed in didactical engineering implies an important depersonalization of the teachers’ actions. To the extent that a teacher is viewed as putting a predetermined design into action, it is that design that is under observation rather than the teacher, who is simply its agent. Nevertheless, “This is often difficult for the teacher. S/he must understand that the participants are speaking of his/her action as that of an actor caught in a network of constraints” (Greslard & Salin, 1999, p. 30).

Moreover, Malara (1999), while acknowledging the complexities of both taking the teaching role and decentring from it, offered a further argument for the shift from detached to participant observation within research for innovation: External observers can never know the pupils as well as the teacher does, and this limits their capacity for interpretation. Indeed, the proponents of reflective teacher research (Hatch & Shiu, 1998) and of “researching from the inside” (Mason, 1998) would make a still stronger case for participant observation as a means of gaining access not just to richer contextual knowledge, but to the teacher’s inner sense. Malara (1999) acknowledged the complexities that this introduces in conducting research for innovation because
“each participant has his/her own point of view which contains implicit beliefs, expectations and even fears” (p. 50). She pointed to the way in which differences in the background of teachers, their teaching styles, their personal preferences, and the differing ways in which they construe the teacher-researcher role, all influence the teaching undertaken in their classrooms and the way in which it is researched. In attempting to bring together the teacher and researcher roles in this way, then, there is a danger of role diffusion.

In championing the role of teacher-researcher, and the method of participant observation by teacher-researchers, research for innovation can be seen as aspiring to a form of colearning agreement. Nevertheless, here again the situation is more nuanced. Although Malara (1999) described the research collaborations as “finding a bottom-up solution to the teachers’ real needs and for responding to their problems in the least academic and most practical way possible” (p. 39), the organizational structure and group dynamics of collaboration create a more familiar process of problem definition:

Even though for a long time in our group, the problem of the passage from arithmetic to algebra had been seen as a possible object of research, the teachers were reluctant, either because of its width/complexity or because of... disagreeing opinions about times and ways of initiating... pupils to the use of letters. For a certain period of time the problem was put aside, but only apparently because... we university-researchers guided the discussion onto some ad-hoc articles... which we suggested... reading. This first phase of slow “underground” work was useful for the awakening of the teachers to the questions linked to this theme. Then the organization of a cycle of meetings with researchers who had already been working in this field for a long time stimulate[d] the teachers to a systematic study of the topic. (Malara, 1999, p. 44)

Hence, it seems that expectations of leadership bring the university-based researchers to take the initiative—at least at the level of project formulation. As already indicated, however, the sometimes diverging purposes and perspectives of the teacher-researchers influence the teaching undertaken and the way in which it is researched—at the level of classroom implementation. Although this gives a coherence to the work taking place within each classroom, Malara pointed to some loss of coherence across classrooms.

The way in which teachers shape the work they carry out in their own classrooms raises the important question of the part played in the research by their craft knowledge of teaching, not just at the stage of implementation, but in the process of formulation. When Malara (1999) referred in the quotation above to the “disagreeing opinions” of teachers, she is probably reporting the language which teachers themselves used to characterize their exchanges. However, it would be surprising if the ideas they presented were not based to some degree on knowledge gained through experience of teaching. More concretely, when she reports the task questions devised by one teacher as being “immediately appreciated by the group for their originality and efficiency,” and comments on the capacity of the teachers “to assess potentialities and difficulties a priori” (p. 45), she is offering direct evidence of how craft knowledge, shaped by their experience of teaching, played a part in the development of the didactical designs.

The eclecticism characterizing research for innovation suggests that it has a strong capacity to be receptive to such knowledge. The craft knowledge of teachers, sometimes seen as an expression of “tacit” theories, might also be considered part of the “kit of tools” to which Arzarello and Bartolini Bussi (1998) refer:

Another feature of this trend of research is that no global coherent theoretical framework is assumed, because of its complexity and the number of interrelated variables to take into account. Rather, instruments are borrowed from various theoretical approaches or
produced inside, and applied as elements of a kit of tools. Local coherence of the framework is necessary, but global coherence is considered impossible or at least irrelevant. (p. 250)

Recognizing and valorizing the contribution of craft knowledge is not simply a matter of acknowledging a distinctive contribution of teachers to the research collaboration, but of creating the conditions under which such knowledge can legitimately come under respectful forms of examination comparable to those applied to scholarly knowledge.

The part played by craft knowledge within didactical engineering is also interesting. Not only do informal mechanisms emerge through which craft knowledge can come into play, but also a form of participant observation. One mechanism is in the approaches that teachers adopt in the lessons between experimental sessions; another lies in the way in which teachers translate the researcher’s didactical designs, however precisely stated, into classroom activity; and yet another arises through the postlesson analysis of such sessions (again featuring as exchange of “opinion” in Brousseau’s earlier description). Greslard and Salin (1999) described how the teacher is encouraged to reflect on the lesson and to explain spontaneous decisions made during its course. Brousseau (1997) reported how insights from such sources provoked revision of the fundamental theory of didactical situations to take greater account of the role of the teacher:

We once thought that we had envisaged all the possible classes of situations. But in the course of our studies…we saw that after a while the teachers needed some more space; they did not want to go on from one lesson to the next, wanting to stop so as to “review what they had done” before continuing; “some students are lost, we can’t go on, something has to be done about it.” It took us some time to realize that they really needed to do some things, for reasons that had to be understood…. This is how we “discovered” (!) what all teachers do all along their courses but which our method of systematization had made unacknowledgeable. (p. 236)

KNOWLEDGE CREATION WITHIN TEACHING: THE SIGNIFICANCE OF CRAFT KNOWLEDGE

“Craft knowledge” refers to the professional knowledge that teachers use in their day-to-day classroom teaching; action-oriented knowledge that is not generally made explicit by teachers, which they may indeed find difficult to articulate or which they may even be unaware of using.

[Craft knowledge describes the knowledge that arises from and, in turn, informs what teachers do. As such, this knowledge is to be distinguished from other forms of knowledge that are not linked to practice in this direct way. Craft knowledge is not, therefore, the kind of knowledge that teachers draw on when explaining the thinking underlying their ideal teaching practices. Neither is it knowledge drawn from theoretical sources. Professional craft knowledge can certainly be (and often is) informed by these sources, but it is of a far more practical nature than these knowledge forms. Professional craft knowledge is the knowledge that teachers develop through the processes of reflection and practical problem-solving that they engage in to carry out the demands of their jobs. (Cooper & McIntyre, 1996, p. 76)

There is, then, a process of knowledge creation within teaching. Through experimenting and problem-solving in the course of teaching and through representing teaching and reflecting on it, craft knowledge is developed. And this can also incorporate a process of knowledge conversion; by contextualizing and activating scholarly
knowledge within teaching, it can be brought to contribute to the development of craft knowledge.

From a cognitive point of view, professional knowledge is developed as a product of professional action, and it establishes itself through work and performance in the profession, not merely through accumulation of theoretical knowledge, but through the integration, tuning and restructuring of theoretical knowledge to the demands of practical situations and constraints. (Bromme & Tillema, 1995, p. 262)

Moreover, knowledge conversion can proceed in the opposite direction, through eliciting craft knowledge and codifying it. Thus articulated through researching, craft knowledge can be brought to contribute to the further development of scholarly knowledge.

The following sections exemplify these processes of knowledge conversion by considering two projects concerned with the teaching of mathematics at the elementary level. The first illustrates an approach to eliciting and codifying craft knowledge, the second, conversely, an approach to contextualizing and activating scholarly knowledge.

ELICITING AND CODIFYING CRAFT KNOWLEDGE: THE EXAMPLE OF EXPERT DIRECT INSTRUCTION

A program of research that has demonstrated the possibilities of eliciting and codifying the craft knowledge of teachers was conducted by Leinhardt and her associates (Leinhardt, 1988a, 1989; Leinhardt, Putnam, Stein, & Baxter, 1991), employing concepts and methods drawn from a strand of cognitive science research that focuses on the analysis of expertise. Instruction was analyzed by observing teachers in action in the classroom and by interviewing them about their thinking. For example, teachers were invited to organize and classify mathematics problems (Leinhardt & Smith, 1985); or to give an account of their plan for a lesson before teaching and of their handling of particular classroom episodes or lesson segments (as captured on video recordings) after teaching (Leinhardt, 1989).

Teachers were identified as “experts” on the basis of their consistency in producing not just high levels of student achievement but substantial gains in achievement over the school year. Compared with novice teachers, the instruction—and underlying cognition—of these expert teachers was characterized in the following terms:

Expert teachers use many complex cognitive skills, weaving together elegant lessons that are made up of many smaller lesson segments. These segments, in turn, depend on small, socially scripted pieces of behaviour called routines, which teachers... use extensively. Expert teachers also have a rich repertoire of instructional scripts that are updated and revised throughout their personal history of teaching. Teachers are flexible, precise and parsimonious planners. That is, they plan what they need to but not what they already know and do automatically. Experts plan better than novices in the sense of efficiency and in terms of the mental outline from which they operate... From that more global plan... they select an agenda for a lesson.... The agenda serves not only to set up and coordinate the lesson segments but also to lay out the strategy for actually explaining the mathematical topic under consideration. The ensuing explanations are developed from a system of goals and actions that the teacher has for ensuring that the students understand the particular piece of mathematics. (Leinhardt et al., 1991, p. 88)

As analyzed here, then, the expertise of outstanding teachers is many layered. Most readily articulated are the processes of deliberate analysis involved in the
preactive framing of a lesson agenda, in its interactive accomplishment—and adaptation—within the lesson, and in postactive review. Most easily neglected are those largely reflex aspects of action and interaction, exemplified by the classroom routines through which the stability and predictability of classroom activity is produced. Leinhardt suggested that “[t]he importance [of routines] is often overlooked because spontaneity, flexibility and responsiveness are so highly valued in our culture, especially by educators” (1988a, p. 49). Equally, one could conjecture that routines receive less recognition precisely because they have become so reflex for expert teachers, in contrast to those aspects of teaching that command their deliberate attention and continue to exercise them.

This body of work analyzes teachers’ pedagogical knowledge and reasoning in terms of constructs of “script,” “agenda,” and “explanation.” A teacher’s “script” for a particular curricular topic is viewed as a loosely ordered repertoire of goals, tasks, and actions, continually developed and refined over time; it incorporates sequences of action and argumentation, relevant representations and explanations, and markers for anticipated student difficulties. The most important feature of a script is the way in which it acts as an organizing structure, coordinating knowledge of subject and pedagogy with reasoning about actions and goals, hence underpinning the efficient and cohesive planning and development of lessons. Such a script provides a matrix of knowledge supporting the setting of a lesson “agenda”: a mental plan including lesson goals, actions through which these goals can be achieved, expectations about the sequencing of actions through the lesson, and important decision points within the lesson. The agendas of the expert teachers studied by Leinhardt showed more developed instructional logic and smoother flow, and they took more account of students’ actions and reasoning and sought more evidence of these. A crucial element of any script is its “explanation” of each new idea. This involves a teachers’ systematic organization of students’ experiences intended to help them construct a meaningful understanding of the concept or procedure, including appropriate verbalization and demonstration by the teacher—or the management of such contributions from students—in support of this goal. A model was developed of the different elements that contributed to the effectiveness of the explanations of expert teachers: anticipation of prerequisite ideas and skills, motivation of the new idea, specification of its conditions of use, principled legitimation of the new idea, integration of different elements of the explanation, and completion of the explanation.

An unexpected finding of the studies concerned the way in which these expert teachers attended to the thinking of students.

[Teachers] did build models, but in different ways that we had anticipated. . . . Teachers seem to construct flags for themselves that signal material that will cause difficulty as it is being learned, and then they adjust their teaching of the topic in response to those flags or to past successes. . . . They seem to diagnose their teaching and its cycle rather than diagnosing the mental representation of a particular student. A major goal of teaching seems to be to move through a script, making only modest adjustments on line in response to unique student needs.’ (Leinhardt, 1988a, pp. 51–52)

This and other characteristics of the teaching observed have led to the value of the model derived from these studies being questioned: “[O]n at least two points is this model lacking: the mathematics that students are being asked to learn and the lack of attention to individuals” (Fennema & Franke, 1992, p. 159). These two points are seen as related: “Although teachers may be able to achieve short-term computational goals without attending to students’ knowledge, they may need to understand students’ thinking to facilitate students’ growth in understanding and problem solving” (Carpenter, Fennema, Peterson, Chiang, & Loe, 1989, p. 502).
Leinhardt acknowledged the need to study other forms of teaching:

Although our experts have been shown to be responsive and supportive of student efforts to learn key concepts and procedures, the content, method, and direction of their lessons are situated primarily with the teacher. Cognitively based learning theories, however, suggest that it is pedagogically sound and cognitively necessary for students to have a role in determining the method and direction of their own learning. . . . A key feature of [future] studies will be the distinction between explanations that are essentially designed by teachers in advance, and those which students play an active role in constructing during classroom dialogue. (Leinhardt, 1991, p. 111)

Conducting such studies is problematic, however, if teachers have not developed pedagogical models compatible with such cognitively based learning theories. A major limitation inherent in simply studying expert teachers within an established pedagogical system is confinement to that system. The development of qualitatively new forms of pedagogy calls for intervention.

**CONTEXTUALIZING AND ACTIVATING SCHOLARLY KNOWLEDGE: THE EXAMPLE OF COGNITIVELY GUIDED INSTRUCTION**

A program of research into Cognitively Guided Instruction (CGI) (Carpenter et al., 1989; Peterson, Fennema, & Carpenter, 1991) has addressed this issue of how new forms of pedagogy might be developed through contextualizing and activating scholarly knowledge. Its central hypothesis has been that “Research provides detailed knowledge about children’s thinking and problem solving that, if available to teachers, might affect their knowledge of their own students and their planning of instruction” (Carpenter et al., 1989, p. 502). This quotation signals the multiple senses of “knowledge” about “children’s thinking” that it is important to distinguish in considering this body of work. First, there is an important distinction between conceptualizing children’s thinking as a whole, as against gleaning information about the thinking of particular children. There is, then, a further distinction between the many viable conceptualizations, as against the particular conceptual framework proposed by the researchers on the basis of their previous work. The researchers’ model classifies arithmetic word problems and the solution strategies adopted by students, describing progression in thinking in terms of the changing use of particular types of solution strategy in response to particular types of problem.

An early study examined what knowledge experienced teachers already had available to analyze such issues (Carpenter, Fennema, Peterson, & Carey, 1988). Teachers were presented with tasks related to teaching, such as creating a word problem corresponding to a given number equation, assessing the relative difficulty of word problems, and—after viewing particular students solving problems—predicting how they would solve others. Most teachers proved relatively successful on such tasks, particularly those involving the types of problem commonly encountered at the grade level at which they taught. Although many teachers found it difficult to articulate the basis on which they made such judgements, they clearly had developed relevant knowledge. Moreover, the form that this knowledge seems to have reflected the circumstances of their teaching. Teachers appear to have been oriented toward helping students to infer the computation expected through identifying features such as cue words within a problem statement, doubtless influenced by a curricular treatment of word problems in which prototypical situations or stereo-typical verbalizations were associated with particular arithmetic operations or solution strategies.
By contrast, the CGI program was based on the conjecture that organizing classroom activity around less structured problem solving and developing pedagogical strategies to focus attention on the solution strategies devised by students themselves, would prove beneficial to student learning. Consequently, the professional development program associated with CGI aimed to familiarize teachers with the model, as a more powerful means of conceptualizing problem and strategy types and relating these to problem difficulty. Carefully chosen videotaped recordings of individual children solving problems were used as the stimulus for discussions aimed at highlighting key distinctions within the model and at clarifying its use to characterize the mathematical thinking of particular children. Teachers were also encouraged to test out the model by presenting agreed-on problems to children in their own class and recording their solutions for further discussion in workshops.

In addition, teachers were invited to reflect on how the model could be exploited in teaching. Although the program emphasized that it was teachers themselves who were best placed to make informed decisions about how the model should and could be used in their classrooms, the researchers acknowledged their influence on teachers’ thinking about such matters:

We do not believe that we did not influence directly what teachers did in classrooms. The mathematical content we showed and discussed with them was based almost exclusively on word problems. The videotapes were of individual interviewers asking a child to solve word problems, waiting while the child solved the problem, and asking questions such as “How did you get that answer?” or “Could you show me what you did?” Teachers were encouraged to ask children to solve word problems and ascertain how the problems were solved. We did not, however, directly prescribe either pedagogy or curriculum for teachers. (Fennema et al., 1996, pp. 408–409)

The double negative in the opening sentence of this quotation, the distinction between influence at the start and prescribe at the close, both signal the complexities and ambiguities of the line that the researchers were treading in their relationships with teachers.

Indeed, to study the impact of the program on teachers, the researchers developed scales of “cognitively guided beliefs” and “cognitively guided instruction” (Fennema et al., 1996, pp. 412–413). At the lowest point, Level 1, of the instruction scale, the teacher “provides few, if any, opportunities for children to engage in problem solving or to share their thinking”; at the middle point in the scale, Level 3, the teacher “provides opportunities for children to solve problems and share their thinking” and is “beginning to elicit and attend to what children share but doesn’t use what is shared to make instructional decisions”; at the highest point, Level 4-B, the teacher “provides opportunities for children to be involved in a variety of problem-solving activities” and “elicits children’s thinking, attends to children sharing their thinking, and adapts instruction according to what is shared” such that “instruction is driven by teacher’s knowledge about individual children in the classroom.” Although almost all of the participating teachers moved up these scales over the course of the study, the results from the final year show that the program did not lead all teachers to the implied ideal: About half of the teachers lay above the middle point on the belief scale, and a third on the instruction scale. Challenging and changing teachers’ beliefs is often portrayed as providing the impetus for them to rethink teaching approaches and develop new teaching skills. However, this study lends further support to previous investigations suggesting that changes in pedagogy may be rather loosely coupled with, rather than directly induced by, changes in beliefs. The beliefs and instruction of a teacher were not always at the same level, and there was no overall pattern as to whether a teacher was at a higher level in beliefs or instruction. There was also no consistency in whether a change in beliefs preceded a change in instruction or vice versa.
Another study identified a subgroup of teachers who, although well disposed toward the ideas associated with CGI, reported difficulties in employing them in their classrooms (Knapp & Peterson, 1995). The barriers that these teachers cited included lack of time for planning, the absence of a curriculum package to support CGI, organizational difficulties in working with individual students or small groups, pressure to cover material in limited lesson time, characteristics of students, an emphasis on computational skills in standardized tests to be taken by students, and expectations of teachers in the following grade. Although these teachers were working in similar circumstances to others participating in the project, they appear to have been less flexible in adapting to these circumstances. As Knapp and Peterson suggested, these teachers might have benefited from greater opportunity for informal coaching through interaction with the researchers and other participating teachers so as to learn how to circumvent what they saw as obstacles. More specifically, such coaching might have given these teachers access to the craft knowledge through which other participants had found ways to manage similar circumstances.

A further study provides insight into how an exceptional teacher had created a classroom culture in which peer interaction supported high expectations and mathematical reflection, and how she managed the learning of students by varying student groupings, scaffolding problem solving and reflection, and monitoring individual progress. Indeed, the researchers were surprised by some features of her approach:

Ms J did not use knowledge of children’s thinking in the way we had anticipated. Because these problems were organized into a hierarchy of difficulty determined by the reasonably well-defined levels that children move through as they learn to solve the problems, we expected that teachers would use the knowledge more or less as a template to assess what students knew and then to systematically select more difficult problems for the children to solve. The hierarchy of problem types and solution strategies would be used systematically to make both daily and long-term instructional decisions. Ms J did not do what we had anticipated. Although at times she made use of the specifics of the hierarchy, we were unable to identify any systematic way in which she selected problems. Instead, she used the knowledge about problem types to dramatically broaden the scope of her curriculum and her expectations of children. She used all problem types from almost the first week of school, and children in her class had many opportunities to solve all types of problems using whatever solution strategy they chose. (Fennema, Franke, Carpenter, & Carey, 1993, p. 578)

What emerges is a picture of a teacher who had already established a powerful social environment for learning in her classroom and who had been able to contextualize and activate the cognitive model for her purposes in managing that environment: to strengthen her capacity to set challenges for her students and to sharpen her understanding of their responses. This seems to have created a virtuous cycle in which success, judged from her perspective, strengthened motivation to use ideas from the CGI program.

A case study of the CGI collaboration provides evidence that encounters with, and the advocacy of, teachers such as this one had an impact on researchers and on the program. As this quotation indicates, assessments of that impact differ:

[Both principal researchers noted changes due to their interactions with experienced practitioners—changes in the problems they posed, as well as in the classification strategies they used in math instruction—but they did not feel that these changes were significant. Nevertheless, that collaborating with teachers had an effect on the researchers’ project is evident in that decisions such as teaching other operations like multiplication to younger pupils, working with older pupils against the “better judgement” of the research team, or changing the nature of instructional designs all came from suggestions by collaborating teachers. (Huberman, 1999, pp. 297–298)]
CGI provides a particularly fully researched example of a program that has enabled teachers to contextualize and activate scholarly knowledge in their professional work, provoking a corresponding adaptation and development of their craft knowledge. It also offers a further illustration of the way in which interaction between researchers and teachers, between the practices of researching and teaching, can change both groups to some degree.

**ESTABLISHING A DIALOGIC CYCLE: COUPLING THE CONSTRUCTION AND CONVERSION OF SCHOLARLY AND CRAFT KNOWLEDGE**

The ideas developed in the preceding sections point toward a dialogic cycle in which knowledge creation within the practices of researching and teaching become more coordinated, and knowledge conversion from one practice to the other is encouraged. In one phase of this cycle, scholarly knowledge is (re)contextualized and activated within teaching, stimulating (re)construction of craft knowledge. In the complementary phase, craft knowledge is elicited and codified through researching, stimulating (re)construction of scholarly knowledge. In both phases, conversion involves the filtering and reformulating of knowledge: Only certain derivatives of scholarly knowledge will prove capable of being productively incorporated within craft knowledge; equally, only some derivatives of craft knowledge will prove able to be fruitfully appropriated as scholarly knowledge (see Fig. 23.1).

Huberman (1993) has pointed to some of the benefits to researchers and researching of “sustained interaction” with teachers and teaching “in which researchers defend their findings and some practitioners dismiss them, transform them, or use them selectively and strategically in their own settings” (p. 34). Reframing ideas to collaborate successfully with teachers appears to trigger a decentring process amongst researchers. In particular, it creates a need to address the counterexamples, qualifications, and outright challenges that arise as ideas are tested out by teachers and within teaching. In so doing, researchers are obliged to go outside the study at hand, to marshal a broader range of scholarly thinking and research experience related to these ideas, and to bring them to bear on these claimed anomalies. Examples of this decentring, and the resulting learning, have been noted in passing within earlier sections of this chapter.

**FIG. 23.1.** Establishing a dialogic cycle.
The development of both researchers and teachers is supported by disruption of their taken-for-granted world. Huberman (1993) argued that

[O]nce they get beyond the initial discomfort of defining common meanings and of working out the social dynamics of their encounters, each party is bound to be surprised or annoyed or even shaken by some of the information and the reasoning put forth by the other party. Both bodies of knowledge are “valid,” albeit on different grounds, and both are contending for salience and prominence. Were the researchers and [teachers] to remain among themselves, there would probably be far fewer instances of cognitive shifts. (p. 50)

Clearly, then, sustained interaction can also make an important contribution to the professional development of teachers. We need to learn more about approaches to professional development in which ideas, methods, and findings from research are tested by teachers in their own classrooms in terms of the insight they provide into teaching and learning processes and of the support they offer in improving the quality of these processes. Without an appropriate renewal of craft knowledge, however, powerful factors act against change in pedagogy. Given that teachers already possess “a highly efficient collection of heuristics . . . for the solution of very specific problems in teaching,” resistance to change on their part “should not . . . be perceived as a form of stubborn ignorance or authoritarian rigidity but as a response to the consistency of the total situation and a desire to continue to employ expert-like solutions” (Leinhardt, 1988b, p. 146). An essential component of the dialogic knowledge-creation cycle outlined above is development in the craft knowledge of teachers participating in the research. Eliciting and codifying this craft knowledge has the potential to improve the effectiveness with which coaching of other teachers can be undertaken by providing more explicit frameworks for analyzing teaching processes, for articulating mechanisms and functions, and for understanding adaptation to different conditions.

SUMMARY

The emergence of mathematics education as an academic subspecialty has been accompanied by concerns that research in the field has not been as successful as many would wish in generating knowledge to illuminate the practice of teaching. Building more strongly reciprocal working relationships between researching and teaching, between researchers and teachers, is an important way of seeking to address this concern. Closer collaboration and deeper interaction lay bare important differences of purpose and perspective, however. Success in such enterprises depends on developing an approach within which the distinctive practices of teaching and researching can accommodate one another, through the cooperation of teachers and researchers or through the coordination of teacher and researcher roles by teacher-researchers.

The tendency in such collaborations has been to highlight—and privilege—the creation of scholarly knowledge within the practice of researching and its application within the practice of teaching. Yet it transpires not only that the craft knowledge of teachers plays an important part in converting scholarly knowledge into actionable form, but that there is a significant, but also largely tacit, process of knowledge creation within the practice of teaching. Equally, it transpires that research processes can play a valuable part in eliciting and systematizing the craft knowledge of teachers. In this chapter, I have argued that coupling the creation of scholarly knowledge within the practice of researching with the creation of craft knowledge within the practice of teaching makes possible approaches to collaboration between researchers and teachers which can contribute to building a more powerful and systematic knowledge base for teaching. This has pointed to a dialogic cycle through which knowledge creation
within the practices of researching and teaching can be coordinated and knowledge conversion from one practice to the other supported.

This chapter provides a view of these issues primarily as they relate to the practice of research, as seems appropriate to this book. In other papers, however, the author has explored the complementary issues of how the practices of teacher education and of teaching could benefit from making fuller—and more critical—use of scholarly knowledge and research processes, through strategies of “practical theorising” (Ruthven, 2001) and “warranting practice” (Ruthven, 1999). Bringing these perspectives together points to the potential—and challenge—of developing a much higher degree of interactivity between the practices of educational research, classroom teaching, and teacher education than is currently typical.

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CHAPTER 24

Linking Research and Curriculum Development

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Commercially published, traditional textbooks predominate mathematics curriculum materials in U.S. classrooms and to a great extent determine teaching practices (Goodlad, 1984), even in the context of reform efforts (Grant, Peterson, & Shojggreen-Downer, 1996). Various standards (National Council of Teachers of Mathematics [NCTM], 1989) and state and local curriculum frameworks are designed to govern or at least guide these materials. However, publishers attempt to meet the criteria, including scope and sequence requirements, of all such frameworks, and thus the educational vision of any one is, at best, diluted. Moreover, teachers’ reliance on textbooks minimizes any effect of such visions. Thus, a primary cause of the poor performance of U.S. students in mathematics (Kouba et al., 1988; McKnight, Travers, Crosswhite, & Swafford, 1985; Mullis et al., 1997) is the curriculum, both in what topics are treated and how they are treated (Clements & Battista, 1992; Porter, 1989). In one main focus of study, geometry, for example, textbooks are not only ineffective in promoting higher levels of geometric thinking (Fuys, Geddes, & Tischler, 1988), they often hinder this development (Jaime, Chapa, & Gutiérrez, 1992; Mansfield & Happs, 1992).

Why does curriculum development in the United States not improve? One reason is that the vast majority of curriculum development efforts do not follow scientific research procedures (Battista & Clements, 2000; Clements & Battista, 2000). In this chapter, I discuss the nature and relationship of science, research, and curriculum; how curricula are usually developed; and recent alternative research and development models. I discuss one model in depth, providing examples of a nascent curriculum.

SCIENCE, RESEARCH, AND CURRICULUM

Science includes the observation, description, analysis, experimental investigation, and theoretical explanation of phenomena. Scientific knowledge is accepted as more reliable than common-sense knowledge because the way in which it is developed is explicit and repeatable. “Our faith [in it] rests entirely on the certainty of reproducing
or seeing again a certain phenomenon by means of certain well defined acts” (Valéry, 1957, p. 1253, as quoted in Glaserfeld, 1995, p. 117). These acts are the method of science. Scientific method, or research, is disciplined inquiry (Cronbach & Suppes, 1969). Inquiry suggests that the investigation’s goal is answering a specific question. Disciplined suggests that the investigation should be guided by concepts and methods from disciplines (for education, this may include psychology, anthropology, and philosophy, as well as a subject area such as mathematics) and also that it should be in the public view so that the inquiry can be inspected and criticized (Kilpatrick, 1992). The conscious documentation and full reporting of the process distinguishes disciplined inquiry from other sources of opinion and belief (Cronbach & Suppes, 1969).

The research process usually involves recursive phases. Summarizing Maturana’s formulation, Glaserfeld (1995, p. 117) described these as follows.

1. The conditions (constraints) under which the phenomenon is observed must be made explicit (so that the observation can be repeated).
2. A hypothetical mechanism is proposed that could serve as explanation of how the interesting or surprising aspects of the observed phenomenon may arise.
3. From the hypothetical mechanism, a prediction is deduced, concerning an event that has not yet been observed.
4. The scientist then sets out to generate the conditions under which the mechanism should lead to the observation of the predicted event, and these conditions must again be made explicit.

One implication is that science is not conceived as producing the “truth” or a single correct view. It provides reliable ways of dealing with experiences and pursuing and achieving goals (Glaserfeld, 1995). Science involves the process of progressive problem solving and advancement beyond present limits of competence (Scardamalia & Bereiter, 1994). Problem redefinition at increasingly high levels is the goal.

A limitation of this description is that it may promote the misconception that scientists are solitary explorers. Science exists and develops in communities. As one example, scholarly journals, with their editors, editorial boards, reviewers, and contributors, are a significant force in the progressive development of knowledge (Latour, 1987). The unique requirement of these periodicals, compared with others, is that each article must advance scientific knowledge.

Thus, science is valued because it offers reliable, self-correcting, documented, shared knowledge based on research methodology (c.f. Mayer, 2000). Let us stand back for a moment and consider curriculum development. Can any of us in the United States imagine curricula being held to such standards? Is it any wonder that U.S. curricula come and go, with little advancement?

The description of science and research proffered so far suffers yet another limitation. It may promote the view of scientists as engaged only in the altruistic pursuit of knowledge. In contrast, scientific research is social and political (Latour, 1987). Researchers have to garner support for their global perspectives, research issues, individual studies, and even results. Science is not divorced from social–historical movements, values, controversies, competition, and egotism. This perspective buttresses the notion that the goal is not to develop the “best” curriculum. The goal is dynamic progress. However, the perspective ironically also buttresses the argument for research-based curricula. Given that all such factors affect curriculum as well—probably to a much greater degree, particularly in the realm of financial gain—the checks and balances of scientific research are essential to support full disclosure as well as progress.

In summary, I view scientific research as essential but do not promote a simple, causal, deterministic view of science. Our view of science for curriculum development
must involve many views, including comutual influences within a social and political context (c.f. Cobb, 2001). How might this be done? Before we describe projects that have attempted research-based curriculum development, we set the background by defining “curriculum” and elaborating how curricula are usually developed in the United States.

**CURRICULUM DEFINITIONS AND HOW TRADITIONAL CURRICULA ARE DEVELOPED**

There are many definitions of curriculum. Consider the following.

- The ideal curriculum is what experts propound; because it is not firmly grounded in relevant experience, it is fundamentally speculative but important in defining directions for change that should be pursued.
- The available curriculum is the one for which teaching materials exist, although these will not always be matched to the capabilities of all teachers.
- The adopted curriculum is the one which some state or local authority says must be taught.
- The implemented curriculum is what teachers actually teach in the classroom; because teachers vary enormously in their capabilities, there is a wide distribution of implemented curricula.
- The achieved curriculum is what the students actually learn; its distribution is even wider in many variables.
- The tested curriculum is determined by the spectrum of tests that carry public credibility, and through that, influence what happens in classrooms. (Burkhardt, Fraser, & Ridgway, 1986, pp. 5–6)

In this chapter, I use the single word *curriculum* to mean the available curriculum (both traditional and innovative). In this meaning, curriculum is an instructional blueprint and set of materials for guiding students’ acquisition of certain culturally valued concepts, procedures, intellectual dispositions, and ways of reasoning (Battista & Clements, 2000). I use adjectives to discriminate other uses of the term (e.g., some use the standards developed by the NCTM [1989, 2000] to define an “ideal curriculum”). The other definitions of curriculum are considered when relevant.

As previously stated, traditional commercial textbooks dominate mathematics curriculum materials and thus have a large influence on teaching practices (Goodlad, 1984). Textbooks are widely used in classroom instruction and structure 75 to 90% of classroom instruction (Grouws & Cebulla, 2000; Woodward & Elliot, 1990). About two-thirds of teachers report they use textbooks almost every day (Grouws & Cebulla, 2000). According to Ginsburg, Klein, and Starkey (1998), the most influential publishers are a few large conglomerates that usually have profit, rather than the mathematics learning of children, as their main goal. This leads them to painstakingly follow state curriculum frameworks, attempting to meet every objective of every state—especially those that mandate adherence to their framework (permitting the curriculum to be listed as acceptable for purchase by state schools). The publishers’ marketing departments determine content and approach as much as the educators do. The writing team is comprised of an editorial staff, writing staff, and the official authors, who increasingly play merely a consultant role, helping to frame philosophy and approach, but having minimal influence on the precise form and content of the final product.

Publishers are also more concerned that their materials appear to meet national standards and state frameworks than that they actually do so (Ginsburg et al., 1998). Teachers’ appeals for textbooks that are easy to use, along with conservative political
forces, usually contradict ideal curriculum guidelines. “Focus groups” of teachers, for example, frequently emphasize that reform movements are not based “in the real world,” that drill and practice should predominate curricula, and that “good textbooks” are those that get one through mathematics as quickly and effortlessly (for both student and teacher) as possible by supplying simple activities and familiar routines (Ginsburg et al., 1998). Thus, publishers give the appearance of meeting standards and frameworks but actually provide traditional lessons.

The result is that publishers produce an incoherent mix of traditional didactic-presentation-plus-drill pages, pages that are designed to give the appearance of higher order thinking but that often do not. They provide a false sense of innovation. This reveals “the skill of publishers in including materials that appear to support the new aspects of the curriculum that are needed for adoption, presented in such a way as not to embarrass those who wish to continue teaching mathematics the way they have always done it” (Burkhardt et al., 1986, p. 16).

This is an unfortunate situation. The following factors and problems are possible contributors.

Social and Political Forces

Already mentioned, but beyond the scope of this chapter to expatiate on, are the diverse forces that—often misinterpreting the standards vision and reform movements and lacking knowledge of mathematics education—work to block reform and maintain traditional conservative practices. Social support for this position is widely available in the U.S. culture’s instrumentalist views of mathematics and knowledge acquisition as simple transmission (Thompson, 1992). Without scientific research as a guideline and a constraint, available and implemented curricula move toward these conservative, instrumentalist, and transmissive views.

Social and political forces at universities also retard progress. Design has less status than other avenues to scholarship and is thus not valued (Wittmann, 1995).

Rejection of All Published Curriculum

From the other end of the spectrum, some educators, discouraged with the lack of depth of traditional materials, often disparage textbooks, leading to the view that good teachers do not use textbooks. Such an overgeneralization limits the role that constructive curricula may play. (Ball & Cohen, 1996)

Lack of Standards for Curriculum Development

There are no established standards for development, peer review, communication, or professional training for curriculum development. Although good curriculum development can benefit from a variety of perspectives and expertise, establishing no standards is unwise. Because curriculum development is too often viewed as not requiring a substantial research component, such standards are not developed, discussed, and applied.

In addition, there is little reflection on or documentation of the process of curriculum development. I and my colleagues have worked for a variety of curriculum development projects, as “authors” or consultants. We have never seen the development team record the reasons for their decisions or otherwise document their process and progress. When we have suggested that they should, about half, usually for-profit organizations, state that this is not their mission. The other half, usually funded by outside agencies, admit that it would be wise but claim that there is insufficient time or funding to do so. As a community, then, we are left with no structure to support the development and sharing of knowledge.
Limited Involvement of and Communication Between Relevant Parties

Because of this lack of structure and the diverse nature of the people and organizations that develop curricula, designers and teachers have few conversations with one another (Ben-Peretz, 1990; Dow, 1991). This has a negative effect on all aspects of education. The main point for this chapter is that curriculum developers do not have standards that require them to interact with teachers. This is unfortunate, given that individual teachers shape the curriculum in fundamental ways. The result is that the relationship between textbooks and teachers has rarely been taken up with much care or creativity (Ball & Cohen, 1996). Instead, developers tend to assume that curriculum materials can be used almost independently (Dow, 1991).

This also limits the contribution of curriculum materials to professional practice. Such a contribution would be enhanced if the materials were created with attention to processes of curriculum enactment (Ball & Cohen, 1996).

RESEARCH-BASED CURRICULUM DEVELOPMENT

Relationships Between Research and Research-Based Curriculum Development

Our position, then, is that the isolation of curriculum development, classroom teaching, and mathematics educational research deleteriously affects each of these three areas of mathematics education. This is not the same as saying that curriculum development should be research. The goal of scientific research is the creation of knowledge, both theories and empirical data. The goal of curriculum development is the production of instructional materials. Although knowledge is usually created during curriculum development, it is usually not explicated (Gravemeijer, 1994b), placed in the context of scientific theory or an empirical research corpus, reviewed, and shared.

In addition, science is necessary but not sufficient for quality curriculum development.

You make a great, a very great mistake, if you think that psychology, being the science of the mind’s laws, is something from which you can deduce definite programmes and schemes and methods of instruction for immediate classroom use. Psychology is a science, and teaching is an art; and sciences never generate arts directly out of themselves. An intermediary inventive mind must make the application, by using its originality (James, 1958, pp. 23–24). James’s formulation is that there exists scientific knowledge, which an inventive mind applies artfully to create teaching materials or acts. I argue that the art should be integrated into curriculum development and that research is present in all stages of the process, from James’s initial scientific base to formative and summative development of the complex enterprise that is education (Brown, 1992). Research should be integrated even (or especially) into the most creative phases, to achieve the documentation of decisions and the ultimate checking of hunches and full reporting of all procedures (Cronbach & Suppes, 1969).

In sum, my colleagues and I do not propose that curriculum development become research. Rather, we propose fusing the two. Research-based curriculum development efforts can contribute to (a) more effective curriculum materials because the research reveals critical issues for instruction, (b) better understanding of students’ mathematical thinking, and (c) research-based change in mathematics curriculum (Clements, Battista, Sarama, & Swaminathan, 1997; Schoenfeld, 1999). Many curricula claim to be based on research; it is therefore necessary to clarify what we mean by this phrase.
Curriculum development might be “based” on research in a variety of ways. Consider the following possibilities in mathematics education:

1. Broad philosophies, theories, and empirical results on learning and teaching are considered when creating curriculum.
2. Empirical findings on making activities educationally effective—motivating and efficacious—serve as general guidelines for the generation of activities.
3. Research is used to identify mathematics that is developmentally appropriate and interesting to students in the target population.
4. Activities are structured to be consistent with empirically based learning models of children’s thinking and learning.
5. Sets of activities are sequenced according to learning trajectories through the concepts and skills that constitute a domain of mathematics.
6. Activities or activity sets are extensively field-tested from their first inception and early intensive interpretive work, to classroom-based studies, and are revised substantially after each iteration.
7. Summative evaluation studies are conducted, including issues of scalability.

Another mode that is spurious but, I suspect, frequent in practice, should be mentioned for completeness.

8. Following the creation of a curriculum, research results that are ostensibly consistent with it are cited post hoc.

Given this variety of possibilities, claims that a curriculum is based on research should be questioned to reveal the exact nature between the two. Furthermore, to realize the full potential of “research and development” for gaining knowledge, we need to add another process.

9. Each phase of the development process is documented, reflected on, analyzed and reported in the scientific literature.

To avoid miscommunication, I state that research, especially psychological research, has played a substantial role in education. It has been used not so much to produce practical materials for teaching, but to interpret the phenomena of mathematics education (Ginsburg et al., 1998). This role is indirect but important and pervasive. Here, in contrast, I examine how research has been directly applied to curriculum development. Begin with early attempts to base curriculum development on research.

Early Attempts at Basing Curriculum Development on Research: The Research-to-Practice Model

Early efforts to write research-based mathematics curricula often were grounded in the broad philosophies, theories, and empirical results on learning and teaching of general theories. For example, early applications of Piaget’s theories often trained children on Piagetian clinical tasks or incorporated materials directly adapted from those tasks (Forman & Fosnot, 1982). These were not particularly successful. Even detailed analyses of Piagetian research failed to guide curricula development in directly useful ways (Duckworth, 1979). Others have based their educational programs on Piaget’s constructivist foundation. For example, Duckworth encouraged children to “have wonderful ideas” (Duckworth, 1973). Such programs have been arguably more successful, although the interpretations varied widely (Forman, 1993). Indeed, the curricula per se were very different. The broad philosophy and theory, unsurprisingly, leaves much room for interpretation and provides little guidance for curriculum construction.
Even theories that are born in instruction, when used as a general framework, may not be successful. For example, in one study, a curriculum based on the van Hielian theory of levels of geometric thinking, featuring informal experiences before formal arguments, was not better that a traditional approach (Han, 1986).

In summary, the research-to-practice model has a less than successful history (Clements & Battista, 2000; Cobb, 2001; Gravemeijer, 1994b). Based on the notion of a one-way translation of research results to principles to instructional designs, it is flawed in its presumptions, insensitive to changing goals in the subject matter field, unable to contribute to a revision of the theory and knowledge on which it is built, and thus limited in its contribution to either theory or practice.

COMPREHENSIVE RESEARCH AND CURRICULUM DEVELOPMENT EFFORTS

Other recent curriculum development efforts form more complete models, incorporating more of the aforementioned methods of basing curriculum development on research. In this section, I briefly describe several of these efforts in Pre-K–12 education. (This is a select list to reflect the international picture; many laudable efforts have not been included specifically, even if they have contributed to the ideas included here, such as didactical engineering (Artigue, 1994), Hoyles and Noss in the United Kingdom, Griffin and Case, Confrey, in the United States, and others.).

One of the longest standing, comprehensive, and innovative projects is taking place in The Netherlands under the name of Realistic Mathematics Education (RME). The curriculum design process is part of a research approach the authors term “developmental research” (Gravemeijer, 1994b). Developmental research is best described as an integration of design and research. The design of instructional sequences serves as research on an instruction theory. Curriculum development is conceived as purposeful and sensible tinkering, guided by theory and producing theory (Gravemeijer, 1994b).

A team begins the process by conducting an anticipatory thought experiment. They formulate a hypothetical learning trajectory that involves conjectures about both a possible learning route that aims at significant mathematical ideas and a specific means that might be used to support and organize learning along this route. These means of support are construed broadly in three categories: (a) resources, including instructional activities, notational schemes, and the physical and computer-based tools that students might use; (b) the classroom social context, including the general structure of classroom participation and the nature of the specific mathematical discourse; and (c) the teacher’s role in supporting the emergence of increasingly sophisticated mathematical reasoning.

The learning trajectory is conceived of through a thought experiment in which the historical development of mathematics is used as a heuristic; more recently, children’s informal solution strategies also have been used as a source of inspiration. The original design is a set of instructional activities with guidelines suggesting an order for the activities and the learning trajectory or the mental activities in which the students are to engage as they work through the instructional activities. This original design is often not worked out in detail because activities are revised extensively during field testing. That is, the activities that are actually used in the classroom are determined on a day-to-day basis considering what was learned from implementing the preceding activities in the classroom.

In the second phase, the educational experiment, this preliminary design is elaborated, refined, and adjusted in a series of intense cyclic processes of deliberations on and trials of instructional activities (Gravemeijer, 1999). In the third and final phase
of developmental research, the knowledge gained is used to construct an optimal instructional sequence. The goal is to develop and describe the local instruction theory (a more general description of the learning trajectories that emerged in specific classrooms) that underlies this entire instructional sequence and to justify it with both theoretical deliberations and empirical data (Gravemeijer, 1994a, 1994b, 1999). The ideal is that such a local instruction theory will provide a framework that teachers can use to construe hypothetical learning trajectories that fit their own classroom situations.

Recent collaborators with The Netherlands developers (McClain, Cobb, Gravemeijer, & Estes, 1999), Paul Cobb and his colleagues have similar philosophical and curriculum development perspectives (Cobb & McClain, in press). Theirs is likewise a methodological approach in which instructional design serves as a primary setting for the development of theory (Cobb, 2001). Like that of The Netherlands, Cobb’s work posits learning trajectories and frequently conducts classroom tests. The learning trajectories are hypothetical and are revised as needed with each test. The goal is not to “prove” that the initial trajectory is correct or that the original instruction plan is effective, but to improve both by modifying them as required by the daily analyses of students’ thinking and the classroom environment (Cobb, 2001).

Cobb argued that their model’s daily cycle of planning, instruction, and analysis is consistent with the practices of teachers who are skilled in nurturing students’ development of deep mathematical understandings (e.g., Lampert, 1988; Simon, 1995; Stigler & Hiebert, 1999). Therefore, the findings and products of such research and development efforts are immediately applicable to other classrooms.

Results are also applicable to knowledge development in larger domains. This is because, although concerns that arise during an experiment relate directly to the goal of supporting the participating students’ learning, the retrospective analysis of an experiment contributes to the development of instructional theory. This theory, emerging from analyses of the several cycles of teaching and learning, explains the relationships between the two and thus generates grounded generalizations.

Some of their recent changes are especially noteworthy. More than a decade ago, the team followed a standard psychological approach that focused on individual students’ internal mental reasoning. The demands of working in the classroom led them to adopt a perspective more consistent with cultural-historical activity theory (Cobb & McClain, in press). This includes considering the overall goal or motive of children’s activity. For example, in building the structure for activities involving data analysis, a goal was for students to participate in discussions of the data creation process. This gave the data a history for the students, reflecting the purposes for which it was initially created. In addition, the team has developed interpretive frameworks that enable the analysis of students’ learning as it occurs in the social context of the classroom, documenting both the developing reasoning of individual students as they participate in classroom practices and the collective learning of the classroom community over extended periods (Cobb, 2001).

A second recent emphasis is on tools, including computer tools. Their basic design principle is to eschew attempts to “build mathematics into” the tools and instead, to focus on how students use the tools and what they might learn in such activity. Thus, the focus is not on the tool as “carrying” meaning, but nevertheless reflects an increased emphasis on the use of tools (internal, not external, to children’s activity) as compared with earlier work (Cobb, 1995). In this experiment, the authors claimed that the idea of data sets as distributions would not have become a significant part of the classroom discourse if the design of the computer tools had been different.

Instructional planning at this level of detail is unusual in the United States. It is typical in Japan, where members of professional teaching communities often spend several years teaching and revising the hypothesized learning trajectories that underpin a sequence of mathematics lessons (Stigler & Hiebert, 1999).
Japanese educators call these “research lessons”—actual classroom lessons taught to ones’ own students, with a set of unique characteristics (Lewis & Tsuchida, 1998). First, these lessons are observed by other teachers and often outside educators as well. Second, they are carefully planned, usually in collaboration with one or more colleagues. Third, they are designed to implement a certain educational vision, similar to the NCTM standards in the United States, and simultaneously to illustrate a successful approach to teaching a certain topic. Fourth, they are recorded and discussed. Videotapes, audiotapes, narrative and checklist observations, and copies of student work document the lesson. This documentation helps later reflection and discussion, including the faculty that developed the lesson, and, frequently, outside educators and researchers. They are a strong part of professional image and development (Lewis reported that one teacher said, “if we didn’t do research lessons, we wouldn’t be teachers”).

Many research lessons follow general steps, often with a group of a half dozen teachers working through the process together (this account is from Stigler & Hiebert, 1999). First, the instructional problem is defined. This may come from the teachers’ practices or may be posed by the National Ministry of Education and addressed by many groups throughout Japan. It may be a general problem, such as motivating students’ interest in mathematics, or a specific one, such as understanding subtraction with regrouping. In either case, the group focuses the lesson on the problem until it can be addressed by one lesson.

Second, the group plans the lesson. They look at books and articles by other teachers to form a hypothesis and a goal that are used to create an effective lesson and to understand why it was effective. They engage in numerous detailed discussions of the problem with which the lesson would begin, including (a) the exact numbers and wording, (b) the materials students would use, (c) the anticipated solutions and thoughts students might develop, (d) the questions that could promote student thinking, (e) how chalkboard space would be used, (f) how to handle individual differences, and (g) how to end the lesson to advance student understanding. The initial plan is presented at a school meeting to solicit criticism. The critical responses are used to revise the lesson. After several months, the lesson is ready to be implemented.

Third, the lesson is taught. One member of the group teaches it, but everyone in the group helps in its preparation, including gathering materials and role playing the lesson the night before. The group observes the lesson as it is taught the next day. Fourth, the group evaluates the lesson, criticizing weak parts (of the lesson, not the teacher who taught it). Fifth, they revise the lesson, often based on specific student misunderstandings.

Sixth, one member of the group teaches the revised lesson. The audience now includes all the members of the school faculty. Seventh, the entire faculty, and sometimes outside experts, evaluate and reflect on the lesson. The original hypothesis is discussed, as are general issues of teaching and learning that were illuminated by the lesson and its implementation. Eighth, the results are shared. A report is published in book form, for the school and sometimes commercially for the nation. Teachers from other schools are invited to observe the teaching of the final version of the lesson.

Yerushalmy (1997) proposed thinking about technology and functions as the foundation of postarithmetic curriculum. She suggested that the major agenda of algebra teaching should be equipping learners with tools for mathematizing the perception of the situation context and that placing function as a central object of the learning could support this evolution of mathematization. To research such an approach demanded new actions, print, and computer materials; a reformulation of classroom structures and discourse including new roles for both student and teacher; and thus a
professional willing to make a long-term commitment to the project. Yerushalmy
prepared an experimental curriculum, including a full sequence of algebra activities,
innovative software tools, specially designed materials for students and teachers, and
workshops and materials for professional development. She found that it is possible
to begin with any of the external representations of the function—symbolic language,
numerical language, graphical language, and natural language—and then proceed to
any other representation.

Across several projects, Yerushalmy became convinced of the need to document
the reform processes intensely. The changes that emerged during a relatively short
period of time are relevant to educational reforms; however, the complexity of the
learning environment makes it difficult for someone from the outset to understand
and replicate the full process. They closely documented the implementation process
along three themes:

1. Following a single classroom with the same teacher and same students for
   3 years and documenting the learning mainly by writing protocols of weekly obser-
   vations, conversations with the teacher and the students and collecting portfolios of
   students.

2. Conducting a longitudinal study, interviewing 12 pairs of students (of various
   ability levels) from four classrooms (two middle schools of different socioeconomic
   background and two teachers in each) twice a year during 3 years on mathematical
   problem-solving tasks that were not directly addressed in the curriculum but instead
   involved conceptual mathematical thinking.

3. Videotaping and analyzing teaching and learning episodes, mainly those that
   involve nontraditional classroom discourse.

These data are used to reflect on and improve the curriculum materials, which are
still undergoing development and revision.

My colleagues and I have been involved intensely in the development of the Inves-
tigations in Number, Data, and Space curriculum, a K–5, reform-based mathematics
program. Various units of this curriculum illustrate a variety of ways curriculum can
be based on research. Some of the units consciously use scientific models (Battista &
Clements, 2000; Clements & Battista, 2000), resulting not only in research-based
curriculum units (Akers, Battista, Goodrow, Clements, & Sarama, 1997; Battista &
Clements, 1995a, 1995b; Clements, Battista, Akers, Rubin, & Woolley, 1995; Clements,
Battista, Akers, Woolley, et al., 1995; Clements, Russell, Tierney, Battista, & Meredith,
1995) but also in various research publications reporting the results of these efforts
(Battista & Clements, 1996, 1998; Battista, Clements, Arnoff, Battista, & Borrow, 1998;
Clements, Battista, Sarama, & Swaminathan, 1996; Clements et al., 1997; Clements,
Sarama, & Battista, 1996, 1998; Clements, Sarama, Battista, & Swaminathan, 1996). In
contrast, most of the other units were built on knowledge of research on the part of the
developers and informal research in classrooms involving field testing the materials.
They may be as or more effective, but there is less documentation of their effective-
ness. Even worse, there is little or no record of the curriculum development process
from which others might learn and on which others might build.

PRINCIPLES FOR COMPREHENSIVE
RESEARCH-BASED CURRICULUM DEVELOPMENT

From projects such as these (I provided no description of my own work because it is
used to elaborate and illustrate the model I describe in the following section), I abstract
several principles for comprehensive research-based curriculum development.
Create and Maintain Connections Between Research and Curriculum Development as Integrated, Interactive, Processes

A synthesis of curriculum development, classroom teaching, and research in mathematics educational is necessary to contribute both to a better understanding of mathematical thinking, learning, and teaching and to progressive change in mathematics curricula. Without curriculum development projects, rich tasks and authentic settings would be unavailable to researchers. Such projects serve as sources and testing sites of important research ideas. Without concurrent research, the curriculum developers and teachers will miss opportunities to learn about the importance of critical aspects of students’ thinking, and the particular features of software, curricula, and teaching actions that engender mathematical development. We believe that development of research-based curriculum such as that presented here will help ameliorate this critical problem (Clements et al., 1997; Schoenfeld, 1999).

This does not mean that all the varieties of ways to “base” curriculum on research, as previously enumerated, must be employed in every project. It does mean that extant research (and curricula) should be studied and used as a foundation on which to build and that curriculum development needs to proceed linked with its own dynamic research.

What is the nature of this research? Schoenfeld (1999) placed research in a two-by-two matrix, asking whether the researcher seeks fundamental understanding on one dimension and whether the researcher considers the application of the findings on the other. The yes–no cell is pure basic research, the yes–yes is use-inspired basic research, and no–yes is pure applied research. Research such as that we espouse here is placed solidly in the yes–yes cell, seeking fundamental understanding and direct application of the findings.

Use a Broad Range of Scientific Methodologies

Scientific research in mathematics education curriculum development is variegated. Some take the stance of traditional aims of science: explanation, prediction, and control. Others take interpretative and other qualitative perspectives, such as those based on anthropological research (Erickson, 1986; Strauss & Corbin, 1990) and seek to understand the meanings that curriculum would have for teachers and children. Taking the perspective of action research, others examine how to help teachers and children gain autonomy and effectiveness in their teaching and learning endeavors. None of these perspectives is irrelevant to research in the service of curriculum development (Mayer, 2000). They underlie, to different degrees, the enterprise of curriculum development integrated with research, the topic of this chapter.

Use Learning Trajectories

Many of the successful approaches use learning trajectories. Such trajectories are descriptions of children’s thinking and learning of a specific mathematical domain and a conjectured route for that learning to follow through a set of instructional activities (Gravemeijer, 1999; Simon, 1995). The route and activities specify the mental actions in which it is hypothesized students engage as they participate in the instructional activities. Significant also is that there is evidence that superior teachers use

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1Schoenfeld left the no–no cell empty; perhaps this is finally a resting place for research conducted only to complete a dissertation or acquire tenure.
learning trajectories. In one study of a reform-based curriculum, the few teachers that had worthwhile, in-depth discussions saw themselves not as moving through a curriculum, but as helping students move through a progression or range of solution methods—a learning trajectory (Fuson, Carroll, & Drueck, 2000).

**Develop or Use Models of Cognition and Models of Mathematics**

Learning trajectories are often based on specific models of children’s thinking and learning. In addition, the instructional activities frequently use a different types of models, models of mathematics used to support that thinking and learning. Gravemeijer (1999) described how these models in RME undergo a transition in which such a model initially emerges as a model of informal mathematical activity (“model of”) and then gradually develops into a model for more formal mathematical reasoning (“model for”). Both types of models are important; in our approach, the two are actually coordinated and synthesized, which we believe provides additional explanatory and instructional power (Clements & Battista, 2000).

**Use Phases and Cycles of Revisions**

Most of the successful approaches have well-conceptualized phases. Often, a preliminary design is created (including a learning trajectory and correlated set of instructional activities), then elaborated and revised through a series of cyclic empirical field tests, leading to final products that include both an effective curriculum and theoretical and empirical research reports (Clements & Battista, 2000; Gravemeijer, 1999; Simon, 1995). Each phase involves the creation or revision of both instructional activities and psychological and instructional theories. The cyclic alternations of curriculum development and research is often considered more efficient and effective when they are as short as possible (Burkhardt et al., 1986; Clements & Battista, 2000; Cobb, 2001; Cobb & McClain, in press; Gravemeijer, 1994b).

**Maintain Close Connections Between Activities and Children’s Mathematical Thinking**

Throughout the phases, it is critical to maintain direct linkages between the instructional activities and children’s mathematical thinking. In the end, if it does not help other people understand children’s thinking and design better activities to promote it, the work has failed a major research goal.

**Curriculum Also Must Be Informed by Ecological Perspectives, Including Research on Teachers and the Social and Cultural Context**

Curriculum does not stand apart from teachers. Teachers’ knowledge, theories, and belief systems influence their instructional plans, decisions, and actions, including their implementation of curricula. Developers must consider these factors, as well as the classroom social context, including the nature of classroom interactions and roles. How do the developers conceive the teacher’s role in supporting the curriculum as it is realized in the classroom? What are the patterns of participation in which teachers and students engage? What supports do teachers need to realize the vision the curriculum embodies?
Document and Describe the Development, Implementation, and Evaluation Procedures in Detail for Each Phase

Any scientific research carefully documents the procedures used. This requirement is especially intense for research-based curriculum development, when myriad decisions of many types are made on a variety of bases. This documentation is required to maintain the connections between instruction and learning and thus to generate grounded generalizations. To accomplish this and all the previous principles, it is important to have the senior researchers directly involved in all aspects of the research and development (Clements & Battista, 2000; Cobb, 2001).

A MODEL FOR RESEARCH-BASED CURRICULUM AND SOFTWARE DEVELOPMENT

As an example of an approach that embodies these principles, I describe the model for integrated research and curriculum development proposed by my colleagues and, emphasizing the development of software. The model is a modification of our previously presented work (Clements & Battista, 2000), taking into consideration our own recent experience and what we have learned from analyzing similar efforts, as described previously. The description that follows omits elaborated reports of the work, both our own (Battista & Clements, 1991; Clements & Battista, 1991) and that of others (Biddlecomb, 1994; Olive, 1996; Steffe & Wiegel, 1994), from which the original model was abstracted; see the original (Clements & Battista, 2000) for these extensive, concrete, phase-by-phase illustrations from these works.

This design model is based on the notion that the state of the art is such that currently we have models of teaching and learning mathematics with sufficient explanatory power to permit design to grow concurrently with the refinement of these models. Thus, curriculum and software design can and should interact with the ongoing development of theory and research, reaching toward the ideal of testing a theory by testing the software. Capitalizing fully on both research and curriculum development opportunities requires the maintenance of explicit connections between these two domains and formative research with users throughout the development process (c.f. Laurillard & Taylor, 1994). It is essential that the entire process be carefully documented. Ideal but difficult is the inclusion of a third-party researcher to study the team as it works and to serve as an external auditor for the team’s research (Lincoln & Guba, 1985). At the very least, the design team needs to document all their decisions and their reasons for these decisions.

Our design model is specific for the instructional use of research-based micro-worlds. With minor revisions, however, it would be applicable to most software based on similar cognitive perspectives. The model moves through phases in a sequence that is as much recursive as linear. The methodologies are complex and interwoven.

Phase 1: Draft the Initial Goals

The first phase begins with the identification of a domain of mathematics that is deemed significant in two ways. First, the learning of the domain would make a substantive contribution to students’ mathematical development. Second, learning about students’ mathematical activity in the domain would make a similar contribution to research and theory.

The first implies both that the domain should play a central role in mathematics per se and that the concepts and procedures of the domain are generative in
students’ development of mathematical understanding. From the beginning, then, there is involvement of a diverse set of experts, including mathematicians, mathematics educators (teachers and researchers), and cognitive psychologists. There is also a presage of the enormity of the challenge for the research community; for example, the generativity criterion requires extensive longitudinal work.

The second appears straightforward, involving the identification of lacuna in research and theory. Here, too, however, there is a harbinger to consider: Although the developers identify a mathematical domain such as motion geometry or addition, the students’ mathematical activity has to be understood from the perspective of the students, which may be distinctly different. Drafting the initial goals includes adults’ conceptions of the domain and also presages developing a model of the concepts and strategies of students as they engage in activities that could be called mathematical.

In establishing mathematical learning goals, study of reform recommendations (National Council of Teachers of Mathematics, 1989, 2000; National Research Council, 1989) and equity issues is recommended. Equity must be addressed throughout the phases. As just one example, thought should be given to the students who are envisioned as users and who participate in field tests; a convenience sample is usually inappropriate. Systemic classroom and home participation patterns and sociocultural issues should be considered as well (Cobb, 2001).

For software developers, such broader perspectives provide balance to the seduction of computer environments as (micro)worlds unto themselves. Nevertheless, a focus on technology can be advantageous because reflecting on the actions and activities that are enabled by a new technology can catalyze a reconceptualization of the nature and the content of the mathematics that could and should be learned. The flexibility of computer technologies can generate visions less hampered by the limitations of traditional materials and pedagogical approaches (cf. Confrey, 1996). For example, computer-based communication can extend the model for science and mathematical learning beyond the classroom, and computers can allow external representations and actions not possible with other media. These actions anticipate Phase 3 and illustrate the nonlinear nature of the design model.

We take an example from our NSF-funded project, Building Blocks—Foundations for Mathematical Thinking, Pre-Kindergarten to Grade 2: Research-Based Materials Development (Clements & Sarama, 1998). One of the domains from the geometry and spatial sense line is composing geometric forms. The basic competence is combining shapes to produce composite shapes. This is one section of the larger composing—decomposing trajectory in geometry (other trajectories include shapes and their properties, transformations—congruence, and measurement). This domain is significant in that the concepts and actions of creating and then iterating units and higher order units in the context of constructing patterns, measuring, and computing are established bases for mathematical understanding and analysis (Clements et al., 1997; Reynolds & Wheatley, 1996; Steffe & Cobb, 1988). There is a lack of research on the trajectories students might follow in the geometric domain, however.

The product of this first phase is a description of a problematic aspect of mathematics. This description should be quite detailed.

**Phase 2: Build an Explicit Model of Students’ Knowledge Including Hypothesized Learning Trajectories**

Developers build a cognitive model of students’ learning that is sufficiently explicit to describe the processes involved in the construction of the goal mathematics. Extant models may be available, although they vary in degree of specificity. Especially when details are lacking, developers use clinical interviews and observations to examine students’ knowledge of the content domain, including conceptions, strategies, intuitive
ideas, and informal strategies used to solve problems. In these experiments, the teacher
tries to set up a situation or task that will elicit pertinent concepts and processes. Once a
(static) model has been partially developed, it is tested and extended with exploratory
teaching (Steffe, Thompson, & Glaserfeld, 2000).

These cognitive models are synthesized into hypothesized learning trajectories
(Cobb & McClain, in press; Gravemeijer, 1999; Simon, 1995). These trajectories ulti-
mately include “the learning goal, the learning activities, and the thinking and learning
in which the students might engage” (Simon, 1995, p. 133). Unlike other approaches
(Gravemeijer, 1994b), we believe that existing research should be a primary means of
constructing the first draft of these learning trajectories (which may, in turn, amelio-
rate the difficulty many development teams appear to have incorporating the research
of others).

As an example, our synthesis of research for the Building Blocks project posits the
following developmental sequence in strategies. The basic structure of this sequence
was determined by observations made in the context of early research (Sarama,
Clements, & Vukelic, 1996) and were later refined through a research review and
a series of clinical interviews and focused observations by research staff and teachers
(Clements, Sarama, & Wilson, 2001).

1. Manipulates shapes as individuals, but is unable to combine them to compose
   a larger shape.
2. Piece Assembler. Similar to Step 1, but can concatenate shapes to form pictures.
   Each shape represents a unique role, or function in the picture. Can fill simple frames
   using trial and error (Mansfield & Scott, 1990; Sales, 1994). Uses turns or flips to
do so, but again by trial and error; cannot use motions to see shapes from different
   perspectives (Sarama et al., 1996). Thus, children at Steps 1 and 2 view shapes only as
   wholes and see no geometric relationship between shapes or between parts of shapes
   (i.e., a property of the shape).
3. Picture Composer. Matches shapes using gestalt configuration or one component
   such as side length (Sarama et al., 1996). If several sides of the existing arrangement
form a partial boundary of a shape (instantiating a schema for it), the child can find and
places that shape. If such cues are not present, the child matches by a side length. The
child may attempt to match corners but does not possess angle as a quantitative entity,
so will try to match shapes into corners of existing arrangements in which their angles
do not fit. Rotating and flipping are used, usually by trial and error, to try different
arrangements (a “picking and discarding” strategy). Thus, there is intentionality and
anticipation (“I know what will fit”), based on shapes’ components.
4. Shape Composer. Matches shapes using angles as well as side lengths. Eventu-
ally considers several alternative shapes with angles equal to the existing arrangement.
Rotation and flipping are used intentionally (and mentally, i.e., with anticipation) to
select and place shapes (Sarama et al., 1996). Can fill complex frames (Sales, 1994) or
cover regions (Mansfield & Scott, 1990). Is beginning to form substitution relationships
among shapes (e.g., two pattern block trapezoids make a hexagon).
5. Substitution Composer. Forms composite units of shapes by trial and error
(Clements et al., 1997). May combine these composite units by simple duplication.
6. Shape Composite Iterator. Constructs and operates on composite units intention-
ally (i.e., children conceptualize each unit as being constituted of multiple singletons
and as being one higher order unit). Can continue a pattern of shapes that leads to a
“good covering,” but without coordinating units of units.
7. Shape Composer with Superordinate Units. Builds and applies units of units
(superordinate units). For example, in constructing spatial patterns, children extend
their patterning activity to create a tiling with a new unit shape—a (higher order) unit
of unit shapes that they recognize and consciously construct.
There may be a misunderstanding of the role of such learning trajectories similar to a misinterpretation of the results of teaching experiments. That is, some believe that experiments that involve a small number of children are not applicable to classrooms. However, from such work we can take the cognitive models, learning trajectories, and potential activities—these can be realized within curricula and again within separate classrooms. The general framework guides such realization, but multiple trajectories and activity sequences are tailored to each situation. This is important to the design of curricula both for students and for teachers (Ball & Cohen, 1996).

The end result of this phase is an explicit cognitive model of students’ learning of mathematics in the target domain. Ideally, such models specify knowledge structures, the development of these structures, mechanisms or processes of development, and trajectories that specify hypothetical routes that children might take in learning the mathematics.

**Phase 3: Create an Initial Design for Software and Activities**

Based on the model of students’ learning generated in Phase 2, developers create a basic design to describe the objects that will constitute the software environment and the actions that may be performed on these objects. These actions on objects should mirror the hypothesized mathematical activity of students.

Continuing the *Building Blocks* example, we wish to allow students to work with shapes and composite shapes as objects. We wish them to act on these objects—to create, duplicate, position (with geometric motions), combine, and break apart both individual shapes (units) and composite shapes (units). Offering students such objects and actions on these objects is consistent with the Vygotskian theory that mediation by tools and signs is critical in the development of human cognition (Steffe & Tzur, 1994). Furthermore, designs based on objects and actions force the developer to focus on explicit actions or processes and what they will mean to the students. This characteristic mirrors the benefit attributed to cognitive science models of human thinking; they did not allow “black boxes” to hide weaknesses in the theory.

Designs are not determined fully by this line of reasoning. Intuition and the art of teaching (Hiebert, 1999; James, 1958) play critical roles in the design of the objects and actions, as well as the activities, to which we now turn.

Thanks to recent technological developments, even children with physical and emotional disabilities can use a computer with ease—if the designers plan for it. Developers should plan for the adaptations the software will need for people with disabilities (e.g., hearing: adjustable volume and register for all speech, simplified captions and visual animation by the agent; visual: high contrast versions of all screens; physical: key press and single switch access); here I emphasize that the environment and activities be designed based on research on specific effective interventions for learning disabled and retarded children (e.g., Baroody, 1996; Kameenui & Carnine, 1998; Mastropieri, Scruggs, & Shiah, 1991; Swanson & Hoskyn, 1998).

The developers next create a sequence of instructional activities that use objects and actions to move students through the hypothesized learning trajectories. They review the professional literature, from reform recommendations to activities from the literature, as well as their own experiences, to create activities. They consider the unique potential of technology for providing cognitive tools, “concrete mathematics,” and “situated abstractions” (Clements, 1994, 2000; Hoyles, 1993). They also seek extensive advice from teachers.

Given the importance yet paucity of student-designed projects in mathematics education and the support that the computer can offer such projects (Clements, 2000), provision for such self-motivated, self-maintained work should be considered.
Open-ended activities using the objects and actions should therefore be a part of the design so that the software environment can be a setting in which students think creatively. Design activity on the part of students is frequently the best way for that to happen. In this, as well as the other activities, developers, teachers, and students should not be constrained by the scientifically based trajectory.

Specific assessment and teaching strategies should be included as part of the plan (c.f. Hoyles & Noss, 1992). Teachers should be encouraged to go beyond the activities and help students to use the environment not just as a “model of” informal mathematical activity but eventually as a “model for” investigating other situations and “esoteric” mathematical problems and relationships.

Returning to the Building Blocks example, we initially created a sequence of activities aligned with the learning trajectory. An essential task is combining shapes to produce composite shapes (e.g., to fill a frame or create an imagined design or picture). Research shows this type of activity to be motivating for young children (Sales, 1994; Sarama et al., 1996). The setting for such tasks will be constantly changing (making pictures, fixing “broken” objects which “work” or are animated when fixed, completing jigsaw-like puzzles with pictures, completing wallpaper patterns or “floor tilings,” etc.). For the purposes of brief illustration of the essential features, only the mathematically significant basic elements are described in the following (furthermore, most activities allow for open-ended projects using the objects and actions).

1. **Piece Assembler.** Child completes a picture given a frame that suggests the placement of the shapes, each of which plays a separate semantic role in the picture and that requires no flips or turns.

2. **Picture Composer.** Child completes a picture given a frame that suggests the placement of the individual shapes but in which several shapes together may play a single semantic role in the picture. As the child succeeds, she is given pictures that include such combinations more frequently and that require applying (small) turn actions to the shapes (note: the computer environment helps bring this action to an explicit level of awareness because the child must consciously choose the turn tool and because sound effects and speech are used to explicate the turning action). The child is challenged to fill an open region and is provided shapes in which matching side lengths is a useful strategy.

3. **Shape Composer.** The child must use given shapes to completely fill a region that consists of multiple corners, requiring selecting and placing shapes to match angles. Later tasks challenge children to fill complex frames or regions in which shape placement is ill defined, allowing for multiple solutions. These tasks require use of turning and flipping and eventually the discrimination of these.
4. **Substitution Composer.** The child is challenged to find as many different ways as possible to fill in a frame or region, emphasizing substitution relationships (as the child is doing to the hexagons) and angle equivalence.

<table>
<thead>
<tr>
<th>![Hexagon Frame]</th>
<th>![Substitution Example]</th>
</tr>
</thead>
</table>

5. **Shape Composite Iterator.** The child works in a toy factory, learning to use the glue and duplicate tools to make several copies of the same (composite) toy. The child then completes a toy puzzle (made completely from multiple copies of a tetromino) using the glue, duplicate, and “do it again” tools to make and iterate composite units in filling space.

<table>
<thead>
<tr>
<th>![Tetromino Puzzles]</th>
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6. **Shape Composer with Superordinate Units.** The child covers regions by building superordinate units of tetrominoes with the glue tool that are then duplicated, slid, turned, and flipped, and iterated systematically to tile the plane. For example, she might fill the rectangle at the right with a strategy that combines four “T” tetrominoes into a superordinate square.

<table>
<thead>
<tr>
<th>![Tetromino Superordinate Units]</th>
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**Phase 4: Investigate the Components**

This phase is especially interwoven with the previous one. Components of the software are tested using clinical interviews and observations of a small number of students. A critical issue concerns how children interpret and understand the screen design, objects, and actions. A mix of model (or hypothesis) testing and model generation (e.g., a microethnographic approach, see Spradley, 1979) are used to understand the meaning that students give to the objects and actions. To accomplish this, developers may use paper or physical material mock-ups of the software or early prototype versions.

In this and the next phase, communication between the developer and programmer is essential. In most of our work, the same people conduct design, programming, and research. If programming is carried out separately, full communication about all of the aspects (e.g., goals, actions, objects, aesthetics, etc.) should be ensured.

A small example from the *Building Blocks* project is our research on children’s initial interpretation of the actions that each icon might engender. For the decomposition of units, we had created a hammer icon. Children did not interpret this tool as breaking things apart, even with minor prompts, but as “nailing down” items (“It will hammer the shapes down harder”) or “hammering it off” the paper or screen. We therefore are testing new icons.
Phase 5: Assess Prototypes and Curriculum

The developers continue to evaluate the prototype, rendered in a more complete form. A major goal is to test hypotheses concerning features of the computer environment that are designed to correspond to students’ thinking. Do their actions on the objects substantiate the actions of the researcher’s mental model of children’s mathematical activity? If not, should the mental model or the way in which this model is instantiated in the software be changed? Do students use the tools to perform the actions, either spontaneously or with prompting? If the latter, what type is successful? In all cases, are students actions-on-objects enactments of their cognitive operations (Steffe & Wiegel, 1994), and as models of informal mathematical activity (c.f. Gravemeijer, 1999), in the way the model posits, or merely trial and error or random manipulation.

Similarly, the developers test the learning trajectories and adjust them as needed. Teaching experiments are used initially. Often, a free exploration phase precedes the introduction of activities. In addition, the developer interprets the children’s contributions, and new tasks or questions are posed. Students’ responses may indicate a need—or, more positively stated, an opportunity—to change the cognitive model, software environment, trajectories, and activities. Some activities and teaching strategies emerge from, and are mutually constituted by, the developer-teacher and the student in the software context. Thus, empirical data may be garnered from the interactions of the students with the software, the activities (writ large), their peers, the teacher-developer, and combinations of these. In addition, responses and advice of teachers playing the role of students are sought.

Throughout this testing period, the ironic goal is to “fail often”; that is, to find gaps or inaccuracies in the cognitive model, the learning trajectories, and the activities, and adjust them through intensive and extensive cycles of testing and reflection. Indeed, this is the most iterative research-design phase; sometimes evaluation and redesign may cycle in quick succession, often as much as every 24 hours (Burkhardt et al., 1986; Char, 1990; Clements & Sarama, 1995; Cobb, 2001). In this way, the computer environment is modified in ways not originally anticipated to fine tune, correct problems, check speed, and add functions as additional needs become known. Similarly, the cognitive model, learning trajectories, and activities are revised. Activities may be completed reconstituted, with edited or newly created ones tried the next day.

Finally, using the cognitive model and learning trajectories as guides and the software and activities as catalysts, the developer creates more refined models of particular students. Simultaneously, the developer describes what elements of the teaching and learning environment were observed as having contributed to student learning. The theoretical model may involve disequilibrium, modeling, internalization of social processes, practice, and combinations of these and other processes. The connection of these processes with specific environmental characteristics and teaching strategies and student learning is critical.

This brings us to another critical point. Many pedagogical issues can and should be addressed in this and the following phases. Space limitations prevent listing them all here, but developers cannot ignore them. Perhaps most important, the focus in this chapter is on the development of software programs, which should not be misinterpreted as designating less importance to the social ecology. Furthermore, the computer tools themselves can contribute to discourse and communication in several ways. For example, students might produce new information in the form of “notes” and enter them into a database that the whole class shares (Scardamalia & Bereiter, 1992). Web-based communication is another palpable avenue with myriad possibilities. In this way, classroom discourse and classroom activity structures (Cobb & McClain, in
press) are considered when planning and assessing the prototype; however, they are developed more completely in the following phases.\footnote{We use the term \textit{classroom}, but it should be noted that there are many types of situations. For example, especially for preschoolers, our software and curriculum are used in day-care and home settings, among others. Often there is little (directly educational) social setting available to the child; although not ideal, considering such situations in the design of educational materials is both more accurate and equitable than the zeitgeist of positing and relying on specific classroom social interactions.}

With so many research and development processes happening, and so many possibilities, extensive documentation is vital. Videotapes (for later microgenetic analysis), audiotapes, and field notes are collected. Computers might store data documenting students’ ongoing activity. Solution-path recording is a particularly useful technique (Gerber, Semmel, & Semmel, 1994; Lesh, 1990). Solution paths can be reexecuted and examined by the teacher, student, or researcher (and analyzed in many ways); they also can be modified. Issues such as the efficiency, simplicity, and elegance of particular solutions—even those that result in the same answer—can be assessed (Lesh, 1990). Techniques such as videorecording a mix of two inputs, traditional camera video, and computer screen output serve similar purposes. This documentation also should be used also to evaluate and reflect on those components of the design that were based on intuition, aesthetics, and subconscious beliefs.

In the \textit{Building Blocks} project, we are just beginning this stage at the time of this writing. For example, we have tested the tools for composition and decomposition. We have found that children using the computer tools develop compositional imagery. Mitchell started making a hexagon out of triangles. After placing two, he counted with his finger on the screen around the center of the incomplete hexagon, imaging the other triangles. He announced that he will need four more. After placing the next one, he said, “Whoa! Now, three more!” Whereas off-computer, Mitchell had to check each placement with a physical hexagon, the intentional and deliberate actions on the computer lead him to form images (decomposing the hexagon mentally) and predict each succeeding placement.

As a second example, consider Alyssa, whose work is illustrated in the first picture of Step 4 of Phase 3. As Alyssa fills the hexagons, she evinces understanding of both anticipatory use of geometric motions and substitution relationships and therefore notions of area, equivalence, and congruence. The second activity challenges Alyssa to finish covering a wall with wallpaper. She is partially successful, but the developer records that she sometimes flounders. When she slides a square near a 60° corner, the developer suggests, “Look at the corners.” By the end of this activity, Alyssa showed that she could completely fill a region and therefore understood covering a plane and did so by matching angles (Alyssa slides a square and later turns and slides the large angle of the rhombus into the angles illustrated). Based on these assessments, the developer decided to move to activity Level 5 for Alyssa’s next session.

Phase 6: Conduct Pilot Tests in a Classroom

Teachers are involved in all phases of the design model. Starting with this phase, a special emphasis is placed on the process of curricular enactment (Ball & Cohen, 1996). Curriculum materials should help teachers interpret students’ thinking about the activities and the mathematics content they are designed to teach; support teachers’ learning of that content, especially which is probably new to teachers; and provide guidance regarding the external representations of content that the materials use (Ball & Cohen, 1996).

There are two research thrusts. First, teaching experiments continue, but in a different form. We conduct classroom-based teaching experiments (including what
we previously called interpretive case studies) with one or two children. The goal is making sense of the curricular activities as they were experienced by individual students (Gravemeijer, 1994a). Such classroom-based teaching experiments serve similar research purposes as traditional teaching experiments but are conducted in a naturalistic classroom setting. Videotapes and extensive field notes are required so that students’ performance can be examined repeatedly for evidence of their interpretations and learning. Developers evaluate whether the objects-and-actions serve not just as a “model of” informal mathematical activity, but also develop into a “model for” more formal mathematical reasoning (Gravemeijer, 1999).

Second and simultaneously, the entire class is observed for information concerning the usability and effectiveness of the software and curriculum. Ethnographic participant observation is used heavily because we wish to research the teacher and students as they construct new types of classroom cultures and interactions together (Spradley, 1980). Thus, the focus is on how the materials are used and how the teacher guides students through the activities (for our preschool materials, child-care providers, and parents are also involved; class dynamics cannot be taken as a given). Attention is given to how software experiences reinforce, complement, and extend learning experiences with manipulatives or print (Char, 1989) as well as the diversity in the practices of students’ homes.

This pilot test phase usually involves teachers working closely with the developers. The class is taught either by a team including one of the developers and the teacher or by a teacher familiar with and intensively involved in curricula development.

**Phase 7: Conduct Field Tests in Multiple Classrooms**

We gradually expand the range of size and scope of our studies (Burkhardt et al., 1986), from studies of students’ learning (1 to 10 students) to studies of different kinds of teaching and their effects on student learning (10 to 100 students) to studies of what can actually be achieved with typical teachers under realistic circumstances (100 to 1,000 students; note that our model does not include the most general level, curriculum change on a large scale (10,000 to 10,000,000 students). These field tests are conducted with teachers not initially connected intimately with development.

In several classrooms, the entire class is observed for information about the effectiveness and usability of the software and the curriculum, but more emphasis is placed on the usability by such teachers. There is too little research done at this level. Innovative materials too often provide less support than the text books with which teachers are accustomed, even when they are teaching familiar material (Burkhardt et al., 1986). We need to understand what the curriculum should include to fully support teachers of all levels of experience and enthusiasm for adopting the new curriculum. Thus, in this phase we wish to know whether the software and its supporting materials are flexible enough to support multiple situations (e.g., variation in the number of computers available), various modes of instruction (e.g., demonstration to a class, class discussion, small-group work), and different modes and styles of management (e.g., how teachers track students’ progress while using the materials, monitor students’ problem solving with the materials, and assess students’ learning). Another question is whether the materials support teachers if they desire to delve more deeply into their students’ thinking and teach differently, such as consistent with the vision of the NCTM standards. Can teachers adapt the materials to their own vision? Again, ethnographic research (Spradley, 1979, 1980) is important, especially because teachers may agree with the curriculum’s goals and approach but their implementation of these may not be veridical to the developers’ vision (Sarama, Clements, & Henry, 1998), and in this phase, we need to determine the meaning that the various
curricular materials have for both teachers and students. In addition, of course, the final field tests may include summative evaluations.

To supplement these data, two additional types of data are collected from numerous other classrooms, many of which are located some distance from the developers. First, surveys of all teacher participants can be used to compare data collected before and after they have used the software. Simultaneously with these surveys, developers analyze data collected by the computer and from paper-and-pencil assessments produced by all participating children. Such data are important in generating political and public support for any innovative materials. Such research is more similar to, but still differs from, traditional summative evaluation. In this design model, a theoretical framework is essential; comparison of scores outside of such a framework, permitted in traditional curriculum evaluation, is inadequate.

The combined interpretive and survey data address whether such supports are viewed as helpful by teachers and other caretakers and whether their teaching practices have been influenced. Do before-and-after comparisons indicate that they have learned about children’s thinking in specific mathematical domains and adopted new teaching practices? Have they changed previous approaches to teaching and assessment of mathematics?

Our hypothesis is that our software developments will have significant positive influence on teachers, thus addressing numerous issues such as equity (adults in diverse settings and possessing various levels of educational background receive research-based, and thus quality, prompts that will help them guide children’s learning), assessment (teachers will have authentic, observational assessment information), and thus scalability (reaches a diverse population and also permits data collection on a wide scale). Moreover, our ideal is that teachers build new activities from the software environments, take control of their curriculum, and develop “research lessons” of their own. Certainly, creating, or at least documenting specific concerns for, materials for professional development should be well underway by this phase.

The first seven phases provide a comprehensive approach to obtaining both advice from users and significant research data. Not every project can or should employ each phase; however, the reasons for omitting any phase and the coherence of the phases that are to be included must be considered and documented.

Phase 8: Recurse

Not really a distinct process so much as a reminder, this phase involves iterative and recursive actions within and between phases. The intensive and extensive cycles of design and analysis and evaluation are critical to the success of both curriculum and research. There are three types of cycles: daily revisions of software environment and activities, longer cycles encompassing an entire learning trajectory or curriculum, and cycles that operate across projects. Substantive progress is often made when a complete project (in our case, Battista & Clements, 1991; Clements & Battista, 1991) is revisited, refined, reconceptualized, and reborn in similar (Clements & Meredith, 1994; Clements & Sarama, 1995) or different (Clements & Sarama, 1998) forms.

Phase 9: Publish

The software and curricula may be disseminated through a variety of channels, from commercial publishers to the Internet. As simple as this seems, this phase is not unproblematic for both curriculum development and research.

On the curriculum side, negotiations and cooperation with a commercial publisher can have a substantive influence on the final software and print materials. The demands on, and of, publishers, were detailed in a previous section. Suffice it to say that
these same pressures are exerted on any curriculum that is commercially published. In addition, multimedia-based materials often require even more support and cooperation from publishers. Therefore, there may be less freedom for developers to publish their own version of their materials. These pressures often are exerted regardless of the research base for the materials, resulting in software, originally designed to support in-depth problem solving and student evaluation of mathematical strategies and products, to shift toward activities characterized by simpler problems and feedback.

On the research side there are constraints to publication. Many interesting pieces of software have been created; however, the expertise developed during the production of that software has not been disseminated. Whether this is because resources are exhausted (finances, time, and emotional energy) or because there is no interest, nonpublication has a strong deleterious effect on the field of curricula development and research.

**CONCLUSIONS**

Commercially published textbooks strongly influence teaching practices in traditional and reform classrooms (Goodlad, 1984; Grant et al., 1996). They constitute an essential resource for many teachers. Although their influence is usually conservative, it is not reasonable to expect teachers to teach well without mathematics curriculum materials. Furthermore, such materials can play positive roles in teaching and reform (Ball & Cohen, 1996; Sosniak & Stodolsky, 1993). This chapter is based on the view that these materials should be based on scientific research. To that end, we have discussed the nature and relationship of science, research, and curriculum and described several models for linking research and curriculum development.

Those implementing such a model assume a responsibility to describe the details of their theoretical and empirical foundations and their design and to conduct the research deemed necessary not only to see if the design is successful, but also to trace whether that success can be attributed to the posited, theory-design connections. Realizing the full potential of both the research and the curricula development opportunities requires consistent, coherent, formative research using multiple methodologies. Some have been discussed, among them clinical interviews, protocol analyses of students’ problem solving, classroom observations, and interviews with teachers, students, and administrators. Others, such as paired teachers’ observations, students’ immediate retrospective reports of their strategies, performance assessments, portfolio development, and content analyses of students’ work, may be more suitable in certain situations. In any case, repeated intensive investigations are required.

I offer several caveats and suggestions.

1. Developers must consider that criteria for success should also include the vision and theory that underlie the curriculum. For example, the following might be evaluated: students’ mathematical power and their opportunities to develop it through creation of their own strategies; students’ expression and communication of their mathematical thinking; and students’ beliefs and attitudes toward mathematics and the tools and resources provided them to do mathematics. Similarly, the ways the curriculum is understood, adopted, and adapted by teachers, as well as the way it affects their beliefs, attitudes, knowledge and future practice, are all relevant. All of these findings inform the curriculum development process and the research base.

2. Technology and its use in our culture are changing rapidly. Designs, research questions, and methodologies should remain sensitive to new possibilities. Research indicates that technological “bells and whistles” should not become a central concern, however: Although they can affect motivation, they rarely emerge as critical
to children’s learning. Instead, the critical feature is the degree to which the computer environment successfully implemented education principles born from specific research on the teaching and learning of specific mathematical topics (Sarama, 2000).

3. Although this model offers comprehensive, rich data collection, the diversity of the methodologies employed could lead to incoherence and confusion between theoretical assumptions. Constant reflection and checks are ever more important in models such as this one. We must consistently ask questions such as the following: What are we attempting to learn? What evidence would convince us? What types of circularity in our design and research work might lead to spurious conclusions?

4. Along a similar vein, our theoretical models and software—to an extent, instantiations of these models—may funnel our perceptions and conceptions. Testing or refining our theories by testing or refining our software has significant advantages: We make our theories more explicit, and we extend our visions of what students can do mathematically. Given the emotional investment in such a complex process, however, we must take precautions that our work does not contain self-gratifying, self-fulfilling circularities.

5. The model and examples described here emphasize one class of effective software (one that is rooted in certain constructivist assumptions). The developmental model would need to be modified for other classes, such as intelligent tutorials with microadaptation assessment. The basic goals and procedures could be quite similar, however.

6. Subtle differences in activities can enhance or sabotage the principles. The basic research principles must be refined and especially elaborated by ongoing research and development work that tracks the effectiveness of every specific implementation. This means that research cannot be considered only something upon which curriculum and software development are a priori based. Research must also be conducted throughout the development process.

**IMPLICATIONS**

In this final section, we briefly describe some of the ramifications of this chapter’s arguments.

**Curriculum Developers Must Accept New Responsibilities**

The most direct implication of this chapter is that curriculum developers must accept new responsibilities. The models described herein make daunting demands. Curriculum developers must expand their knowledge to include scientific research procedures and ideas—and a wide range at that. They must consider issues of mathematics, psychology, instruction, and implementation in turn (Gravemeijer, 1994b). In our vision, curriculum development is painted as an extremely creative, complex enterprise in which multiple demands must be met and multiple resources used.

**Developers Should Study All Research**

The position taken here is that theoretical purity is less important than a consideration of all relevant theories and empirical work. The complexity of the field often creates a Babel of disciplines (Latour, 1987) in which the lack of communication prevents progress. This is one conceit curriculum developers can ill afford. Instead, they must meld academic issues and practical teaching demands no less than a serious consideration of what researchers and teachers from other philosophical positions experience and report. This does not imply inconsistent positions. It does imply that overzealous applications (often misinterpretations and overgeneralizations) can limit practical
effectiveness. As merely one illustration, constructivism does not imply that practice is not necessary and does not dictate (Clements, 1997; Simon, 1995).

Along a related vein, this chapter presented my and my colleagues’ design model, based on principles abstracted from our own and others’ work. These other models, however, and still more not described here, have unique features and advantages that any curriculum developer should also investigate.

Developers Should Remain Receptive to the Successes of All Approaches

Given its scientific basis, can research-based curricula be outperformed? Of course. We discussed previously that curriculum development is an art as well as a science. James (1958) had more to say on this subject:

The science of logic never made a man reason rightly, and the science of ethics (if there be such a thing) never made a man behave rightly. The most such sciences can do is to help us catch ourselves up and check ourselves, if we start to reason or to behavior wrongly; and to criticise ourselves more articulately after we have made mistakes. A science only lays down lines within which the rules of the art must fall, laws which the follower of the art must not transgress; but what particular thing he shall positively do within those lines is left exclusively to his own genius. One genius will do his work well and succeed in one way, while another succeeds as well quite differently; yet neither will transgress the lines. . . . And so everywhere the teacher must agree with the psychology, but need not necessarily be the only kind of teaching that would so agree; for many diverse methods of teaching may equally well agree with psychological laws. (p. 24)

Thus, there are many approaches, but each should be consistent with what is known about teaching and learning. Researcher-developers should be amenable to the lessons learned by any curriculum that leads to desirable outcomes. If such approaches are not based on research, they should use research methodologies to document these outcomes and investigate why the approach is successful. Without such research, the curricula will be limited in their contribution to all succeeding curriculum development projects.

Developers Should Support Professional Development and Systemic Change

Curriculum has a large effect on teaching and learning in the United States. This does not mean, however, that this “intended curriculum” determines classroom practice (Sosniak & Stodolsky, 1993). Beliefs and former experiences influence how teachers interpret an innovation (Haimes, 1996; Sarama et al., 1998). If research-based curricula are developed, teachers will not necessarily adopt their philosophy, especially if it conflicts with their traditional beliefs and practices. Teachers may instead give priority to curriculum content coverage, emphasize methods and procedures, and adopt teacher-focused pedagogical practices (Haimes, 1996). Changing teacher beliefs is incredibly difficult, but necessary (Prawat, 1992). Essential, then, is the provision of meaningful and accessible support materials and pre- and inservice training (Haimes, 1996; Sarama et al., 1998). These efforts must acknowledge that teachers face many competing requests for reforms in many different content areas (Grant et al., 1996; Sarama et al., 1998), that they are not adequately knowledgeable about teaching practices consistent with reform standards (Gravemeijer, 1994b; Kemis & Lively, 1997), and that “teachers who take this path must work harder, concentrate more, and embrace larger pedagogical responsibilities than if they only assigned text chapters and seatwork” (Cohen, 1988, p. 255, as cited by Prawat, 1992, p. 357). Also important are
issues of systemic change, and thus studies and curriculum change efforts a much larger levels than the curriculum development process described here (Burkhardt et al., 1986).

**The Education Community Should Support and Heed the Results of Research-Based Curriculum Development**

Given the grounding in both comprehensive research and classroom experience, the curricular products and empirical findings of such integrated research and development programs should be implemented in classrooms. Curriculum developers should follow models and base their development on the findings and lessons learned from these projects. Administrators and policymakers should accept and promote curricula based on similar research-based models. Educators at all levels should eschew software that is not developed consonant with research on students' learning of mathematics and that does not have the support of empirical evaluation. This would eliminate much of what is presently used in classrooms. This is a strong position but one that may avoid a backlash against the use of computers in education and that will, I believe, ultimately benefit students.

Fortunately, the design models discussed here, with their tight cycles of planning, instruction, and analysis, are consistent with the practices of teachers who develop broad conceptual and procedural knowledge in their students (Cobb, 2001; Lampert, 1988; Simon, 1995; Stigler & Hiebert, 1999). Therefore, the curriculum and findings are not only applicable to other classrooms but also support exactly those practices.

**Universities Should Legitimize Research-Based Curriculum Development**

There is a long history of bias against design sciences.

As professional schools, including the independent engineering schools, are more and more absorbed into the general culture of the university, they hanker after academic respectability. In terms of the prevailing norms, academic respectability calls for subject matter that is intellectually tough, analytic, formalizable, and teachable. In the past, much, if not most, of what we knew about design and about the artificial sciences was intellectually soft, intuitive, informal, and cookbooky. Why would anyone in a university stoop to teach or learn about designing machines or planning market strategies when he could concern himself with solid-state physics? The answer has been clear: he usually wouldn’t. (Simon, 1969, pp. 56–57)

In particular, the more that schools of education in prestigious research universities “have rowed toward the shores of scholarly research the more distant they have become form the public schools they are bound to serve” (Clifford & Guthrie, 1988, p. 3, as quoted in Wittmann, 1995). This is a dangerous prejudice, and one we should resist. Mathematics education might be seen largely as a design science, with a unique status and autonomy (Wittmann, 1995). “Attempts to organize mathematics education by using related disciplines as models miss the point because they overlook the overriding importance of creative design for conceptual and practical innovations” (Wittmann, 1995, p. 363).

The converse of this argument is that universities benefit because the approaches described here will prove practically useful, they will legitimize academic research per se.

Some argue that curriculum should be carried out only by experts (Battista & Clements, 2000). “A teacher can be compared more to a conductor than to a composer or perhaps better to a director . . . than to a writer of a play” (Wittmann, 1995, p. 365).
In this chapter, I mitigate this argument to welcome creative efforts but argue forcibly that research methodologies be used to evaluate every curriculum offered to others and that specific curriculum development projects follow research-based models.

**Funding Agencies Should Reconsider Time Frames and Funding Requirements for Curriculum Development**

Until a much larger body of research-based development is created, greater funding opportunities for research-based curriculum development are needed. Software is even more costly. Multimedia components, speech production and recognition, well-designed tools, interactive diagrams, and the like are expensive. They greatly increase the cost of software, with which, even in its traditional forms, it is difficult to make a profit.

Such funding should also consider the time period such development requires. In the development of traditional curricula, there are deadlines, but any extra time that might exist is used to improve the product, rather than for reflection and research (Gravemeijer, 1994b). Curriculum projects that are funded usually are given implausible time frames that make such reflection and research nearly impossible, such as 5 years to develop 5 years of curriculum (Schoenfeld, 1999).

**Policymakers Should Support and Insist on Research-Based Curricula**

To garner this type of support, curriculum developers need to be proactive, particularly in the political arena and especially when they are reform oriented. “Decisions about educational reform are driven far more by political considerations, such as the prevailing public mood, than they are by any systematic effort to improve instruction” (Dow, 1991, p. 5). The proportion of funds presently allocated to research in education is abominably inconsistent with virtually any other enterprise (Dow, 1991; President’s Committee of Advisors on Science and Technology—Panel on Educational Technology, 1997; Schoenfeld, 1999).

State policymakers, especially those with strict criteria for getting on the “list” of approved curriculum materials, should change their criteria to require a research basis for curricula and for the criteria themselves (and not artificially and unjustifiably limited subsets of research). This should avoid the political (in the pejorative sense of the term) swings typified by California’s recent transition from one end of the pedagogical spectrum to the other. Done well, it can serve as a partial antidote to the pervasive anti-intellectualism and fundamentalism of American politics that eschews honest reflection and research (Ginsburg et al., 1998). Partial is the best that can be expected; however, values, more than science, fuel such debates (Ginsburg et al., 1998; Hiebert, 1999).

**All Groups Should Collaboratively Address Implementation Barriers in the United States**

Software development requires cooperation from publishers who are more connected to research and development than is the present norm. All curriculum development, however, benefits from informed publishers who put the needs of children higher, relative to profit considerations, than is presently done.

Systemic issues must be addressed. As just one example, while the Japanese research lessons do not have extensive connections to theoretical and empirical research, they have several unique advantages that should be considered by countries whose integration of research has been problematic, such as in the United States. They create demand. According to Lewis and Tsuchida (1998), the United States suffers not from
a low supply of good educational programs but from a low demand for those programs. Demand occurs when teachers want to improve their practice—and when they can see the possibility of doing so. Principals say that research lessons build momentum for improvement more effectively than direct leadership by the principal. There are also lessons for U.S. policymakers and for curriculum leaders and developers. Supporting conditions for research lessons include a shared, frugal curriculum, collaboration among teachers, critical self-reflection, and stability of educational policy. A common, coherent vision such as that of the new NCTM Standards could provide a useful framework for such work in the United States. We all need to cooperate to change the whole system (Stigler & Hiebert, 1999).

Traditional research is conservative; it studies “what is” rather than “what could be.” When research is an integral component of the design process, when it helps uncover and invent models of children’s thinking and build these into a creative curriculum, then research moves to the vanguard in innovation and reform of education.

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CHAPTER 25

Historical Conceptual Developments and the Teaching of Mathematics: from Phylogenesis and Ontogenesis Theory to Classroom Practice

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1. INTRODUCTION

More than a century ago, Hieronymus Georg Zeuthen wrote a book about the history of mathematics (Zeuthen, 1902). Of course, this was not the first book on the topic, but what made Zeuthen’s book different was that it was intended for teachers. Zeuthen proposed that the history of mathematics should be part of teachers’ general education. His humanistic orientation fitted well with the work of Cajori, 1894 who, more or less by the same time, saw in the history of mathematics an inspiring source of information for teachers. Since then, mathematics educators have increasingly made use of the history of mathematics in their lesson plans, and the spectrum of its uses has widened. For instance, the history of mathematics has been used as a powerful tool...
tool to counter teachers’ and students’ widespread perception that mathematical truths and methods have never been disputed. The biographies of several mathematicians have been a source of motivation for students. By stressing how certain mathematical theories flourished in various countries, the diverse contributions of various cultures to contemporary mathematics becomes evident. Specialized study groups have emerged in the past years as a result of the increasing interest in the history of mathematics in educational circles. Two of these are the Commission INTER-IREM Épistemologie et Histoire des Mathématiques in France and the International Study Group on the Relations between History and Pedagogy of Mathematics, which is related to International Commission on Mathematical Instruction (ICMI). In addition, regular conferences are organized, such as the European Summer Universities on the History and the Epistemology in Mathematics Education (see Lalande, Jaboeuf, & Nouazé, 1995, and Lagarto, Vieira, & Veloso, 1996, for proceedings). Concomitantly, an important number of books are now available to help teachers use the history of mathematics (Calinger, 1996; Chabert, Barbin, Guillemot, Michel-Pajus, Borowczyk, Djebbar, & Martzloff, 1994; Dhommes, Dahan-Dalmedico, Bkouche, Houzel, & Guillemot, 1987; Fauvel & van Maanen, 2000; Katz, 2000; Reimer & Reimer, 1995; Swetz, Fauvel, Bekken, Johansson, & Katz, 1995).

Instead of offering an overview of the different domains in which the pedagogical use of the history of mathematics is now ramified, we want, in this chapter, to focus on something that Cajori started and in which mathematics educators interested in the history of mathematics are still involved. That is, in considering history not only as a window from where to draw a better knowledge of the nature of mathematics but as a means to transform the teaching itself. The specificity of this pedagogical use of history is that it interweaves our knowledge of past conceptual developments with the design of classroom activities, the goal of which is to enhance the students’ development of mathematical thinking.

Cajori’s 1894 ideas have led us to developments that he could not have suspected. Indeed, Cajori adopted a positivistic view of the formation of knowledge. He saw knowledge as an objective entity that grows gradually and cumulatively. His reading of the history of mathematics was framed by viewing history as an unfolding process. The direction or completion of the process guaranteed by the idea of progress—an idea underpinning the Enlightenment philosophy and attitudes toward life from which modern thought arose. Nonpositivistic views about the formation of knowledge were later elaborated by philosophers and epistemologists such as Bachelard, Foucault, and Piaget, among others, and by anthropologists such as Durkheim, Levy-Bruhl, and Lévi-Strauss, to mention but a few. Bachelard presented an interpretation of the formation of knowledge in terms of ruptures and discontinuities. Piaget was interested in explaining genetic developments in terms of stages and the intellectual mechanisms allowing the passage from one level to another. Foucault was opposed to the conception of history as a date-labeling practice and investigated the problem of the constitution of knowledge in terms of its emergence, which he related to the different spheres of human activity. Bachelard, Foucault, and Piaget had different goals, and thus their projects differed. But what is important for our discussion here is that, contrary to what Cajori and many other positivist thinkers believed, knowledge in general and mathematical knowledge in particular cannot be taken as an unproblematic concept. Behind a concept of knowledge there is an epistemological stance, and this epistemological stance conditions our understanding of the formation of students’ mathematical thinking as it conditions the interpretation of historical conceptual developments (Grugnetti & Rogers, 2000; Radford, Boero, & Vasco, 2000). Nevertheless, the study of the development of students’ thinking and of the conceptual development of mathematics belong to two different domains—the psychological and the historical, respectively. Each has its specific problems as
well as the tools with which to investigate them. Students’ conceptualizations can
be investigated through classroom observations, interviews, tests, and so forth. The
same cannot be done in the historical domain, where historical records are the only
available material for study. The difference in methodologies in both domains is, in
fact, a token of more profound differences. These cannot be ignored in the context
of a pedagogical use of the history of mathematics as a useful tool to enhance the
development of students’ mathematical thinking. Despite their differences, the psy-
chological and historical domains need to be weighed and articulated in a specific
way. One of today’s more controversial themes concerns the terms in which such an
articulation must be understood. More specifically, the question is how to relate the
development of students’ mathematical thinking to historical conceptual mathemat-
ical developments. Psychological recapitulation, which transposes the biological law
of recapitulation, claims that in their intellectual development our students naturally
traverse more or less the same stages as mankind once did; it has been taken as a guar-
antee (sometimes implicitly) to ensure the link between both domains. In its different
variants, however, psychological recapitulation has been subject to a deep revision
recently, in part because of the emergence of new conceptions about the role of culture
in the way we come to know and think.

The purpose of this chapter is to discuss in some detail the basic problems referred
to in this introduction. In the next section, we deal with psychological recapitulation
and mention some of the current arguments against it. In section 3, we examine key
ideas about ontogenesis and phylogenesis as found in the works of Piaget and in the
works of Vygotsky. In section 4, we present some paradigmatic examples of mathe-
maticians who commented on phylogenesis and its relation to ontogenesis. Section 5
focuses on a particular interpretation of the recapitulation law that led to the so-called
“genetic approach”, which had an obvious impact on early mathematics education.
In section 6, we discuss some examples of teachers who take into consideration the
history of mathematics to improve their teaching; determining how interpretations of
the recapitulation law articulate the teachers’ goals and actions guides our discussion.
Section 7 provides a brief account of a few current approaches in contemporary math-
ematics education that relate to the history of mathematics regarding either theoretical
or practical links between the development of students mathematical thinking and
historical conceptual developments. In the last section, we offer a critical assessment
of the law of recapitulation and recommend ideas for conceptual and applied research
in the 21st century regarding historical and ontogenetic developments in mathematics
education.

2. FROM BIOLOGICAL TO PSYCHOLOGICAL
RECAPITULATION

The way in which people perceived psychological recapitulation at the beginning
of the 20th century was linked to the way they perceived themselves in the overall
view of the world. As long as humans thought of themselves as essentially differ-
ent from animals and plants, no relation in terms of ancestry could be advocated.
Even in the early 18th century, a common scholarly view to explain the origin of
species and to understand the formation of living things was that species came from
those beings fortunate enough to survive the deluge, as indicated in the Genesis (see,
e.g., Osborn, 1929), by finding refuge on Noah’s ark. But with the appearance of the
early 19th-century philosophy of nature, humans came to join the greater kingdom
of species. In their broader sense, however, recapitulationist ideas data back, to the
pre-Socratic thinkers. They did not state them in terms of a telescoping or condensed
process of lower life that culminates with humans. Often their reference point was
the cosmos. Thus, Empedocles believed that the growth of the embryo echoes in a foreshortened way the cosmogonic process: The embryo is submerged into amniotic fluid that evokes the originally fluid earth (de Santillana, 1961, p. 114). During the 18th and early 19th centuries, a vigorous debate separated two opposing schools with regard to the concept of recapitulation. One of them, which became known as preformation theory, stated that ontogenesis was the unfolding or growing of preformed structures, whereas the other school adopted a more dynamic stance, arguing that the embryo was neither the exact miniature of the developed species nor the unfolding of preformed structures, but a being in a state of development. The “causes” originating embryo’s the unfolding or the changes were variously interpreted. Charles Bonnet (1720–1793), usually recognized as one of the leaders of the preformationists, saw change as coming from an affectionate God who had ordered the world according to increasing perfection and progress. Whereas in the early-19th century Naturphilosophen attributed development to a “natural” final cause, Lamarck and Darwin envisioned a new theory that replaced the philosophical idea of final cause with an efficient cause—individual development. (For a detailed discussion of preformationist and Naturphilosophen ideas, see Gould, 1977.) Indeed, from the mid-19th century onward, the “causes” were seen in the context of the theory of evolution. “Heredity and adaptation are, in fact, the two constructive physiological functions of living things,” wrote Haeckel (1912, p. 6), who, in one of the most famous statements ever made in the realm of anthropogenesis (which he modestly called the fundamental law of biogeny), declared that

The series of forms through which the individual organism passes during its development from the ovum to the complete bodily structure is a brief, condensed repetition of the long series of forms which the animal ancestors of the said organism, or the ancestral forms of the species, have passed through from the earliest period of organic life down to the present day. (pp. 2–3)

Haeckel’s law was more than the simple statement of a condensed repetition of steps. What he was suggesting was that embryos of man and dog, at a certain stage of their development, are almost indistinguishable. Indeed, to take one of Haeckel’s favorite examples, “the human gill slits are (literally) the adult features of an ancestor” (Gould, 1977, p. 7).

How, then, was the discussion about the biological growth of humans transferred to the psychological domain? It was Haeckel who, after discussing the nervous system, said “we are enabled, by this story of the evolution of the nervous system, to understand at length the natural development of the human mind and its gradual unfolding” (1912, p. 8, italics as in the original). A sharper formulation was the following: “the psychic development of the child is but a brief repetition of the phylogenetic evolution” (Haeckel quoted by Mengal, 1993, p. 94). The adoption of the psychological version of biological recapitulation served as a general framework to conceive the functioning of child psyche as something traveling the same path as his or her ancestors. For instance, the child was seen as behaving as humans in previous stages of the chain of evolution (e.g., such as having, in an early stage of his or her development, an “animist” view of nature, that is, that immaterial forces animate the universe).

Psychological recapitulation endorses a peculiar view of history and development. Concerning development, for Bonnet and the preformists, there was no development, strictly speaking, but only growing or unfolding. Environment cannot alter the preformed structures and their growth. For evolutionary-based recapitulation theories, in contrast the environment is supposed to play in the development of species a role. The individual is seen as an organism adapting to his or her environment; in the interplay between individual and environment, some of the biological and
psychological functions may develop, whereas others may be lost according to the natural selection.

As for history, in contrast to views that conceived a world that underwent different creations, Bonnet saw the world as created at one time, with its whole history encapsulated within it. History was therefore the unfolding of a predetermined plan. The concept of history is much more problematic for recapitulationists. Indeed, from a theoretical point of view, history and recapitulation become difficult to reconcile because, on one hand, Haeckel’s psychological recapitulation supposes that present intellectual developments are to some extent a condensed version of those of the past. On the other hand, natural selection is presented as a function of the environment against which individuals act. For recapitulation to be possible, therefore, such an environment must remain essentially the same, which obviously is not the case. Given that the environment changes, it becomes difficult to maintain that the children’s intellectual development will undergo the same process as the ones children experienced in the past. The variability that natural selection imposes on the course of events in history conflicts with the idea of recapitulation as condensed repetition of some intellectual aspects registered in past history. Indeed, this point was recognized as a weakness. Werner (1957), for instance, advocated contextual factors and argued that it is impossible to equate a certain intellectual stage of a child in a modern society to the stage an adult could have reached in an ancient society because the respective environments, as well as the genetic processes involved in them, are completely different (see Radford, 1997a). Elias also mentioned the differences that necessarily result as a consequence of variations in cultural settings. Whereas in traditional societies children participate directly in the life of the adults earlier and their learning is done in situ (as apprentices), “modern” children are instructed indirectly in mediating institutions, or schools (Elias, 1991, pp. 66–67). Consider memory, an example that is addressed neither by Werner nor Elias but which conveniently clarifies the previous ideas. As many anthropological accounts clearly show (see e.g., Lévy-Bruhl, 1928), memory plays a central role in illiterate societies. In contrast, sign systems related to writing in literate societies dispense with memory to a certain and fundamental extent. They create a different way to handle and distribute knowledge and information between the members of the society and shapes attitudes about how to scrutinize the future (see Lotman, 1990).

The theoretical difficulties encompassing the crude version of psychological recapitulation encouraged new reflections to find more suitable explanations concerning the relations between phylogenesis and ontogenesis. In the next section, we will discuss two different views that have been influential in the use of history in mathematics education.

### 3. PIAGET AND VYGOTSKY ON ONTOGENESIS AND PHYLOGENESIS

Piaget was interested in understanding the process of the formation of knowledge. To do so, he considered knowledge as something that can be described in terms of levels and strived to describe those levels, as well as the passage from one level to a more complex one. He said, “The study of such transformations of knowledge, the progressive adjustment of knowledge, is what I call genetic epistemology” (Piaget cited in Bringuier, 1980, p. 7). As a reaction to the simplistic psychological version of recapitulation and the positivist view of knowledge that we mentioned in the introduction, Piaget and García elaborated the concept of genetic development. They envisioned the problem of knowledge in terms of the intellectual instruments and mechanisms allowing its acquisition. According to Piaget and García, the first of those mechanisms is a general process that accounts for the individual’s assimilation and integration of
what is new on the basis of his or her previous knowledge. In addition to the assimilation mechanism, they identified a second mechanism, a process that leads from the *intraobject*, or analysis of objects, to the *interobject*, or analysis of the transformations and relations of objects, to the *transobject*, or construction of structures. This epistemological viewpoint led them to revisit the parallelism that recapitulationists had emphasized. Therefore, Piaget concluded, “We mustn’t exaggerate the parallel between history and the individual development, but in broad outline there certainly are stages that are the same” (Bringuier, p. 48). The two mechanisms were hence considered as invariables, not only in time but also in space. That is, we do not have to specify what they are in a certain geographical space at a particular time because they do not change from place to place or from time to time. They are exactly the same, regardless of the period of history and the place of the individuals.

In modern mathematics, at the level of algebraic geometry, of quantum mechanics, although it’s a much higher level of abstraction, you find the same mechanisms in action—the processes of the development of knowledge or the cognitive system are constructed according to the same kinds of evolutionary laws. (Garcia in Bringuier, 1980, pp. 101–102)

Thus, when Piaget and Garcia investigated the relations between ontogenesis and phylogenesis, they did not seek a parallelism of contents between historical and psychogenetical developments but of the mechanisms of passage from one historical period to the next. They tried to show that those mechanisms are analogous to those of the passage from one psychogenetic stage to the next.

The two mechanisms of passage discussed by Piaget and Garcia have a different theoretical background. The second, that of the intra-, inter- and trans-objectual relations, obeys a structural conception of knowledge and reflects the role that mathematical and scientific thinking played in Piaget’s work. As Walkerdine noted, “In the work of Piaget, an evolutionary model was used in which scientific and mathematical reasoning were understood as the pinnacle of an evolutionary process of adaptation” (Walkerdine, 1997, p. 59). The first one, the assimilation mechanism, has its roots in the conception of knowledge as the prolongation of the biological nature of the individuals: “The human mind is a product of biological organization, a refined and superior product, but still a product like another” (Piaget in Bringuier, 1980, p. 108).

Both intellectual mechanisms of knowledge development embody a general conception of rationality that has been contested by some critics who find missing, among other things, a more vivid role of the culture and the social practices in the formation of knowledge. For instance, the epistemologist Wartofsky, who has stressed an intimate link between knowledge and the activities from which knowledge arises and is used, said:

> We are, in effect, the products of our own activity, in this way; we transform our own perceptual and cognitive modes, our ways of seeing and of understanding, by means of the representations we make. . . . Theoretical artifacts, in the sciences, and pictorial or literary artifacts, in the arts constitute the a priori forms of our perception and cognition. But contrary to the ahistorical and essentialist traditional forms of Kantianism, I propose instead that it is we who create and transform these a priori structures. Thus, they are neither the unchanging transcendental structures of the understanding, nor only the biologically evolved a priori structures which emerge in species evolution (as, for example, Piaget and the evolutionary epistemologists suggest). Piaget’s dynamic, or genetic structuralism is important here, of course. His dictum, “no genesis without structure, no structure without genesis,” suggests the dialectical interplay of the practical emergence and transformation of structures with the shaping of our experience and thought by structures. But the domain of this genesis I take to be the context of our social, cultural and scientific practice, and not that of biological species-evolution
alone. . . . In a sense, then, our ways of knowing are themselves artifacts which we ourselves have created and changed, using the raw materials of our biological inheritance. (Wartofsky, 1979, p. xxiii)

Vygotsky, in many writings, dealt with the problem of recapitulation and, like Piaget, believed that the understanding of ontogenesis and phylogenesis had to be based on a deep understanding of our biological nature. (This is clear, for instance, in his book *Speech and Thinking,* as well as in the influence he had on his student Luria and the huge amount of physiological research that the latter conducted.) Instead of posing the problem of the formation of knowledge in terms of universal and atemporal mechanisms functioning beyond culture, however, he saw the cognitive functions allowing the production of knowledge as inevitably overlapping with the context in which individuals act and live. His basic distinction between lower and higher mental functions is reinforced by the idea that the former belong to the sphere of the biological structure, whereas the latter are intrinsically social. Thus, in a passage from *Tool and Symbol in Child Development,* when discussing the problem of the history of the higher psychological functions, Vygotsky and Luria commented:

Within this general process of development two qualitatively original main lines can already be distinguished: the line of biological formation of elementary processes and the line of the socio-cultural formation of the higher psychological functions; the real history of child behaviour is born from the interweaving of these two lines. (Vygotsky & Luria, 1994, p. 148)

The merging of the natural and the sociocultural lines of development in the intellectual development of the child definitely precludes any recapitulation:

In the development of the child, two types of mental development are represented (not repeated) which we find in an isolated form in phylogenesis: biological and historical, or natural and cultural development of behavior. In ontogenesis both processes have their analogs (not parallels). . . . By this, we do not mean to say that ontogenesis in any form or degree repeats or produces phylogenesis or is its parallel. We have in mind something completely different which only by lazy thinking could be taken to be a return to the reasoning of biogenetic law. (Vygotsky, 1997, p. 19)

For Vygotsky even the elementary intellectual functions of the individual are intrinsically human, acquired through the activities and actions on which are based the intercourse between individuals and between people and objects. One of the central reasons for this is that human activities are mediated by diverse kinds of tools, artifacts, languages, and other systems of signs which, Vygotsky argued, are a constitutive part of our cognitive functions. Most important, these systems of signs, as well as tools and artifacts, are much more than technical aids: They modify our cognitive functioning. The knowledge produced by the individuals hence becomes intimately related to the activities out of which knowledge arises and the conceptual and material “cultural tool kit” (to borrow Bruner’s expression, see Bruner, 1990) with which the individuals are equipped. Of course, it does not mean that with every new generation, all knowledge must be constructed anew. As Tulviste (1991) noted, whereas rats are still doing what they did centuries ago, humans have, from one generation to the next, assimilated, produced, and passed on their knowledge. During this process, humans have changed their activities and the way in which they think about the world. In Vygotsky’s view, knowledge appears as an individual and social creative reappropriation and coconstruction carried out using conceptual and material tools that culture makes available to its individuals. In turn, in the course of this process, the previous tools and signs may become modified, and new ones may be created. It is in this
sense that tools and concepts have embodied the social characteristics from which they arose, and their insertion into other activities allows their transformation and eventually their growth. Because activities, sign use, and attitudes toward the meaning of scientific inquiry do not necessarily remain the same throughout time, changes are effected in phylogenetic lines (and the plural of lines needs to be emphasized here) serving as the historicocultural starting point to new genetic developments. Epistemological reflections have then to evidence the relation between cognitive context and action. As Wartofsky pointed out:

If, in fact, our modes of cognitive practice change with changes in our modes of production, of social organization, of technology and technique, then the connection between cognition and action, between theoretical and applied practice, between consciousness and conduct, has to be shown. (Wartofsky, 1979, p. xxii)

One implication of the previous remarks for the use of the history of mathematics in education is that the study of recapitulation can be advantageously replaced by the contextual study of the social elements in which the historical geneses of concepts are subsumed. This can be accomplished through a careful investigation of the cultural symbolic webs shaping the form and content of scientific inquiry and the ways in which mathematical concepts are semiotically represented (Radford, 1997a, 1998, 1999a, 2000a). We return to this point in section 7.

4. INTERPRETATION OF RECAPITULATION LAW BY MATHEMATICIANS

In the period when the treatises of Zeuthen and Cajori appeared, the history of mathematics was growing as a scientific discipline. The first journals dealing exclusively with the history of mathematics were appearing in that period. We have extensive evidence that mathematicians and mathematics educators were both looking at the history of mathematics with great interest. Mathematics educators were creating new areas of work in their field linked to changes in societies. As discussed in Furinghetti (2000) and in Furinghetti and Somaglia (1998), the history of mathematics was considered a suitable means to find efficient ways of teaching in different situations. Among mathematicians, the axiomatization and the foundational works were undertaken. These themes were addressing mathematicians’ attention to reflections on the nature of mathematics and on the activity of doing mathematics. The history of mathematics was considered a field that offered inspiration to discuss these kinds of problems. In this context, we consider some interpretations of recapitulation law made by important mathematicians.

In the first issue (1899) of *L’enseignement mathématique*, an important journal devoted to the teaching of mathematics, the eminent mathematician Henri Poincaré clearly stated his position on the relations between conceptual and historical developments:

Without a doubt, it is difficult for a teacher to teach a reasoning that does not satisfy him completely. . . . But the teacher’s satisfaction is not the sole purpose of teaching . . . above all one should be concerned with the student’s mind and of what we want him to become.

Zoologists claim that the embryonal development of animals summarizes in a very short time all the history of its ancestors of geologic epochs. It seems that the same happens to the mind’s development. The educators’ task is to make children follow the path that was followed by their fathers, passing quickly through certain stages without eliminating any of them. In this way, the history of sciences has to be our guide. (Poincaré, 1899, p. 159; our translation)
Poincaré gave examples of concepts to be taught at an intuitive stage before presenting them rigorously. Among these examples were fractions, continuity, and area. As far as we know, Poincaré never used his ideas on the efficacy of recapitulation law directly with teachers. This makes Poincaré’s position different from that of Felix Klein, another supporter of the use of history in mathematics in teaching. In contrast, Klein applied his ideas in courses for prospective teachers and in related texts that he wrote.

Klein supported the German translation of the famous book A study of Mathematical Education by Benchara Branford (1921) in which, according to Fauvel (1991, p. 3), the theory of recapitulation “reached its apogee.” This can be considered evidence of Klein’s agreement to the recapitulation law (Fauvel, 1991, p. 3). Nevertheless, from what Klein wrote in his articles and books (see Klein, 1924), we understand that the application of the law was not advocated in a literal sense. As in the case of Poincaré, his opinion on the use of history was born of his wish to abolish the use of mathematical logic and the excesses of rigor advocated by some of his colleagues. Klein was interested in the dichotomy of “intuition versus rigor” and, as far as school is concerned, was in favor of intuition. He singled out the history of mathematics as being the suitable context for bringing intuition back into the teaching and learning process:

I maintain that mathematical intuition . . . is always far in advance of logical reasoning and covers a wider field. . . . I might now introduce a historical excursus, showing that in the development of most of the branches of our science [mathematics], intuition was the starting point, while logical treatment followed. This holds in fact, not only of the origin of the infinitesimal calculus as a whole [this issue was discussed at the beginning of Klein’s paper] but also of many subjects that have come into existence only in the present [19th] century. (Klein, 1896, p. 246)

Klein claimed that in school, as well as in research, the phase of formalization must be preceded by a phase of exploration based on intuition.

We find an analogous statement in a secondary school geometry book written by a famous Italian mathematician, Francesco Severi, which clearly refers to school practice:

We need to take inspiration from the principle that in learning new notions, the mind tends to follow a process analogous to that according to which science has developed. One who is aware of the value of foundation theories [in Italian critica dei principi] does not make the dangerous mistake of giving to the elementary teaching a critical and excessively abstract style. It is necessary to know foundation theories for personal intellectual maturity; but in the elementary teaching they are not to be considered as a pedagogical means. (Severi, 1930, p. IX; our translation)

Both Klein and Severi do not clearly state what “intuition” means for them, but both state to what intuition is opposed: rigor, excessive abstraction, and formal logic used at the beginning of the presentation of a mathematical notion. (It may be interesting to note that Severi, famous during the first half of the 20th century, is one of the scholars of the Italian school of algebraic geometry who based his results on intuition to such a degree that these were published without being careful verified by a mathematical proof, as reported by Hanna, 1996).

5. THE GENETIC APPROACH

Using the history of mathematics in teaching does not necessarily entail a direct assumption of the recapitulation law; it also may be used in the so-called genetic approach to teaching. The term “genetic” is an ambiguous one because it is used with different meanings. In particular, in the foundation literature, the term genetic method is used
in contrast to *axiomatic method*. David Hilbert probably introduced this term, which was popularized by Edward V. Huntington. Before Hilbert, we find other uses of the word “genetic.” Immanuel Kant stated that all mathematical definitions are genetic; after Kant, the term “genetic definition” is present in all major logic treatises.

In addition to its use among mathematicians and philosophers, we find the word “genetic” in other fields of research. Piaget and Garcia used it in their epistemological studies. As to mathematics education, Ed Dubinsky, who dealt with genetic decomposition, used the word.

Here we are concerned with the word “genetic” as it is used in connection with history. In the 1920s the idea of a genetic principle was taking shape, as evidenced by the work of N. A. Izvolsky.

Gusev and Safuanov (2000) report that, according to Izvolsky, nor teachers nor textbooks try to explain the origin of geometrical theorems. He suggested that, when attempts to do this are done, students see geometry in a different way. Moreover sometimes students themselves guess that a given theorem was not originated by a mere wish of the teacher or textbooks’ authors, but by questions arisen in previous works. It happens that students try to imagine the origin of a theorem. According to Izvolsky, even if their hypotheses are not correct from the historical point of view, this approach to the teaching of geometry is valuable.

The idea of a genetic approach later took a definite form in a work by Otto Toeplitz that he wrote to describe a method of presenting analysis to university students. The following passage illustrates the ideas underlying the genetic method:

Regarding all these basic topics in infinitesimal calculus which we teach today as canonical requisites, e.g., mean-value theorem, Taylor series, the concept of convergence, the definite integral, and the differential quotient itself, the question is never raised “Why so?” or “How does one arrive at them?” Yet all these matters must at one time have been goals of an urgent quest, answers to burning questions, at the time, namely, when they were created. If we were to go back to the origins of these ideas, they would lose that dead appearance of cut and dried facts and instead take on fresh and vibrant life again.

Burn explains in this way Toeplitz’s ideas:

The question which Toeplitz was addressing was the question of how to remain rigorous in one’s mathematical exposition and the teaching structure while at the same time unravelling a deductive presentation far enough to let a learner meet the ideas in a developmental sequence and not just in a logical sequence. While the genetic method depends on careful historical scholarship it is not itself the study of history. For it is selective in its choice of history, and it uses modern symbolism and terminology (which of course have their own genesis) without restraint. (Burn, 1999, p. 8)

It is not by chance that this alternative approach developed in the domain of teaching calculus. It is in this domain where the notion that learning mathematics takes place in a sequence predetermined by mathematical logic has shown its pedagogical

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1Nikolai Alexandrovich Izvolsky was born in 1870 in Tula, Western Russia. He worked as a teacher at the 2nd Moscow Military School, and from 1922, he was a professor at the 2nd Moscow State University (now Moscow State Pedagogical University). He wrote papers on mathematics education and some textbooks in arithmetic, algebra, and geometry. Izvolsky died in Yaroslavi in 1938. The authors are grateful to Professor Idlar Safuanov from the Pedagogical Institute of Naberezhnye Chelny for the information he kindly provided concerning the life of Izvolsky.

2A complete study of the genetic method as intended by Toeplitz can be found in Schubring (1978).

limitations. Indeed, when organized around their logical basis, the definitions of the main concepts of calculus (integrals, limits, derivatives) are abstract, and therein lies the burden of formal rules and theorems. Students have difficulty grasping the meaning of that with which they are asked to work. At present there are projects (not based on history) that take into account these difficulties and organize the teaching of calculus according to different patterns. (See, for example, the Harvard project based on giving an informal, operative approach to concepts in Hughes-Hallet et al., 1994).

What Toeplitz proposed is realistic and may be considered a compromise between the two ways of thinking about teaching mathematics, the logical versus developmental sequences. Toeplitz’s historically based approach aims to provide a slow process of understanding that the student performs through a sequence of steps. Because Toeplitz’s aim is to provide teaching materials that facilitate the learning of calculus, the main concern of the author is not to teach history, but to find learning sequences. Burn (1999) elaborated on these ideas. If we analyze Toeplitz’s proposals or the more recent ideas of Burn, we may find an example of the history of mathematics used as a key element in the construction of a teaching sequence (on calculus) from intuition to logical deduction. The role of history is therefore that of providing materials on which to develop intuition. The presentation of the historical materials is not shaped according to recapitulationist principles because it uses modern symbols, verbal expressions, and cultural tools that are different from those of past authors.

An older example of the use of the genetic method (intertwined with a naïve heuristic approach) is in the treatise on geometry by Alexis-Claude Clairaut (1771). The preface of his book is an early example of predidactic literature. Its importance lies in the traces of Clairaut’s thought that can be found in works on mathematics education through the 20th century. Clairaut wrote:

Even if geometry is abstract in itself, we nonetheless must agree that the difficulties suffered by beginners come mostly from the way it is taught in usual treatises. They always start with a great deal of definitions, questions, axioms, and preliminary principles, which only seem to promise dry issues for readers…. To avoid this dry quality that is naturally linked to the study of geometry, some authors put examples after each proposition to show it is possible to do them; but in this way, they only prove the usefulness of geometry without making it any easier to learn. Because each proposition is presented before its use, the mind reaches concrete ideas after having toiled with abstract ideas. Having realized this fact, I intended to find out what may have given birth to geometry and tried to explain principles with the most natural methods, which I suppose were adopted by the first inventors, while trying to avoid the wrong attempts they had necessarily made. (Clairaut, 1771, pp. 2–4; our translation)

According to Glaeser (1983), Clairaut contributed greatly to the introduction of the genetic method. Glaeser commented on Clairaut’s work with the following observations: “Giving up the dogmatic exposition, and to follow the true historical development of discovery, this method consists on imagining a road that learned peoples “could have followed”! Thus this is pretense education”. (Glaeser, 1983, p. 341, our translation)

In spite of Glaeser’s criticism, Clairaut’s attempts present interesting features, even more so if we consider that in the period when this author conceived his project, the paradigm of geometrical teaching was based on the hypothetical-deductive Euclidean method. If we compare the passage from Toeplitz’s book and Clairaut’s passage, we see an extraordinary coincidence of intentions and didactic observations (i.e., the idea of “dryness” that is present in the work of both authors).

Freudenthal (1973) provided an interpretation of the genetic method:

Urging that ideas are taught genetically does not mean that they should be presented in the order in which they arose, not even with all the deadlocks closed and all the detours
cut out. What the blind invented and discovered, the sighted afterwards can tell how it should have been discovered if there had been teachers who had known what we know now... It is not the historical footprints of the inventor we should follow but an improved and better guided course of history. (Freudenthal, 1973, pp. 101, 103; our italics)

Freudenthal termed this way of using history “guided reinvention.” It implies an active and aware participation of the teacher in designing and carrying out teaching experiments with history.

6. THE HISTORY OF MATHEMATICS IN THE CLASSROOM FROM THE TEACHER’S POINT OF VIEW

We have argued elsewhere (Furinghetti, 1997) that to study the applications of the history of mathematics in the classroom, we need a systemic net of experiments to analyze. For this reason, one of the authors (F. F.) has constituted a permanent monitor to keep track of the use of history in mathematics teaching in Italy. This means that teachers experimenting with the use of the history of mathematics, or only wishing to do so, are invited to contact the monitor and to discuss their ideas. In this way, it has been possible to create a file containing a range of different situations. The examples that we shall present in what follows come from these data.

First, we report on a workshop of teachers held by Jan van Maanen in Italy to present and discuss the ICMI Study document, “The role of the history of mathematics in the teaching and learning of mathematics,” together with Italian researchers in mathematics education and high school teachers. Teachers participating in the workshop were asked if they use history in their classrooms. The answer, in general, was negative because of the constraints of the school system. Nonetheless, all the teachers expressed the strong interest in using it if they were given the opportunity. When asked to explain why they consider the use of history fruitful, the answer was something echoing—usually unintentionally—the recapitulation law. Some of the paradigmatic statements (quoted literally) include the following: “The students’ development of concepts follow the historical sequence,” “The historical genesis of the concept may help teachers understand the genesis of the concept in students’ minds,” and “If I present the students with how algebra developed in history, they feel differently about their difficulties in learning it.”

Although not necessarily in a conscious or explicit way, the answers exhibit an understanding of the relation between ontogenesis and phylogenesis that is close to Haeckel’s psychological version of the law of recapitulation. The following three examples illustrate, in a more detailed way, some teachers’ positions about recapitulation.

We will see that in these cases the initial stimulus to consider the history of mathematics in their teaching is the vague idea that some parallelism between child development and mathematical development exists. Nonetheless, the kind and amount of adaptations that result from changes due to differences in historical periods and their cultural contexts are so significant that it is not possible to talk about some form of genuine recapitulation.

6.1. First Example

The first teacher is a mathematics instructor in a middle school (students aged 11 to 13), who studied biological sciences in college (and hence does not have a substantially deep understanding of mathematics) but is fond of mathematics and of teaching. She confesses her difficulties in teaching because of students’ lack of motivation and
her personal incapacity to interpret their difficulties. She has never carried out experiments in the classroom encompassing the use of history in mathematics teaching; nonetheless, she wrote (see also Gallo, 1999):

I feel that my mathematical preparation lacks a historical perspective. I think I could find in history some answer to my teaching problems.

In my opinion, to follow the evolution of the mathematical thinking could help the teacher understand how learning mathematics develops in children and preadolescents.

As an example, I mention the use of fractions by the Egyptians: It is closer to the intuitive concept held by a primary pupil. I gave my 10-year-old daughter an Egyptian problem of dividing loaves among men taken from a seventh-grade mathematics textbook. She solved the problem in the way that the Papyrus Rhind solves it.

I think it could be interesting to show students other issues taken from history: the geometrical representations of numbers, the geometrical representations of algebraic situations offered by Euclid. I think that the latter are more illuminating than the usual modern presentations.

The division problem the teacher used is the following problem in the Rhind Papyrus (ca. 1650 BCE): “Example of reckoning out 100 loaves for 10 men, a sailor, a foreman and a watchman with double” (see Peet, 1923, p. 109). Here we have an example of a teacher who does not have historical preparation; she only has some scattered ideas taken from notes in books and articles. She never carried out experiments using history in the classroom. Her experience is based on anecdotal facts. We interpret what she writes about history as being representative of the ideas that teachers in similar situations have about the use of history in teaching: There is a parallel between history and the way students learn.

### 6.2. Second Example

Other examples of the relationship of teachers with history that are more precise focus on experiments performed in the classroom. In these cases, the ideas expressed by the teacher are not mere intuition but are based on fact. The first case concerns a class of twenty-one 15-year-old high school students. We only briefly report on this experiment. (For a wider account, see Paola, 1998.) The teacher has an extensive experience in instruction and research in mathematics education. In the experiment, he acted as a teacher and as an observer. His purpose was to work with students on the concept of probability, which they had already encountered in previous school years. He chose to work with history to return to the concept of probability using a different (historical) approach. The work in the classroom was centered on a problem that is treated in many books of arithmetic from the Middle Ages “How can the stake be divided in a game where the two players are of the same value (in modern terms, have the same probability of winning) if the game is interrupted before one of the two players has realized the winning score?” This problem is known as “the problem of partition.” Luca Pacioli gave his solution (based on proportionality) to this problem in his famous treatise *Summa de arithmetica geometria proportioni et proportionalità* (Printed in 1494). The classroom activity was developed through discussion of the problem between students divided into groups. The teacher not only orchestrated the discussion but also acted as an observer and reported all that happened in the classroom. Initially all students agreed that the best way to solve the problem would be to divide the stake in parts that were proportional to the scores earned by each player. The teacher easily refuted this solution by proposing that one of the two players had a score of zero when the game was interrupted. After a discussion on this particular case, another group of students proposed other ways of solving
it that did not satisfy their classmates. At this point, the teacher read Pacioli’s solution, which is similar to that of the students, allowing them to see that an important historical personage followed the same process they did. The students seemed ready to approach the concept of fair division of the stake. Additional classes were dedicated to discussing this concept, but the students did not arrive at effective results on their own (i.e., they were not able to grasp the concept of probability). The teacher expounded Pascal’s solution to the problem, as reported in (Pascal, 1954), and thus introduced students to the concept of probability.

As we said previously, the teacher acted as an observer, and he accurately reported the activity in the classroom (Paola, 1998). Even if some elements of probability had been taught to these students in the previous school years, it is clear from the chronicle of the classroom activity that their strategies were based on proportionality, as Pacioli’s were. The teacher believes this experiment shows that students follow the path of history:

The voice of history is again evoked by the teacher to give dignity to the students’ solutions which actually follow the path hinted by mathematicians before Pascal and Fermat. (Paola 1998, p. 34)

There are many passages suggesting that the teacher is concerned with the mistakes in the ancient attempts of solving Pacioli’s problem. For example: “The incursion into history had the goal of giving dignity to the mistake made by students: it was not a trivial mistake if a mathematician made it” (p. 33).

The teacher showed interest in the parallels between the strategies his pupils and Pacioli used, but he did not draw general theoretical conclusions concerning the recapitulation law. From his conclusions, we see only that he has a certain confidence in the validity of following the stages of the historical development for didactic purposes:

With another session I could have read and commented on the Pascal–Fermat letters in the classroom and thus I would have stressed the role of history [in helping students to bypass some obstacles in constructing concepts of probability theory] (Paola, p. 35).

6.3. Third Example

The last case we present concerns a high school mathematics and physics teacher who works with students ranging in age from 16 to 19 years. The teacher has researched the history of mathematics. She is interested in proof and tries to develop students’ abilities on this subject using historical examples. To this end, she uses the method of analysis and synthesis, found in the Pappus’s Collectiones Mathematicae. We describe this method with the following passage taken from Hintikka and Remes (1974):

Now analysis is the way from what is sought—as if it were admitted—through its concomitants [the usual translation reads consequences] in order to something admitted in synthesis. For in analysis we suppose that which is sought to be already done, and we inquire from what it results, and again what is the antecedent of the latter, until we on our backward way light upon something already known and being first in order. And we call such a method analysis, as being a solution backwards. In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequents the former antecedents and linking them one with another, we in the end arrive at the construction of the thing sought. And this we call synthesis. (p. 8)

The method of analysis is described in a manual for teachers (Smith, 1911) as follows:
I can prove this proposition if I can prove this thing; I can prove this thing if I can prove that; I can prove that if I can prove a third thing, and so the reasoning runs until the pupil comes to the point where he is able to add, “but I can prove that.” This does not prove the proposition, but it enables him to reverse the process, beginning with the thing he can prove and going back, step by step, to the thing that he is to prove. Analysis is, therefore, his method of discovery of the way in which he may arrange his synthetic proof. (Smith, 1911, pp. 161–162)

Historically this method originated in the field of geometry, but it has since been used in other branches of mathematics. For example, the method of analysis is at the heart of algebra: The introduction of symbols made by Viète in the 16th century did not arise spontaneously but was a consequence of having adopted the method of analysis for solving algebraic problems (Charbonneau, 1996). The method of analysis also is not specific to mathematics; for example, in Marchi (1980), it is applied to chemistry. The method of analysis represents a link between history and education. In their chapter on proof, Alibert and Thomas (1991) proposed a method of proving that is similar to the method of analysis, probably without considering the history of mathematics.

The teaching experiment with this method that the teacher in this example carried out lasted for many years. We report on only briefly this experiment; for a lengthier account, see Somaglia (1998). At the beginning of the lesson, the teacher presents her students with the method of analysis in the field of Euclidean geometry. Students experience the application of this method in different problems until the method is mastered and recognized as a tool for attacking geometrical problems. Afterward, the teacher has the students apply the method to other parts of mathematics (algebra and calculus) so that they become aware of the transversality of the method (i.e., that the method is not linked to a particular domain of knowledge but can be generalized). Students are then ready to attack problems in physics and in chemistry using this method (see Clavarino & Somaglia, 2001).

In the description of her work, the teacher never mentioned any parallel between the strategies of her students and those of past mathematicians, nor the persistence of errors. In our experience, this fact is unusual among teachers dealing with mathematics history. There are two developments in the work of this high school teacher, the historical and the educational, that interact, and her way of looking at these processes is very positive. The teacher looks for what can give students the means to realize the condensation of concepts (see Sfard, 1991). This teacher has an excellent knowledge of mathematics history, and moreover it is quite natural for her to work with original sources. Thus, history is an integral part in her mathematics teaching. Her contact with the past is not that of someone who looks at the past with the eyes of the present but one who sees the concepts of the past as real and important content—as foundations in an architectonic sense—upon which our modern concepts and methods are based. She puts in action Gadamer’s way of looking at the past, that is, “a dialogical process in which two horizons (the past and the present) are fused together” (Radford, 1997a, p. 27).

7. THE RECOURSE TO HISTORY IN CONTEMPORARY MATHEMATICS EDUCATION

In the previous sections, we discussed some interpretations of recapitulation law made by past mathematicians and teachers. Let us now examine a few examples of contemporary mathematics educators, confining our discussion to two specific cases. The first emphasizes (mainly although not exclusively) a theoretical interest. The second appears closer to specific contexts arising from the needs to enhance teaching and learning processes in mathematics instruction. In the first case, the history of mathematics appears as a theoretical tool to understand developmental aspects of
mathematical thinking. The purpose of the second case is to facilitate, through explicit pedagogical interventions, students’ learning of mathematics by attempting to relate the development of students’ mathematical thinking to historical conceptual developments.

7.1. The Interface Between History and Developmental Aspects of Mathematical Thinking

The work of Sfard (1995) provides a clear example of contemporary views on the relation between history and the developmental aspects of mathematical thinking. She analyzed the development of algebra by blending historical and psychological perspectives. At the beginning of her article, she claimed that

there are good reasons to expect that, when scrutinized, the phylogeny and ontogeny of mathematics will reveal more than marginal similarities. At least, this is what follows from the constructivist view according to which learning consists in the reconstruction of knowledge. (p. 15)

The similarities between the phylogenetic and ontogenetic domains result in this account from “inherent properties of knowledge.” For Sfard, who follows a Piagetian epistemological perspective, knowledge can be theoretically described in terms of genetic structural levels, and it is precisely the nature of the relationship between the different levels that accounts for the similarity of phenomena appearing in the historical and in the individual’s construction of knowledge. As she noted, “difficulties experienced by an individual learner at different stages of knowledge formation may be quite close to those that once challenged generations of mathematicians” (Sfard, 1995, pp. 15–16). A large part of the text is devoted to the discussion of the development of algebraic language. Indeed, using Nesselmann’s (1842) distinction between rhetorical, syncopated, and symbolic algebra, Sfard endeavored to locate those “constants” (more precisely, those “developmental invariants”) that ensure the passage from rhetorical and syncopated algebra to symbolic algebra. Rhetorical algebra refers to the reliance on nonsymbolic, verbal expressions to state and solve a problem, as it appears, for instance, in Arabic, Hindu, and Italian Medieval texts. Syncopated algebra is seen as a more elaborate algebra in that, although still relying heavily on verbal expressions, it introduces some symbols, the work of Diophantus being the canonical example. Viète’s systematic introduction of letters epitomizes symbolic algebra. After confronting experimental classroom results with the traditional view of the historical development of algebra, Sfard concluded that one of the development invariants underpinning the passage from rhetorical and syncopated algebra to symbolic (Vietan) algebra is the precedence of operational over structural thinking. Operational thinking, in this context, means a way of thinking about algebraic objects in terms of computational operations. Structural thinking is related to more abstract objects conceived structurally on a higher level.

As we can see, the use of history in Sfard’s approach is an attempt to corroborate parallelisms between ontogenetic and phylogenetic developments. As she said, “history will be used here only to the extent which is necessary to substantiate the claims about historical and psychological parallels” (Sfard, 1995, p. 17). Although she stressed the importance for teachers to be aware of the historical development of mathematics, the intention is not that of creating an historically inspired classroom activity. This is the goal of another perspective in contemporary mathematics education, discussed in section 7.2. For the time being, we want to mention a sociocultural approach that shares Sfard’s use of history for epistemological reasons but, in contrast, emphasizes the crucial link between cognition and the practical human activity in which
cognition is embedded. This approach (see Radford, 1997a; Radford et al., 2000), inspired by key ideas of the Vygotskian and cultural perspectives alluded to in section 3 of this chapter, is driven by a conception of knowledge that differs from Piagetian genetic structuralism, particularly in that knowledge and the individuals’ intellectual means to produce it are seen as intimately and contextually related to their cultural setting. Knowledge, in fact, is conceived as the product of a mediated cognitive reflexive praxis (see Radford, 2000b). The mediated character of knowledge refers to the role played by artifacts, tools, sign systems, and other means to achieve and objectify the cognitive praxis. The reflexive nature of knowledge is to be understood in Ilyenkov’s sense, that is, as the distinctive component that makes cognition an intellectual reflection of the external world in the forms of the individual’s activity (Ilyenkov, 1977, p. 252). Knowledge as the result of a cognitive praxis (praxis cogitans) emphasizes the fact that what we know and the way we come to know it is framed by ontological stances and by cultural meaning-making processes that shape a certain kind of rationality out of which specific kinds of mathematical questions and problems are posed.

Theoretically, however, this does not mean that the study of knowledge is determined by social, economical, and political factors because these are also historically produced. Certainly, the link between culture and cognition is more subtle than the distinction between the “internal” and “external” realms employed in many historiographic approaches that see the external as mere stimulus for conceptual changes and development. Methodologically, this means that the study of the historical development of mathematics cannot be reduced to the sociology of knowledge. This also means that such a study cannot be done through the analysis of texts only. The “archive” (to borrow Foucault’s expression), as a historical repository of previous experiences and conceptualizations, bears the sediments of social, economic, and symbolic human activities. Therefore, understanding the rationality within which a mathematical text was produced requires relocating the text within its own context. The goal of this kind of epistemological reflection is not to find a parallel between phylogenetic and ontogenetic developments. In the sociocultural approach that we advocate, mathematical texts from other cultures are investigated while taking into account the cultures in which they were embedded. This allows the researcher to scrutinize the way mathematical concepts, notations, and meanings were produced.

Through an oblique contrast with the notations and concepts taught in contemporary curricula, we seek to gain insights about the intellectual requirements that learning mathematics demands of our students. We also seek to broaden the scope of our interpretations of classroom activities. In designing classroom activities, we aim at eventually adapting conceptualizations embedded in history to facilitate students’ understanding of mathematics. Our work on Babylonian algebra and the teaching of second-degree equations (Radford & Guérette, 2000) is an example of the latter. Our classroom research on the strategies students use to deal with the algebraic generalization of patterns and the way they conceive relations between the concrete and the abstract (see Radford, 1999b, 2000c)—research based on our investigation of pre- and Euclidean forms to convey generality (Radford, 1995a)—is an example of oblique contrast between past developments and contemporary students’ conceptualizing processes.

Our classroom research on the introduction of algebraic symbolism also benefited from our epistemological inquiries based on editions of original texts from Medieval and Renaissance Italian mathematics (Radford, 1995b, 1997b). Space constraints do not allow us to go further, but this anthropological approach to the epistemology of mathematics offers a new view of the rise of symbolic algebra in the 16th century. The difference from traditional views stressing the passage from syncopated to abstract algebra in terms of abstractive processes is that, in our account, changes in development are related to changes in societal practices and the way in which mathematical
conceptualizations are subsumed in them. Briefly, what we find in our analysis is that there were two main mathematical practices in the early Renaissance, that used by merchants and abacus mathematicians and that used by humanists and court mathematicians. While the latter were busy with the restoration of Greek texts, the former were applying Arabic algebraic techniques to practical as well as nonpractical problems (e.g., problems about numbers). Symbolic algebra was a timeconsuming effort made by Italian humanist and engineer mathematicians, such as the priest Francesco Maurolico, who eradicated all commercial content in his *Demonstratio Algebrae*, which was completed October 7, 1569 and edited by Napoli in the 19th Century (Napoli, 1876). Another example is the engineer Rafael Bombelli, who, after having learned that the first books of Diophantus’ *Arithmetic* were on the shelves of a Roman library, studied them and ended up eliminating the commercial problems in his *Algebra*. Bombelli provided a final version of it that conformed much more to the humanist understanding of Greek mathematics. In France, a similar effort was made by the humanists Jacques Peletier and Guillaume Gosselin (although in this case, the promotion of French as a scientific language was an important drive; Cifoletti, 1992). The underlying reason for the effort to introduce a specific symbolism in algebra was not due to the limitations of vernacular language. Mathematicians working within the possibilities offered by rhetorical algebra produced many difficult problems involving several unknowns, as can be seen in Fibonacci’s *Il Flos* (Picutti, 1983). These problems could not be simplified by the introduction of letters because what was symbolized in the emergence of symbolic algebra did not include all of the unknowns mentioned in a problem but only one of them. (See, for instance Bombelli’s symbolism or the neogeometrical example in Piero della Francesca’s *Trattato d’abaco*, edited by Arrighi, 1970.) It was only later that some in Germany began using letters for several unknowns (see Radford, 1997b). In our approach, the emergence of algebraic symbolism appears to be related to the effort made by humanists and court-related mathematicians to render the merchant’s algebra noble and Court worthy (details in Radford, 2000b). This was accomplished by the lawyer and mathematician François Viète, at the French court, who followed the prestigious Greek traditions typified by Diophantus’ *Arithmetic* rather than the multitude 15th- and 16th-century of abacus treatises.

We now discuss a second reference to the use of history in contemporary mathematics education, that which aims at enhancing, through explicit pedagogical interventions the students’ learning of mathematics.

### 7.2. Enhancing Students’ Mathematical Thinking Through Historically Based Pedagogical Actions

Boero and collaborators (see Boero, Pedemonte, & Robotti, 1997; Boero, Pedemonte, Robotti, & Chiappini, 1998) made use of the mathematics history to investigate the nature of theoretical knowledge and the conditions by which it emerges. Their historico-epistemological analysis aims at looking for elements considered typical of mathematical thinking, such as organization, coherence, and systematic character. They have investigated the role played by definitions and proofs, as well as by the type of theoretical discourse. The framework draws from Bakhtin’s theory of discourse, mainly from the theoretical construct of “voice” (Bakhtin, 1968, Wertsch, 1991) and from Vygostky’s distinction between scientific and everyday concepts (Vygotsky, 1962). The historico-epistemological inquiry is subsequently invested in the design and implementation of teaching settings based on a careful selection of primary sources of which the main objective is to allow the students to echo the voice of past mathematicians. In the students’ echoing process, the students bring their individual subjective and cultural backgrounds to build from it a “voices and echoes game,” which proves
to be fruitful for the acquisition of theoretical knowledge. The voices from the past are not listened to passively but actively appropriated through an effort of interpretation. Usually the students’ echoes may take various forms. Boero and his team have provided a categorization of some of the ways in which the students enter the dialogical game. For instance, a “mechanical echo” consists in precise paraphrasing of a verbal voice, whereas an “assimilation echo” refers to the transfer of the content and method conveyed by a voice to other problem situations. A “resonance” is a student’s appropriation of a voice as a way of reconsidering and representing his or her experience.

Among the concrete instance of theoretical knowledge examined by the authors are the theories of the falling bodies of Galileo and Newton, Mendel’s probabilistic model of the transmission of hereditary traits, and theories of mathematical proof and algebraic language, all of which feature aspects of a counterintuitive character.

Another example of the contemporary use of history in the classroom is the research of Sierpinska and collaborators. One of the goals of this research is to provide an alternative, based on the use of the Cabri-Géomètre software, to the traditional axiomatic approach to the teaching linear algebra in undergraduate courses. A problem examined in this research, which underlies important aspects of the learning of basic linear algebra, is that of understanding key differences in the representations of mathematical objects. In this line of thought, Sierpinska has emphasized the distinction between a “numerical” and “geometrical” space. The objects of the arithmetic spaces are sets of n-tuples of real numbers defined by conditions (in the form of equations, inequalities, etc.) on the terms of the n-tuples belonging to the sets. It stresses the fact that these objects can be represented by geometric figures (e.g., lines, surfaces). Geometric objects, in contrast, are defined as a locus of points verifying some conditions (e.g., the “geometric circle” means the locus of points equidistant from a given point). The geometric objects can be represented by sets of n-tuples defined by conditions on their terms (e.g., by equations). Thus, in the case of arithmetical spaces, the geometrical aspect is derived from the numerical one; in the case of geometrical spaces, the numerical aspect results from the geometrical one. A suitable understanding of elementary linear algebra requires the students to establish a convenient relation between the geometrical and the numerical views of the objects of linear algebra and to grasp that the roles of objects and representations are reversed.

The difference between geometrical and numerical space is clear in the history of linear algebra. Sierpinska, Defence, Khatcherian, and Saldanha (1997) identified three modes of reasoning, which they labeled “synthetic-geometric,” “analytic-arithmetic,” and “analytic-structural.” As they noted (a more detailed report is in Bartolini Bussi & Sierpinska, 2000), the concepts of linear algebra do not all have the same meaning and, in the classroom, they are not equally accessible to beginning students. The design of the teaching activities as well as the understanding of students’ answers took into account the modes of reasoning as determined in the historico-epistemological analysis. (An extended account of the teaching activities can be found in Sierpinska, Trgalová, Hillel, & Dreyfus, 1999a and Sierpinska, Dreyfus, & Hillel, 1999b.)

8. SYNTHESIS AND CONCLUSION

In this chapter, we dealt with one of the many uses of the history of mathematics in mathematics education, namely, a use that can be characterized as an attempt to investigate historical conceptual developments to deepen our understanding of mathematical thinking and to enhance the students’ conceptual achievement. In the first part of the article, we saw how psychological recapitulation was imported from biological recapitulation and gave rise to a discourse that framed much of the discussions about child development since the beginning of the 20th century. Psychological
recapitulation was adopted by some eminent mathematicians who, in one form or another, supported the idea that in developing their mathematical thinking, children would traverse similar steps as those followed by humans. Within this conception, children will supposedly find during their development some similar problems, difficulties, or obstacles as those encountered by past mathematicians. Recapitulationism, we argued, served the cause of some mathematicians as a means to counter the teaching orientation based on commitments to rigor and logical structures arising in the flow of the research on the foundations of mathematics at the turn of the 20th century.

Nonetheless, one of the problems with the recapitulationist approach is that conceptual developments are seen as chronologically self-explanatory, and psychological evolution is taken for granted. Furthermore, knowledge is conceived as having little (if any) bond to its context, and the idea of history is reduced to a linear sequence of events judged from the vantage point of the modern observer. In all likelihood, the extremely low number of studies that attempt to check the validity of recapitulation law is evidence of the impossibility of reproducing the conditions in which ideas developed in the past. As Dorier and Rogers noted, "naive recapitulationism" has persisted in many forms and now we accept that the relation between ontogenesis and phylogenesis is universally recognized to be much more complex than was originally believed" (Dorier & Rogers, 2000, p. 168).

This statement corresponds well with recent nonpositivist epistemological and anthropological trends. Indeed, in emphasizing the relation between knowledge and social practices, these trends have raised some criticisms to the acultural stance conveyed by the general and universal character of the recapitulation law, thereby opening new ways to reconceptualize the relations between historical conceptual developments and the teaching of mathematics.

In the course of our discussion, we mentioned two different and critical stances toward the relation between ontogenesis and phylogenesis as elaborated by Piaget and Garcia on one hand and by Vygotsky and his collaborators on the other. The way Piagetian and Vygotskian epistemologies have inspired current work on contemporary mathematics education was made clear in the brief presentations of specific traits in the works of Sfard, Radford, Boero, and Sierpinska, works that attempt to contrast (with different purposes and in different senses) ontogenetic and phylogenetic developments to shed light on the nature of mathematical knowing as well as on the teaching and learning of mathematics.

Regarding recommendations for future research, it can be suggested, in light of the previous discussion, that a pedagogical use of the history of mathematics committed to enhance students' conceptual achievements requires a critical reflexion about the conceptions of ontogenesis and phylogenesis and, of course, of knowledge itself. But to be fruitful in practical terms, such a critical reflexion must be clear about its classroom implications. In particular, efforts to include teachers in the reflexive enterprise must be made. The work of Furinghetti suggests that to reach effectiveness in using history, teachers' willingness is not enough. To use history productively, teachers need to gain an appropriate understanding of differences between ontogenetic and phylogenetic developments and to bear a critical stance toward recapitulation views. As the sophisticated methodology of Boero's approach suggests, this requires teachers to be amply comfortable in handling cognitive and historical aspects. Let us make three suggestions concerning actions for research.

1. On a theoretical level, discussions about recapitulation and its different meanings should be promoted among historians, epistemologists, psychologists, anthropologists, and mathematics educators.
2. On a practical level, models of contrasts and conceptualizations between ontogenetic and phylogenetic developments also should be considered further. Models
of contrast may help us to better grasp specific traits of mathematical thinking, its relation to the cultural settings, and the mathematical concepts thus produced. This can lead to a better understanding of the kind of practical pedagogical interventions that can be envisioned.

3. Theoretical reconceptualizations of recapitulation and contrasts and comparisons between ontogenetic and phylogenetic domains should be explicit as to how they can frame the engineering of material and teaching sequences.

We consider these related research topics as being interactively fed by theoretical enquiries, historical studies, and also classroom observations.

The course of the three aforementioned actions for future research will ultimately depend on the very conception of mathematical knowledge to be adopted. At this point, two main contrasting trends seem to be emerging. In the first trend, what makes the specificity of mathematical knowledge is its systemic, objective, and logical nature (see Fujimura, 1998). In the second trend, which is much more anthropologically driven, knowledge is conceived as a kind of culturally framed activity enabling individuals to enquire about their world and themselves. Here “systematicity” and “logicality” are seen as circumscribed characteristics of knowledge that can be different from culture to culture (see Radford, 1999c). Between them, of course, many possibilities can be envisaged. To theoretically elaborate on some of those possibilities, to build practical and conceptual reflexions about historical and contemporary “developments,” and to deepen our understanding of mathematics and facilitate the way students learn is a challenge for the years to come.

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REFERENCES


SECTION IV

Influences of Advanced Technologies
CHAPTER 26

Mathematics Curriculum Development for Computerized Environments: A Designer–Researcher–Teacher–Learner Activity

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The goal of this chapter is to shed light on the development of mathematics curricula integrating interactive computerized learning environments. Rather than describe and analyze one of the components in isolation from the others, we attempt to give a comprehensive picture of the compound and long-term activity of curriculum development.

Curriculum development is the process of developing a coherent sequence of learning situations, together with appropriate materials, the implementation of which has the potential to bring about intended change in learners’ knowledge. The term knowledge may be understood and interpreted quite differently by various parties such as decision makers, curriculum project team members, subject matter specialists, researchers, teachers, students, and their parents. This is one reason why curriculum development may lead to tensions between some of these parties.

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The situation is especially complex when the activity of curriculum development is aimed at learning mathematics in an environment in which the benefit from the potential of computerized tools has a central role. In their comprehensive chapter on ‘computer-based learning environments in mathematics,’ Balacheff and Kaput (1996) explained why they think that the technology’s power is primarily epistemological, and added, “While technology’s impact on daily practice has yet to match expectations from two or three decades ago, its epistemological impact is deeper than expected” (p. 469).

In this chapter, we aim to show how such epistemological impact may lead to the design and realization of a curriculum, which does impact on the daily practices of teaching and learning mathematics in many classrooms. Although any curriculum development project is embedded in its own sociocultural context, there are also many common features between them. In this chapter, we describe and illustrate these common features. Rather than deriving them from theoretical deliberations, we will give meaning to them via specific examples of curriculum development. In other words, the specific examples will serve as appropriate windows through which curriculum development is seen as a comprehensive, theoretically and practically consistent activity. These windows belong to CompuMath, a large-scale curriculum development, implementation, and research project for the junior high school level. The CompuMath project has been active for the past 6 years. It is the most recent cycle in a long-term process that was initiated more than 20 years ago and propagated in subsequent cycles of curricula. These cycles are all based on the same national syllabus, and in each of them our main goal was to design and create a learning environment in which students are engaged in meaningful mathematics.

By meaningful mathematics, we mean that students’ main concern are mathematical processes rather than ready-made algorithms. The following mathematical processes serve as a representative sample: Inductive explorations—generalizing numerical, geometrical, and structural patterns, making predictions and hypotheses; and Explaining, justifying, and proving these hypotheses. These processes arise for the students in familiar problem situations as natural means for investigating and solving the problem, rather than as ritual procedures that are imposed by the teacher or the textbook.

In each cycle of curriculum development, we took into account the lessons learned from research and development in previous cycles, theoretical frameworks and relevant cultural artifacts which were available at the time (for example computerized tools in the CompuMath cycle) and, above all, our sociocultural view about mathematics and the learning of mathematics. Such a curriculum development cycle is a comprehensive process, which consists of three stages. The first stage involves design considerations, before starting the actual development and research work; the second consists of a first design of the activities and their implementation in a few classrooms, accompanied by classroom research on learning and teaching practices (observations, data collection, and analysis); and the third stage comprises the creation of coherent sequences of redesigned activities forming a complete curriculum and its implementation, including the dissemination of the curricular aims and “spirit” on a national scale.

When discussed theoretically, the potential of computerized learning environments refers to what might happen in such environments. “Developmental research” (Cobb, 1988) involving computerized learning environments tries to show an existence example of what can happen, and to serve as a window for investigating how it happens. Practice-oriented educators are faced with the challenge to make things happen for large populations of students and teachers. In the CompuMath activity of curriculum development, we had all of these goals in mind at all times. The team thus has to deal with many faces of theory, research, and practice of development and implementation,
where practices are fed by theory and research and vice versa. The team functions as a huge cell eager to live and develop, but the life of which is in large part determined by its interaction in an unknown world. Through this interaction, the curriculum development activity constantly redefines its own components.

This chapter has two parts. In the first part, we describe the specific characteristics of the three stages in terms of various components of the curriculum development process, as listed below. Particular attention will be paid to the issues related to the use of computerized tools. In the second part, we take a more longitudinal view and present three narratives that are representative of the process of curriculum development. Although these narratives are based mainly on our own experience, they do relate strongly to theories and work done by others and thus contribute to the provision of a comprehensive view of mathematics curriculum development for computerized environments. Each narrative focuses on a small number of major concerns in curriculum development including the role of research (the section on geometry), the choice and potential problems of computerized tools (the section on algebra), and project work and learning trajectories (the section on statistics).

The main issues in all three stages of curriculum development, fall into four groups. First, mathematical content and syllabi as given by external agents, explicitly or implicitly, as well as possible national standards and international trends. Second, the participants in the process, from project team members to students, teachers, classrooms, principals, and other functionaries of the school system, each with possibly different roles at different stages. Third, the theoretical, sociocultural and technological background of different participants, including their knowledge and experience of research; and fourth, the actual process of design, development, and implementation. The sections of the chapter are organized around these issues.

STAGE I: PREDESIGN CONSIDERATIONS

Syllabus, Curriculum, and Standards

In the introduction to his section Curriculum, Goals, Contents, and Resources, Kilpatrick (1996) claimed: “In most of the countries of the world, the 20th century has witnessed a rather strong stability in the structure of the school and university mathematics curriculum even as waves of reform have swept across the surface” (p. 7).

We propose to refine this diagnosis by distinguishing between the terms syllabus and curriculum. In many countries, a central syllabus is prescribed by some authority. This syllabus is usually expressed as a list of contents, skills, or both which students at a specific age or level should know. Often an external, central examination with a crucial role in the students’ academic future is imposed, based on this syllabus. It seems that in the above claim about curriculum stability, Kilpatrick related to what we call here, for purpose of clarity, a syllabus.

In contrast, a curriculum, as we understand it, is a far more comprehensive notion. Its goals are intended changes in learners’ knowledge (in the widest possible sense), and it is expressed as a coherent sequence of learning situations, together with the necessary materials such as textbooks, teacher guides and many other components created to implement the intended changes. Hence, a successful curriculum mediates teaching and learning in actual classroom practice in such a way as to bring about intended change in learners’ knowledge.

One of our goals in this chapter is to show that even in places where an official prescribed syllabus does exist, and even when this syllabus stagnates for a long time, far-reaching change is possible. Such change can be achieved through a carefully designed and developed curriculum by means of approaches to problem solving,
classroom practices, the incorporation of sociocultural tools such as computers. In this manner, genuine change in mathematics learning and teaching (official or not), and even in the mathematical content itself, can be achieved. An example is provided by the “calculator-aware number curriculum” in England (Ruthven, 1999) in which a government-supported curriculum development project had a strong influence on the design of a national curriculum (what we would call a syllabus) in mathematics.

Syllabus and curriculum (as products) may be seen as two poles between which the curriculum development activity is taking place. In many countries, it is common that materials for classroom use are produced by a single writer, usually supported by a publisher. Such materials tend to reflect the bare syllabus, presumably because a lone writer lacks the resources or the motivation for change. In other cases, as we demonstrate here, the curriculum is immensely richer than the syllabus. In these cases, there exists no direct and easy translation of the syllabus into a curriculum that supports the intended changes in learning and teaching processes in the classroom. There are two crucial reasons for this. One is that the syllabus is a static list that deals only with the questions of what contents are to be learned, whereas the curriculum leads the practice of doing mathematics in the classroom, and as such it also deals with the how they are to be learned. The second reason is that unlike a syllabus, the curriculum relates to mathematical processes such as visual reasoning, hypothesizing, and investigating.

To bridge the gap between syllabus and curriculum, documents intended to inform and lead reform efforts have been published. The most impressive of these are the National Council of Teachers of Mathematics (NCTM) standards (NCTM, 1989), published as the result of a 5-year effort by leading mathematics educators in the United States and revised on the basis of a decade-long follow-up effort (NCTM, 2000). The NCTM standards go far beyond the bare list of mathematical topics; on the other hand, they are still far from constituting a curriculum that can be implemented in classrooms.

If no appropriate guidelines such as the NCTM standards are available to curriculum developers, they have to develop “internal” standards, to guide their curriculum development work. Such internal standards may be implicit or explicit. In the context of the CompuMath project we dealt with the reality of a rather rigid official syllabus on one hand, and long-term government support for innovative curriculum development projects on the other. In the absence of external standards, we developed internal standards. Some of these were adopted or adapted from previous cycles of curriculum development, others were decided explicitly during Stage 1, and still others that may have been implicitly influencing some of the work at Stage 1 became explicit only during Stage 2. The standards of the CompuMath team at Stage 1 were as follows:

1. Inquiry (observing, hypothesizing, generalizing, and checking) is a desirable mathematical activity.
2. Mathematical activity should be driven by the goals of understanding and convincing.
3. Proving is not only the central tool for providing evidence that a statement is true but should also support understanding why it is true.
4. Mathematical activity should take place in situations that are meaningful for the students.
5. Mathematical activity must stem from previous knowledge (including intuitive knowledge).
6. Mathematical activity should be largely reflective.
7. Mathematical language (notation systems) fosters the consolidation of mathematical knowledge; it should be introduced to students when they feel the need for it.
8. Technical manipulation is not a goal in itself but a means to do mathematics.
9. Computer tools support and foster the above and beyond.

Some of these standards were already well formulated at this stage, whereas others were not. For example, Standards 1, 3, and 6 deal with the character of students’ mathematical activity and with forms of their mathematical knowledge, topics with which the team had extensive experience from previous cycles; therefore, we were able to give them a definitive formulation. On the other hand, Standard 9 relates to computer tools that were rather new to us at the time; it is optimistic but vague because, at that time, so was our knowledge about the potential of computers as a regular and integral part of classroom activity. The development and elaboration of this standard in the course of the project is discussed in the following sections.

Participants in the Curriculum Development Activity

At the predesign stage, only a few people are actually participating in the activity of curriculum development, namely the members of the research and development team. For the CompuMath project, these were mostly members of the mathematics group at the Department of Science Teaching of the Weizmann Institute. The members of the team included designers who specialized in producing written materials, experienced teachers working with the team part time, and researchers in mathematics education. At that stage, the team functioned as “designers” of the future curriculum; they regularly imagined how a particular design would play out in classrooms with students and teachers, virtual participants in the activity of curriculum development, which the designers had in mind. One of the team members described this as follows:

I am thinking about the big mathematical questions to be addressed by means of a classroom activity; when doing this, I think about the children intuitively, based on my teaching experience. This is why I am much better at developing junior high school materials than, say, elementary school materials: I have many years of teaching experience at the junior high school level. I am playing things through with virtual children—the kind I know from experience—and this often leads me to reconceptualize or reformulate, before a particular activity is even tried out in the classroom. Moreover, classroom trials of activities have often failed because teachers acted differently from what I had expected. Therefore, I now imagine a virtual teacher and often write a brief teacher manual for her before I ever send an activity to be tried in a classroom. This again has frequently led to changes in the design of planned activities before they were tried out.

Tools, Theories, and Research

At this stage of the CompuMath project, we invested a considerable amount of time and effort in analyzing various computerized tools and establishing criteria for choosing the technology to be incorporated in our future work. We list here the main criteria that determined our choices, together with the underlying theoretical considerations, and explain in a general manner how the tools we chose actually satisfy the criteria. In the sections on geometry, algebra, and statistics, we complete this discussion by means of evidence for the potential of the chosen tools to support curricula that live up to our internal standards in specific content areas.

The primary consideration we used in choosing a piece of software for a specific mathematical topic was the degree of adaptation of the software to the deductive nature and the content structure of the topic. This led us to define the following three more specific criteria:
1. The generality of the tool, its applicability in different content areas, its availability, and its cultural status. Most tools have multiple uses. For example, a spreadsheet such as Excel may be used to store and analyze data, to create sequences of numbers from other sequences of numbers by manipulating general symbolic rules, and to represent numerical data graphically. Similarly, function graphers, as parts of more comprehensive software packages, are used in many domains in science and mathematics. More broadly speaking, we considered the cultural nature of the tool. Spreadsheet programs and graphers are ubiquitous in various places and domains. Such tools may be used again by students later, even in their professional life.

2. The potential of the tools to develop and support mathematization by students working on problem situations. This can take the form of amplification and reorganization (Pea, 1985; Dörfler, 1993) and of experiencing new “mathematical realism” (Balacheff & Kaput, 1996). For example, the power of a grapher to smoothly transform a function from its algebraic to its graphical representation, and the availability of the corresponding numerical data directly from the graph make it possible to deal with problem situations involving complicated functions at an early stage of learning. Similarly, the capabilities of spreadsheets enable students to explore the meaning of trends in data and to use different representations to exhibit these trends. They thus provide students with opportunities to relate data mathematically. And finally, the dragging mode in dynamic geometry environments provides the means to investigate geometric features as invariants of a changing figure.

3. The third criterion is what we call communicative power (or the semiotic mediation power) that is the power of the tool to support the development of mathematical language. This concerns the nature of the symbol system used by the tool, and its relation to the symbol system more commonly used in mathematics. The symbol systems of graphers and dynamic geometry programs are in one-to-one correspondence with the symbol systems of mathematics. The symbol system of Excel, on the other hand, is intermediate between the formal algebraic symbol system and an informal verbal notation system.

All three criteria are closely related to the multirepresentational nature of the tools, the support they give to transformations between representations, and to the manipulation of mathematical objects (drawings, graphs, tables). They also imply that the curriculum developer pay attention to the fact that every representation admits many different representatives of the same mathematical object (such as many different graphs of the same function) and that students may choose to transform between these (Schwarz & Dreyfus, 1995). Efficient problem solving in mathematics depends on the flexible manipulation of objects in different representations and notation systems (e.g., the algebraic, graphical, and tabular representations in algebra, the pie chart or the frequency stick chart in statistics, geometrical drawings). Actions or operations one can undertake in each of the representations are different in nature when one is limited to paper, pencil, and ruler. Kaput (1992) called a notation system such as the algebraic representation an action notation system because it allows calculations and transformations. In contrast, he called notation systems, such as the graphical and tabular representations, display notation systems because the activity of the user is generally confined to interpretation. This theoretical distinction between action and display notation systems does not hold any more when one uses computer tools that provide, in addition to the representations themselves, the option of passage among representations and user based manipulations. In this case, all representations become action notation systems: It is possible to “walk” on a graph, to stretch graphs (scaling), to change geometrical shapes by dragging, to rearrange a table, and so on.

The three criteria led us to decide on a type of tool for each of the main topics in the syllabus: spreadsheets (for statistics and algebra), graphers (for functions), and dynamic geometry. The selection of a particular piece of software within these
types was based on various additional criteria including user-friendliness, didactic power (e.g., how many graphs can be shown simultaneously), and more mundane considerations such as affordability and availability. Detailed discussions of the nature and potential of the chosen tools appear below.

**Design and Development Plans**

At this stage, design and development existed only as plans for future work. Our attitude during this stage, and during much of the initial development in Stage 2, is expressed well in a few lines by Balacheff and Kaput (1996): “In challenging most traditional assumptions about teaching and learning, technology forces us to think deeply about all aspects of our work, including the forms of the research that need to be undertaken to use it to best advantage. Clearly, our most important work lies ahead of us” (p. 495).

Although we were experienced curriculum developers, we faced a new learning arena. This challenged us to rethink our day-to-day work while creating new learning environments. We reflected on integrating and translating into realizable plans the standards discussed above, given the computer tools and given limitations imposed by the educational system. As in the case of the “virtual student” and the “virtual teacher,” this “virtual design” emerged, at least partially, from our established practices and shaped our new ones. The decisions made at this stage focused on the following:

1. The nature of mathematics—we decided to broaden the learned mathematical contexts.
2. The scope—we decided to create a curriculum for all the central topics in the Junior High School syllabus (Grades 7, 8, and 9).
3. The ways in which the tools would be incorporated—we decided to base the teaching–learning process on the regular use of tools, rather than to use them sporadically only.
4. The order in which we would develop the different topics—we decided, for example, to start with functions and to base the learning on the use of multi-representational tools (graph plotters) because their potential seemed obvious and relatively easy to adopt. Moreover, the possibility to use a large variety of functions (including linear and quadratic that appear in the syllabus) was an opportunity to create rich problem situations.
5. The characteristics of teaching–learning processes—we decided to amplify processes, which were started in previous curriculum development cycles, such as investigations of open problem situations, in which groups of two to four students deal with a broad variety of mathematical phenomena.
6. We were developing a nontraditional curriculum for a large-scale population of teachers and students—we thus decided to educate and train teachers in the spirit of the project’s goals from the beginning.
7. We were fully aware to the novelty of the curriculum development as well as of our limited experience with it—we thus decided that sound research was going to be an integral part of our work.

**STAGE II: INITIAL DESIGN–RESEARCH–REDESIGN OF ISOLATED ACTIVITIES**

The goal of stage II was the first realization of the plans and predesign considerations, elaborated in line with the internal standards agreed on in the previous stage and the knowledge and beliefs of the participants. This first realization consisted of the design
of mathematical activities and the investigation of their impact in trial classes. Because so many new objects and actions of curriculum development had to be taken into consideration and because the team members could grasp them only gradually, the activities designed in this stage were isolated rather than in sequence. The overall continuum served as a somewhat vaguely envisaged background against which the isolated activities were designed. This stage was characterized by the following features:

1. Dilemmas concerning the translation of the contents prescribed by the syllabus into first trial activities, which conform to the standards of the emerging curriculum.
2. The new community of participants in the curriculum development activity.
3. The development of isolated activities as a dialectic process of design–research–redesign cycles, the theoretical framework of the research, and what we can learn from it.
4. First models of implementation with a view to the larger population.

**Dilemmas on the Way from Syllabus to Curriculum**

During the first steps of development, developers cannot avoid thinking and rethinking the basic approaches to the specific mathematical content. Often, the dilemmas arise from conflicts between the content as it appears in the syllabus and the approaches that were adopted to make full use of computerized tools. Such dilemmas serve as catalysts for rethinking approaches and methods and for innovative solutions in the curriculum development work. We demonstrate this issue by relating to three content areas.

**Euclidean geometry** is one of the main topics in the prescribed syllabus, according to which it is to be taught in the second half of the 8th grade and during all of 9th grade. Within Euclidean geometry, proof and proving are central. In our section on geometry, we discuss how familiarity with dynamic geometry software, and the awareness of its potential, put in question the role of proving because invariant properties of geometric figures can be observed visually and justified inductively. This caused us to rethink the various roles of proof. As a consequence, we started to develop activities where proof is considered as a tool for explanation, by means of which students may understand and explain why the conjectured invariant attribute, which they discovered in the dynamic geometry environment, is really invariant. Similarly, we developed activities in which the crucial point cannot be discovered by dynamic geometry, for example, because it is an impossibility; in such a case, proof is the only way to be certain about the conclusion. Examples for each of these types of activity are given in the section on geometry.

Within the topic of **functions**, the possibility of obtaining the graph of any function from its symbolic representation, to “walk” on the graph, and to read from the graph the coordinates of special points like extrema, considerably enrich the teaching–learning of functions in junior high school. But this power puts in doubt what is commonly presented as a main motive for learning calculus in high school: the ability to find the main features of a given function’s graph. The dilemma arises of whether to reduce the teaching of derivatives in high school or to give it new motives, such as modeling by differential equations.

Before starting the design and development of **algebra** activities, we had to make a decision about the approach. Several considerations led to the selection of a functional approach (Yerushalmy, 1997). First, the main focus of beginning algebra in the 7th grade is the generation of symbolic generalizations of number patterns, which can naturally be seen as the discovery of the symbolic rule of a function. In addition, the use of a spreadsheet emphasizes the transition between dynamically varying numbers and their symbolic rules. Moreover, graphs can be produced when desired. We provide an example of the rich activities developed for beginning algebra in the
section on algebra. The functional approach and the use of graphing software appear
to create an appropriate opportunity to broaden the concept of solving equations.
Solving equations was presented as finding the intersection points of the graphs of two
functions, linear or not. Thus, the graphical solution of a given situation became more
fundamental than the algorithmic–symbolic one. On the other hand, students were
required by the official syllabus to master the algorithm for solving linear equations,
and we needed to develop suitable activities for teaching it. One of these activities
dealt with the transformation of a given equation into an equivalent one. Students
in the trial classrooms, who were already familiar with the intersection point view
of a solution, were asked to conjecture the graphical representation of an equivalent
equation. Students, and even some of the teachers, were quite surprised to discover,
by means of a computerized tool, that the equivalent equation is represented by
functions that have a different intersection point (with the same x-coordinate). Thus,
we discovered that the functional approach does not well support the algorithm for
solving a linear equation and presented this topic differently.

Through these and similar cases, we learned that we needed to use different ap-
proaches and points of view flexibly in curriculum development just as in problem
solving.

Participants in the Curriculum Development Activity

The curriculum development team members continued, of course, to form the core,
but additional participants were added at this stage: teachers and students in trial
classes, who were not “virtual” any more. The team members as a group were in-
volved with a large and complex array of interrelated tasks: design and development
of activities, in-depth learning of the mediating potential of the computerized tools,
and teaching in trial classrooms, including observation, investigation, and analysis
of teaching–learning processes. The role of several members of the team expanded
considerably during this stage. For example, one central member of the team, an
experienced teacher as well as developer, became part of the research team that in-
vestigated the trial teaching of activities on function in her own classroom (Resnick,
Schwarz, & Hershkowitz, 1994). During this stage, many team members went through
intensive processes of introspection and reflection on their own products and actions.
This issue will be discussed in more detail in the next subsection.

On the other hand, teachers and students in trial classes began to have an impact
on the development process and were thus integrated into the “community of partici-
pants.” The common denominator among all these new participants is that they were
highly motivated to realize the goals of the project and aware of their potential impact
on the curriculum. For example, in our “lab school,” the teachers (some of whom
are members of the core team) used to meet regularly to create activities together,
produce worksheets, and try them with their students. The students were aware that
their role was important in evaluating the new approaches and activities. It often hap-
pened that teachers, and even students, suggested modifying the activities. At this
stage, the researchers in the team videotaped some classroom activities. The students
were generally willing to be videotaped because they felt that they were part of an
adventure and that the difficulties they encountered were precious data that served
to improve the curriculum.

The Dialectic Process of Isolated Activity
Development Through Research

The core of the development work at this stage consisted of developing isolated activ-
ities in design–research–redesign cycles. The process was thus a dialectic one during
which design and research influenced each other.
The various dimensions considered in the design of each activity include the content and the overall mathematical approaches, the intended mathematical thinking processes (generalizing, hypothesizing, reflecting, and justifying), the potential of the tool, the classroom organization (including redistribution of learning responsibilities between students and teacher), practices, and sociomathematical norms.

In trials of these early isolated activities, we were carried away by the exciting and surprising processes we observed and by the extent to which they differed from what we had observed during the previous two decades of development and research. What started as naïve observation and documentation by taking field notes was transformed into coherent research with videotape documentation and detailed analysis and interpretation. The need to describe, understand, explain, and analyze what was going on in these classrooms naturally brought us closer to the concerns of sociocultural psychology. Like many others (c.f. Perret-Clermont, 1993; Yackel & Cobb, 1996), we felt the shortcomings of cognitive theories, methodologies, and tools we had at our disposal to describe and interpret learning and teaching processes in the classroom. We adopted activity theory (Kuutti, 1996) as the theoretical frame for the interactionist approach. The construction of knowledge was analyzed while students were investigating problem situations in different contexts. Research became a crucial component in the curriculum development activity.

Two types of research were interwoven in these design–research–redesign cycles. Both types might be called developmental research (Cobb, 1998) in the sense that they involve instructional development with research. The first used interviews with pairs of students, interlaced with development cycles. In the section on geometry, we present an example comprising several research–redesign cycles.

The second type is classroom research, which focused on investigating the ways in which the goals and standards of the intended curriculum were implemented. We illustrate this classroom research by means of an example in which we investigated students’ processes of hypothesizing and reflecting, as well as the role of teacher in the orchestration of these processes. The example deals with the design–research–redesign cycles of an activity called Overseas at the end of the year-long functions course in Grade 9.

We had been working in a particular trial classroom from the beginning of the school year. The teacher was a member of the CompuMath team and the main designer of the functions’ activities. As team member, she was eager to try the new activities in her classroom, but as teacher, she also had to follow the official syllabus of the Grade 9 functions course with her class. Hence the activities, despite being isolated and innovative, formed an integral part of the official syllabus.

Two team members observed each new activity in this classroom throughout the year. In this way, we accumulated experience concerning the development of learning opportunities through the power of the computerized tool and inquiry during problem-solving processes. At the same time, changes in classroom practices were noted. The observations, which were at first unstructured, became focused in the course of the year. The analysis of the observations and the conclusions we were able to draw served as the basis for the design of other activities as well as for improving the observed activities themselves at the next stage of curriculum development (Stage III).

Hence, in Overseas, we already inserted all the knowledge we had gathered from the development–research experiences in previous activities. The classes were carefully planned as a research arena with precise documentation. Specifically, we observed and videotaped a group of four girls during group work and collected all their products; we also collected the written products of the other groups and carefully recorded all whole-class discussions. On the whole, we had a twofold goal: to develop and structure an “ideal” activity and at the same time to examine and check
its realization in the classroom. In particular, we examined students’ ability to make hypotheses, their awareness of the quality of hypothesizing processes, and the nature of different reflective processes in different phases of the activity. In addition, we investigated the orchestrating role of the teacher during the activity.

The research has been reported in detail in Hershkowitz and Schwarz (1999). Some of the conclusions are the following:

1. The power of the tool to deal with a large variety of functions makes rich problem situations possible; for example, in Overseas, students found a local maximum of a function that is a composition of a rational and quadratic function—a type of task not accessible to ninth graders without a graphical tool.

2. In rich problem situations, inquiry is a natural process. Students have the opportunity to move among representations to progress.

3. Asking students to make hypotheses about possible solutions before solving the problem is a valuable didactic technique. The students were able to delay the actual solution and accept hypothesizing as a valuable activity.

4. Reflection does not usually occur spontaneously but has to be initiated, for example, by requiring students to write a group report on their inquiry process.

5. A teacher-led synthesis in a session with the entire class is useful for many reasons. Students can be given an opportunity to report on their work and to practice participation in classroom debates, in which they can give, as well as obtain, critique. The teacher can use their reports to raise criticism and evaluation, as well as for a synthesis of the main processes students went through. The session thus affords another opportunity for reflection. Last, but not least, such a synthesis allows the teacher to define the common knowledge, which she expects the students to have gained.

6. The teacher’s role during the synthesis session is crucial. In Overseas, for example, she made it clear that the goal was not to present results but to reflect on the process they had gone through, in particular, how they had hypothesized possible solutions. Thus, she conveyed that hypothesizing before arriving at a solution is a sociomathematical norm for her classroom.

7. It is advantageous to let students carry out the inquiry and reporting on the inquiry in groups because social interaction in the group supports mathematical argumentation: Students complete, oppose, and criticize others’ proposals, progressing toward agreement among the group.

Classroom research thus gave us a large amount of input in an area with which we had little experience from prior cycles of curriculum development: How to design extended activities based on rich problem situations into multiple phases, including inquiry by groups of students, reporting on the inquiry, and teacher-led synthesis. We note in passing that a more detailed model for an activity contains, in addition, preparatory individual homework before the group inquiry and summary individual work after the synthesis. We also learned about the teacher’s role in inviting students to act differently in each of the phases, so as to have different opportunities for reflection.

During this stage we accumulated several activities in each topic, which incorporated the research results about rich problem situations, activity design, use of computer tools and teachers’ and students’ roles in the learning process. These model activities later served as milestones for the further development (see Heid, Sheets, & Matras, 1990, for parallel experiences).

First Models of Implementation

Implementation takes place at different stages of curriculum development activity. Teachers who chose to teach with the project materials needed a lot of support both before and during their teaching because every component was radically new—the
When a sufficient number of activities on functions had accumulated, we organized an inservice course (about 60 hours during the summer vacation). Teachers from various schools went through the same learning practices, in the same learning environment, as students in the trial classes. They then reflected on each activity as students and as teachers. About half of them volunteered to use these activities and others that would be developed during the following academic year. They received intensive support, mainly in bimonthly meetings with team members. They received the new materials and discussed ways of teaching and the rationale and practice of evaluating students learning in the new environment. An additional, quite different role of this group of teachers was to provide feedback from their classes. This feedback was invaluable in the next stage of curriculum development as well as in the establishment of curriculum implementation practices.

At the same time, a different model of implementation was used; a long-term implementation model, which we call the fan model because it propagates from lead teachers to other teachers. This model formed part of a larger project concerned with the use of computers in teaching and learning the mother tongue, foreign languages, and mathematics in all elementary and junior high schools of a particular midsize town.

Two or three teachers from each school were chosen as leaders to introduce computers for teaching mathematics in their school. They received personal computers as well as instruction in basic literacy in the most common uses of computerized tools; they were also connected via an electronic network and met at a municipal teacher center 1 day per week. Within this framework, they took a 2-year course on mathematics teaching with computerized tools. Although we had no influence on the choice of the participating teachers, some established very creative ways of teaching. For example, some teachers asked their students to work in pairs and to create their own investigation project. They encouraged the students to reflect on choosing the subject, the ways of posing questions, the openness of the project, and so on. In some classes, each pair of students was asked to give their project to another pair for criticism, after evaluation criteria had been established in whole-class discussions. As a result of our involvement in this project, all junior high school classes in this municipality now use the CompuMath curriculum.

Already during the second stage, we thus had opportunities to learn about the potential of different implementation models and different types of inservice courses. We came to know some of the difficulties and limitations in leading reforms in general, and reforms involving computerized tools in particular, in a large-scale population. This issue is taken up in detail in the next section.

STAGE III: EXPANSION

Having an impact on a large number of students and teachers is the raison d’être of curriculum development projects in general and of CompuMath in particular.

The metaphor of throwing a stone into still water is quite appropriate here. The stone causes waves to propagate outward in circles. These waves are forceful near the place where the stone fell, but become progressively smaller with distance. If only a single stone is used, the waves also fade out with time. To preserve the intensity of the waves, one should continue throwing stones into the water. In CompuMath, we found that the best way to do this is to offer the teacher a large and varied collection of activities, which combine together to cover a whole continuum with a considerable breadth...
of choice. The breadth allows each teacher to choose from the already-developed materials, her own trajectory along the continuum in a way that is appropriate for her class. We were thus led to expand the project along three main lines. The first is the “democratization” of the teacher and student population using the curriculum: Teachers and students in classes unknown to the project team, with their own motives, drives, needs, and school conditions. These may contain elements that are quite orthogonal to the approach and underlying philosophy of CompuMath; as other agents throw other stones, the waves of which interfere with those of the CompuMath project. The second line of expansion is the design and development work, which turned the project materials from a collection of isolated activities in Stage II into a broad and flexible continuum. The third is research, of a more global nature, which emerged as an integral part of Stage III. These three lines of expansion are discussed in the following three subsections.

**Expansion of the Population**

The social texture of the participating teacher and student populations in the third stage was radically different from the previous ones. The expansion to a wider population implied a large measure of democratization for both learners and teachers. Regarding learners, it consisted in the adaptation of the “intended curriculum” to a wider spectrum of possible learners. Some information on possible extensions and needed adaptations was gleaned from the experience during the second stage. For example, the students who participated in the Overseas activity belonged to the upper 60% ability level of the general population. During this activity, groups that finished the inquiry phase early were asked by the teacher to prepare a worksheet about Overseas for students whose level was a bit lower. In the whole-class session at the end of the activity, one student presented her worksheet. It was a sequence of coaching questions such as “draw the surface area graph using the graphical calculator,” “walk on it,” and “read off the minimal point.” The prescriptive character of this worksheet contrasts with the openness of the original activity. This worksheet presentation triggered a class discussion about learning modes suggested by activities such as Overseas. Some students, even though they were able to solve the open version of the activity, preferred a more closed version or felt that students in other (lower level) classes would prefer it. Such a preference reflects a different learning norm. Extending learning situations to a wider spectrum of learners should and did take the existence of such norms, as well as the abilities and motivation of students in the lower 40% of the population into consideration.

With the expansion of the project waves to the heterogeneous general population of learners, to an anonymous population of teachers, principals, inspectors and even parents, less support and monitoring is flowing from the team to each classroom, and less information is coming back from classrooms to the team than in Stage II. The teachers in these classrooms have varying degrees of commitment, and as a consequence the degree of implementation of the project varies from including only sporadic activities to the adoption of the entire approach and set of project materials. The team initiates new ways to encourage and support innovations in schools, without impinging on the schools’ autonomy.

**From Isolated Activities to a Continuum**

The second line of expansion in Stage III is the creation of chains of activities that have the potential to lead to long-lasting cognitive gains. These chains combine and shape the isolated activities developed in Stage II into a whole. They were elaborated according to the following perspectives:
• The structure of the content to be learned
• The standards of the project (see Stage I)
• The power as well as the limitations of the computerized tools (see the section on algebra)
• The available time for computer use in school
• The lessons drawn from the design–research cycles in the pilot classes in Stage II
• The use of activities that offer opportunities to capitalize on lessons from previous activities.

A typical chain of activities in the functions course, for example, constitutes a unit of six to eight lessons, which includes a key activity, a consolidation activity, several additional activities, and several follow-up corners, each with its purpose to be described below.

A unit starts with a multiphase activity based on an open problem situation in the computer lab. This activity was typically designed and investigated in Stage II. In Stage III, it was revised in the light of lessons we had learned from research and observations in Stage II, as well as its role as the key activity of the whole unit. One of the main goals of this opening activity is to create opportunities for students to deal informally, from the start with all concepts, relations, and representations to be learned in the unit as a whole. During the inquiry phase, tasks are set that allow these concepts to arise, without requiring a full treatment. We have thus chosen a holistic, frontal, and bold approach to learning rather than a linear one.

Immediately after the key activity, students are presented with a follow-up activity, which elaborates, formalizes, and consolidates the informal knowledge constructed in the key activity. In other words, the teacher and the follow-up activity function as agents to facilitate a process of mathematization. The follow-up activity may contain some new elements, such as an efficient strategy, a new algebraic technique, or more formal language for one of the concepts. Very often observations in Stage II are the source of such elements. For example, “interesting mistakes,” such as wrong verbal or graphical hypotheses, incomplete strategies, or inefficient representation, are presented as possible solutions to be accepted or refuted. The intention is to create opportunities for students to reflect on these “mistakes,” through dialectical processes. We hope that teachers will pick up these ideas, accumulate interesting responses from their own classroom, and invite their students to reflect on them.

Additional activities are typically different problem situations, each giving rise to a multiphase activity, with or without a computerized tool. The teacher may select two or three for her class or even replace the opening activity with one of them. These activities are not intended to require basically new content or processes. One of their goals is to lead students to more autonomy; there are fewer instructions (such as to use a certain representation) than in the opening activity.

We have observed that during the inquiry phase, especially when students are working with computerized tools, they have strongly varying strategies and rhythms. Two additional corners serve to keep the students “together.” The “beacon” section provides clues and support in various places in each of the open problem situations (the opening one as well as the additional ones) for students who may need such support. The “see if you can” corner offers challenging questions on the problem situations for students who finish the inquiry earlier than others.

A typical unit also contains a collection of homework tasks (without the computerized tool), a summary, usually in verse, in the “poet’s corner,” which summarizes the most important issues in the unit, and a “keep fit” corner of short tasks to practice algebraic techniques.

Finally, two additional corners are devoted to interactive reading of mathematical texts connected to the unit and to reflection on actions, content, and learning processes.
Throughout the unit, reflection is encouraged, either as a monitoring action during a problem-solving process or soon after it. In the “looking back” corner, students have opportunities to reflect globally on the whole unit.

In summary, each unit is organized so as to generate a dialectic process in which students deal with the key concepts and processes of the unit in different contexts—from different angles and with different purposes. Such a dialectic process fits the cognitive accumulation of knowledge well. It emerges from the key activity, in which the concepts and contents are linked mainly through the context of the problem situation, rather than through the mathematical structure. Each of the following tasks in the unit deals with a specific aspect, such as the formalization of the mathematical knowledge or links between different elements; moreover, quite a few of the tasks at the same time stress different aspects based on the standards, for example the students’ mathematical autonomy or the development of their reflective ability.

Research

When the isolated activities were redeveloped and integrated into a whole curriculum, new research questions, such as the construction of the knowledge of individuals over an extended period of time, became important (Hershkowitz, 1999). The construction of knowledge via learning trajectories of students in the space of the classroom can be examined by tracing individuals as well as groups within and across activities. For example, the need arose to investigate the ways in which hypothesizing processes develop in the first few months of algebra. The researcher created a “diary” for each student and for every task from the beginning of the year. In this way, she expected to achieve three things in parallel: trace for each individual student how the notion of conjecture was constructed, investigate how conjecturing processes develop in different types of students, and follow the entire class through an activity to learn about the conjecturing potential of the activity and to improve it if necessary. From this combination of observing individuals and the entire class, we expected to learn about the continuum and to be able to adapt it to fostering the ability of conjecturing.

Another researcher has focused on the progress over time of the relationship between shared knowledge and individual knowledge construction. She has begun observing and documenting the common as well as the separate work of a pair of students in activities scattered along the 7th grade algebra course. She is attempting to understand and interpret each student’s construction of knowledge as well as the social interaction between the pair. The intention is to focus on relationships between the changes in individual knowledge and the changes in shared knowledge as well as changes in the way the students interact. The problem of individual versus shared knowledge is central to our curriculum. Many questions arise, such as to what extent the composition of student groups can influence the type and effectiveness of the interaction and the ensuing learning. Such research has feedback on Stage III development. Interaction in a group and with the entire class, as well as with the computer tool, have a strong influence on building the continuum because curriculum development also includes the planning of the type of collaboration within each phase of a multi-phase activity.

Finally, some of our research focuses on outcomes, in particular, on the effect of the curriculum on knowledge structures. In one such study, we characterized students’ function concept images at the end of the CompuMath course on functions (Schwarz & Hershkowitz, 1999). Specifically, this research does not show how to do curriculum development but rather what is its effect. This was done partly to satisfy our curiosity and partly to provide data to principals, inspectors, and other interested parties.

This concludes the first part of the chapter, in which we have described the characteristics of the three stages of curriculum development. In the second part, we consider
selected aspects of the development process for the various mathematical topics. For each topic, we focus on a few issues particularly salient for this topic. Because many of the illustrations in the above description of the three stages were chosen from the topic of functions, we present below narratives from the three remaining topics: geometry, algebra, and statistics.

GEOMETRY: CONCEPTS AND JUSTIFICATIONS

General questions such as “why teach geometry?” or “what should the study of geometry entail?” have been treated extensively in Lehrer and Chazan (1998). In this section, we focus on two main issues in learning geometry: the construction of concepts and the role of proof and proving. We concentrate on these issues because they appear in the syllabus in many countries and are decisive for geometry learning in many classrooms.

When learning geometry, students are usually presented with a geometrical figure—the mathematical object—represented by a drawing. The geometrical figure is the concept itself, and the drawing is a representative of the geometrical concept(s). Laborde (1993) expressed this as follows: “drawing refers to the material entity, while figure refers to a theoretical object” (p. 49). She made it clear that there is always a gap between drawing and figure for the following reasons:

1. Some properties of the drawing are irrelevant. For example, if a rhombus has been drawn as an instance of a parallelogram, then the equality of the sides is irrelevant.
2. The elements of the figure have a variability that is absent in the drawing. For example, a parallelogram has many drawings; some of them are squares, some of them are rhombuses, and some of them are rectangles.
3. A single drawing may represent different figures (Yerushalmy & Chazan, 1990). For example, a drawing of a square might represent a square, a rectangle, a rhombus, a kite, a parallelogram, or a quadrilateral.

When a single drawing is used as representative of a figure, these three differences between figure and drawing may turn into three connected obstacles to learning.

The isolated drawing is thus ambiguous as a representative of a figure, especially when students are engaging in geometrical situations using pencil and paper. A proper representation of a figure should include an infinite set of possible drawings. Dynamic geometry tools (e.g., the dynamic version of the Geometric Supposer, Cabri, the Geometric Sketchpad, and the Geometry Inventor) offer this feature. They allow the student to “drag” elements of a drawing and thus enable the production of an infinite set of drawings for the same figure, all of which have the generic attributes of the figure (thus overcoming the problem raised by problem 2). This “variable” method of displaying a geometrical entity solves problem 3) of the ambiguity of a unique drawing as a representative of the entity. Moreover, it stresses the intrinsic attributes, the invariants of the entity, thus going a long way toward overcoming Problem 1. It is for these reasons, that we chose to adopt a dynamic geometry environment as a computerized tool for the geometry component of CompuMath. We decided to use the Geometry Inventor (1994), mainly for reasons of availability; an additional reason was the ease with which the geometric variation of the elements of a figure could be represented in a Cartesian graph.

Research indicates that students engaged in dynamic geometry tasks are able to capitalize on the ambiguity of figures in the learning of geometrical concepts (Hoyles & Jones, 1998). In addition, Goldenberg & Cuoco (1998) claimed that
Dynamic geometry offers an interesting arena within which to watch students construct or reconstruct definitions of categories of geometry objects, because it allows the students to transgress their own tacit category boundaries without intending to do so, creating a kind of disequilibrium, which they must somehow resolve. Confusions can be beneficial or destructive. Understanding how students resolve such conflicts will help us devise educationally better uses of the software, ones that maximize the opportunities and minimize the risks of the confusions created by such transgression of accepted definitions. (p. 357)

The above discussion is relevant not only to the choice of a particular technological tool but also to the two main issues to be discussed in this section, the construction of concepts and the role of proving. In each case, we will start with a short presentation of research that had an impact on our work and then demonstrate how research and development interweave in the development of an “isolated activity.”

Concept Construction

A considerable part of the research on concept construction in geometry focused on the role of different examples in the construction of concept images. Among the examples of a concept one can often identify a “prototype,” which is a popular example typically drawn by many students when they are asked to provide a representative of the concept. For example, Hershkowitz (1987) found the following prototypical behavior concerning altitude in a triangle: Students usually draw the altitude inside the triangle, even the altitude to a leg of an obtuse angle. More than that, during the process of learning the altitude concept in a traditional setting, more students constructed the prototypical concept image than a flexible (correct) one. Yerushalmy and Chazan (1993) also reported that teachers and students were unwilling to draw exterior altitudes for obtuse triangles.

Dynamic geometry software enables the design of activities in which students investigate the relevant properties of the figure by means of dragging. Such activities can support students in constructing a more appropriate concept image. More specifically, such an activity can lead students to construct an altitude in an acute triangle and then drag the vertices in such a way that they see the altitude move outside the triangle. During Stage II of the CompuMath project, such an activity was designed (see Fig. 26.1). The team member who proposed and elaborated this activity described the process as follows:

I knew that students were likely to develop a limited concept image of altitude. I needed an activity that would work on this point. So first I sat down at the computer and played around with triangles and their altitudes. Obviously, you can drag the altitude outside the triangle, but simply dragging will not achieve much; before dynamic geometry students also saw obtuse triangles with altitudes outside. I needed a task that would engage them. So I imagined some [virtual] students; I know from my teaching experience how they might think, and what would catch their attention. Suddenly, I noticed something I hadn’t explicitly thought about before: It never happened that a single altitude was outside the triangle. This was what I needed: The students would try to generate a triangle with one altitude outside; they would be surprised at their inability to find one, but not immediately understand why. On the other hand, the reason is clearly accessible to them, and more important, it is directly connected to my main aim, namely, to help them establish the connection between the sides of the obtuse angle and the altitudes to these sides, which are always outside the triangle. The option to combine the dragging mode of the object itself, with the surprise which raises a need for explanation, may influence the creation of more accurate concept images.

Dragging the vertices in Task 1 (Fig. 26.1) enables students to see many different cases, including some with outside altitudes or altitudes that coincide with one of the sides of the triangle.
**Task 1.** Draw a triangle with an altitude to the side \( AC \). Investigate the connection between the position of the altitude and the two angles \( A \) and \( C \).

Consider three cases: the altitude is inside the triangle, outside the triangle, or coincides with one of the sides.

![Diagram of a triangle with an altitude](image)

Explain the connection between the position of the altitude and the sizes of the angles \( A \) and \( C \).

**Task 2.** Construct the two other altitudes and predict, without dragging the triangle, whether each of the following results, is possible. Then check and explain.

- Two altitudes are inside the triangle, and one is outside.
- Two altitudes are outside the triangle, and one is inside.
- All the three altitudes are outside the triangle.

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By describing the connections between the position of the altitude and a relevant attribute of the triangle’s angles, students may create visual and verbal connections. Such an activity may help students to generate empirical evidence to make transition from the particular to the general case and from empirical to analytical reasoning and thus to overcome the tendency to construct a limited concept image.

In Task 2 (Fig. 26.1), students make use of the connections they have established to investigate more complex situations in which all three altitudes are involved. While manipulating the triangle on the screen trying to observe and explain the relationships, students have the opportunity to visualize many representatives of the figure and connect them to the triangle’s attributes; thus, they have opportunities to construct a more sophisticated concept image. By dragging, students realize that in an obtuse triangle, two altitudes are drawn to the legs of the obtuse angle, and therefore both these altitudes fall outside the triangle (see Fig. 26.2).

In a preliminary questionnaire to this activity, a group of students was asked to draw altitudes to a marked side of triangles drawn on paper. Their performance was similar to those described in Hershkowitz (1987). After the activity, students answered the same questionnaire with few or no mistakes.

The altitude task is an example of what Laborde (1999) has characterized as new kinds of tasks that arose out of yearlong evolution of development work in dynamic geometry environments: “tasks in which the environment allows efficient strategies which are not possible to perform in a paper and pencil environment” and “tasks raised by the computer context; i.e., tasks which can be carried out only in the computer environment” (p. 306).

As Laborde pointed out, such tasks may create “intriguing visual phenomena that are not expected by students. The only way of explaining those phenomena is recourse
to theory” (1999, p. 300). This is exactly the intention of the altitude task. Experiencing the impossibility of one outside altitude, while dragging the vertices of the triangle with its altitudes on the computer screen, creates surprise and leads students to a dilemma: Should they continue the empirical search (for a triangle with one altitude outside), or should they attempt to understand the impossibility? This dilemma lets students experience a mathematical need for proof, as described in the next subsection.

The Role of Proof in Dynamic Geometry Environment

For generations proofs were considered as tools for verifying mathematical statements and showing their universality. Hanna (1990) mentioned Leibniz, who believed “a mathematical proof is a universal symbolic script which allows one to distinguish clearly between fact and fiction, truth and falsity” (p. 6). Thus, the two classical roles of teaching proofs are to teach deductive reasoning as part of human culture and to verify the universality of geometric statements. Experimenting, visualizing, measuring, inductive reasoning, and checking examples were not included for this purpose. Recently there has been a change in this approach for several reasons, as follows:

The Failure of Teaching Proving Tasks in School. The teaching of mathematical proof appears to be a failure in almost all countries (Balacheff, 1988). Only 30% of the students in full-year geometry courses that teach proofs (in the United States) reach a 75% mastery in proving (Senk, 1985). Every teacher of traditional geometry courses can confirm these findings.

Moreover, students rarely see the point of proving. Balacheff (1991) claimed that if students do not engage in proving processes, it is not so much because they are not able to do so but rather that they do not see any reason for it (p. 180). High school students, even in advanced mathematics and science classes, do not realize that a formal proof confers universal validity to a statement. A large percentage of students state that checking more examples is desirable (Fischbein & Kedem, 1982; Vinner, 1983). Many do not distinguish between evidence and deductive proof as a way of knowing that a geometrical statement is true (Chazan, 1993). After a full course of deductive geometry, most students don’t see the point of using deductive reasoning in geometric constructions and remain naïve empiricists whose approach to constructions is an empirical guess-and-test loop (Schoenfeld, 1986). They produce proofs because the teacher demands it, not because they recognize it as necessary in their practice (Balacheff, 1988).

The Role of Proof and the Goals of Teaching Proofs. For mathematicians, proofs play an essential role in establishing the validity of a statement and in enlightening its meaning. An analysis of teaching materials indicates that the social and practical importance of proofs in mathematical activities remains hidden, and it is important to create classroom activities in which the student becomes aware of that aspect of proofs (Balacheff, 1988, 1991).
Hanna (1990) distinguishes between two kinds of proof: proofs that show only that the theorem is true, by providing evidential reasons, and proofs that explain why the theorem is true, by providing a set of reasons that derive from the phenomenon itself. Hanna (1995) asserted that the main function of proof in the classroom is to promote understanding. Similarly, Hersh (1993) believed that in mathematical research, the purpose of proof is to convince but in the classroom the purpose of proof is to explain.

The Existence of Dynamic Geometry Environments. The advent of dynamic geometry environments raised a question about the place of proof in the curriculum because conviction can be obtained quickly and relatively easily: The dragging operation on a geometrical object enables students to apprehend a whole class of objects in which the conjectured attribute is invariant, and hence to convince themselves of its truth (De Villiers, 1998). The role of proof is then to provide the means to state the conjecture as a theorem (Yerushalmy & Houde, 1986), to explain why it is true, and to enable further generalizations.

Researchers have investigated how students function in open inquiry activities in computerized learning environments that support experimentation, conjecturing, and checking invariant properties of a figure, and thus lead to conviction (Yerushalmy, Chazan, & Gordon, 1993). Yerushalmy and Chazan (1987) pointed to the importance of problem posing in the design of activities in geometry. De Villiers (1997, 1998) illustrated how one can enrich investigations in dynamic geometry environments by asking “what if” questions and using them to make generalizations and discoveries. He claimed that in this case, the search for proof is an intellectual challenge, aimed at understanding why the conclusion is true, not an epistemological exercise in trying to establish truth. In addition Goldenberg, Cuoco, and Mark (1998) stated that “a proof, especially for beginners, might need to be motivated by the uncertainties that remain without the proof, or by a need for an explanation of why a phenomenon occurs. Proof of the too obvious would likely feel ritualistic and empty” (p. 6).

Dreyfus and Hadas (1996) argued that students’ appreciation of the roles of proof can be achieved by activities in which the empirical investigations lead to unexpected, surprising situations. Activities of this kind let students experience the need for proof to explain the surprising findings, and sometimes even to be convinced what are the correct conclusions. Different kinds of activities in this spirit were developed in the CompuMath project:

1. Comprehensive inquiry activities, where the geometric fact discovered as invariant of a geometric feature is surprising. This surprise is the trigger for the question why and for the proof as answer to this question. For example, If students are asked to draw the three angle bisectors of a triangle on paper, many draw them intersecting in one point. They are thus not surprised to realize, while checking with the dynamic geometry software, that they were right and do not feel a need for proof. When the same result is obtained as a by-product of a nontrivial investigation into the number of points of intersection of the angle bisectors of a quadrilateral, however, opportunities for surprise are created. The investigation of the quadrilateral’s angle bisectors already offers many surprises, and then, when going over to the case of the triangle, students are surprised again because they do not expect the triangle’s bisectors to intersect in a single point. (Dreyfus & Hadas, 1996).

2. Constructions under uncertainty conditions, in which students try to construct, a figure satisfying given conditions. For an example of such an activity, see Hadas and Hershkowitz (1999).

3. Activities in which the measurement and graphical options of the dynamic geometry software are used to present the dynamic variations of a geometrical
phenomenon in real time and also in the graphical and numerical modes. In such a context, questions and hypotheses raised in one mode may be answered in other modes, and in this way the investigating itself is enriched (Arcavi & Hadas, 2000).

4. Activities in which one cannot find any example for a conjecture one has made. Situations of uncertainty like this lead to the dilemma mentioned above: Should one continue the empirical search or attempt to understand the impossibility? Hadas and Hershkowitz (1998) claimed that, by careful design, based on experimentation and cognitive analysis of students' actions, situations can be constructed in which students will feel the need for proof. We conclude this section with an example of this type of activity, which illustrates how the dialectic process of the design of an activity with certain pedagogical purposes is integrated with cognitive research, as mentioned in Stage II.

**Are These Three Equal?**

The development process of this activity had three cycles.

1. We started with a “preresearch” version of the activity (see Fig. 26.3), which was tried in a Grade 9 classroom. Most of the students guessed that the three angles were equal. After measuring, all students hypothesized that there are some cases in which the angles are equal. They struggled to find such a case and failed to explain why they could not. Finally, the explanation was formulated in a whole-class discussion.

2. To help students with the explanation, we designed and added two preliminary tasks (see Fig. 26.4). This version formed the basis for a first semistructured interview with two Grade 9 students. They were not satisfied by visual considerations while dragging the vertices and changing the triangle, and therefore searched for a deductive explanation based on the preliminary tasks (Hadas & Hershkowitz, 1998).

3. We decided to add an additional investigating tool—the graphical representation of the varying angles as a function of one third AC, for given AB and angle A (see Fig. 26.5). This version (Fig. 26.5), with the graph tool available to the students, was tried in a second interview with two other students. After matching the intersection points on the graph with the corresponding geometrical situations, the students started looking for an explanation why the angles must be different, rather than

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**FIG. 26.3.** Preresearch version of the activity, "Are these three equal?"

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a. The side AC of a triangle is divided into 3 equal segments, by D and E. What can you say about the 3 angles created at the vertex B?

![Diagram](image)

b. Construct the above as a dynamic figure on the computer. Investigate, by dragging and measuring, the relationships between the sizes of the 3 angles <A<BD, <DBE, and <EBC. Explain!
Task 1. Which values can the base angle of an isosceles triangle have? Explain.

Task 2a. Investigate the median and the angle-bisector from the same vertex, in a "dynamic triangle." What can you say about the triangle when both segments coincide?

Task 2b. Draw the median and the angle-bisector from the other two vertices. Try to find a situation in which two pairs coincide and the third pair does not. Explain.

FIG. 26.4. Preliminary tasks for the activity: "Are these three equal?"

FIG. 26.5. An additional investigating tool.

searching for an example when they are equal. They ended up using deductive reasoning (Hadas & Hershkowitz, 1998). In a third interview, after checking the three graphs for situations with equal angles, a pair of students tried to make the three intersection points as close as they could. They then realized that this could not resolve their uncertainty and that only the understanding of why the three angles will never be equal has the power to convince.

In the final version of the activity, which is now in the student materials for the entire population, the graph is suggested as an option to students who insist on finding an example with equal angles.

Activities of the four types mentioned above were developed in the second stage and served as the basis for the third stage, in which the entire geometry curriculum was shaped. In this curriculum the construction of concept images is rich and dynamic, and the need for proof emerges in comprehensive inquiry activities.

WHAT DOES A SPREADSHEET CONTRIBUTE TO BEGINNING ALGEBRA?

Several experiments studying computers at the beginning algebra level were carried out during the 1990s (Kieran, 1992). Heid (1995) described the present and expected shifts in a computer-intensive algebra as follows.

What was once the inviolable domain of paper-and-pencil manipulative algebra is now within easy reach of school level computing technology. This technology demands new
visions of school algebra that shift the emphasis away from symbolic manipulation toward conceptual understanding, symbol sense, and mathematical modeling.

No longer can the main purpose of algebra be the fine-tuning of techniques for by-hand symbolic manipulation or the acquisition of a predefined set of procedures for solving a fixed set of problems. Lesson after lesson of "simplify these expressions" or "solve these equations" will no longer characterize the school algebra experience. Students will spend far less time on many of these techniques, will execute a majority of them with computing technology, and will completely forgo the study of others. Although some of the attention now paid to symbolic representations will be rededicated to developing "symbol sense," most class time will be spent in helping students develop a sense for how algebra can be used to explain the world around them. Applications of algebra will no longer be synonymous with "age," "coin," "mixture," and "distance–rate–time" word problems. Students will leave their school algebra experience with answers to such questions as "What good is algebra?" (p. 1)

These trends served as background for the development of a computer-based beginning algebra course as part of the CompuMath project.

Pre-Design Considerations

The design of this algebra course was preceded, and hence influenced, by the design of the course on functions (aimed for Grade 9, see Stages II and III) and a course in statistical data analysis (aimed for Grade 7, see the section on statistics). As a result, the algebra course naturally adopted the basic characteristics of these earlier courses, such as basing most of the process of the construction of student knowledge on the investigation of problem situations, conducted in an environment of active peer interaction. However, in addition to the earlier developed basic assumptions, we had to consider the particularities and needs of the domain of beginning algebra and of the students learning it.

An Intuitive Functional Approach. The decision to base student activities on complex "real-life" or mathematical situations led us to investigations of processes of quantitative variations, such as measures of geometrical shapes, series expressed in a numerical or geometrical form, variation of weight, price, savings, distance, and so forth. We also learned from other algebra curriculum projects (Heid et al., 1990; Yerushalmy, 1997) that the dynamic aspect of technological tools provides additional relevance to this approach. Our intention was to keep the formal aspect of the concept of function (definition, notation, mappings, etc.) at a minimal level and to require students to investigate variation, as expressed by numerical series, algebraic expressions, and graphs (see discussion in Stage I).

A Gradual and Smooth Transition Between Arithmetic and Algebra. Traditionally, the transition from numbers to algebraic expressions is made in a sudden and arbitrary manner, causing a "didactical cut" and consequently many cognitive difficulties (Ainley, 1996; Sutherland & Rojano, 1993). Our intention was to allow students to make this transition at a slower pace, that is, promoting a gradual and meaningful introduction of algebra in parallel to the numerical representation of the investigated models.

Emphasis on Generalizations and Justifications. Generalizing is one of the central activities in algebra and is traditionally interpreted as expressing patterns, structures, or processes symbolically. The traditional approach considered the translation of routine word problems into symbolic equations or expressions to be the main manifestation of this thinking skill, whereas our intention was to focus on the generalization of variations and patterns. Justification belonged traditionally to the domain
Launch
The students are presented with specific examples or specific examples are produced by them.

Towards a working generalization
Producing additional examples
Producing and solving examples with large numbers
Solving “reversal” tasks

Towards an explicit generalization
Verbal description of the observed pattern
Symbolic description of the observed pattern

Towards a justification

FIG. 26.6. Stages in generalizing a pattern.

of geometry and was seriously neglected in algebra. Justifications are unnecessary as long as the main concern is translation to a symbolical representation and the manipulation of symbols—both activities that are often disconnected from any context or meaning. We surmised that the investigation of meaningful situations and the use of computers would increase considerably the role and significance of justification in algebra.

Previous research on beginning algebra students (Friedlander, Hershkowitz, & Arcavi, 1989) provided a notion about stages in generalizing a pattern (see Fig. 26.6), stages that could be implemented in our case as a guiding scheme for the structure of an algebraic activity, involving processes of generalization and justification. This scheme includes transitions from the investigation of particular cases to generalizations, then to the justification of the generalized pattern, and later to its implementation in additional cases.

Choice of a Technological Tool. The following considerations, based on the criteria discussed in Stage I, led us to choosing spreadsheets as the technological tool for this course:

- Studies of students working with spreadsheets on arithmetic or beginning algebra problems (Ainley, 1996; Sutherland & Rojano, 1993) describe interesting and powerful thinking and strategies, evolving from students’ use of spreadsheets as a problem-solving tool (communicative power).
- Spreadsheets seem to provide satisfactory answers to most of the project’s general requirements and in particular to those relevant to the domain of beginning algebra, such as mathematization.
- Spreadsheets (Excel) were successfully used as a technological tool for other area in mathematics (see the next section on statistics) as well as in science, thus satisfying the generality criterion.

The Initial Development Stage

Observations in the experimental classes in Stage II produced many encouraging results. We describe some of the more relevant ones:

- high student satisfaction and motivation to work (usually in pairs) on the designed activities;
• a considerable extension of the mathematical concepts compared with the traditional beginning algebra curriculum (e.g., early and natural introduction of algebraic variables, expressions, and recursion formulae, an introduction to exponential and quadratic variation, arithmetical and geometrical sequences), most of them presented informally;

• a considerable extension of the range and level of students’ mathematical activities, such as algebraic modeling (which became the basis for any activity) and monitoring and justifying the results produced by the developed mathematical discourse conducted during peer interaction, teacher interventions, and class discussions;

• a lack of significant technical difficulties in students’ handling of the algebraic aspects of spreadsheets, such as the syntax of formulas or basic spreadsheet operations; and

• an evolution of spreadsheet elements in the mathematical discourse, such as discussing the spreadsheet operations or discussing the solution of a problem, both orally and in writing, in an “Excel language,” that is, using Excel notation as a means of communication.

Moreover, the team located the following general issues related to the use of Excel as a mediator in the process of learning algebra:

**Generalization by Recursion versus Generalization by Position or Spreadsheet Formulae versus Algebraic Expressions.** Frequently, a number sequence can be obtained on a spreadsheet by either using position numbers or by relating recursively to the previous number in the sequence. From a mathematical point of view, expressions that use the position as a variable reflect the underlying relation in a global way, whereas a recursive formula usually emphasizes a local aspect of the same relationship. Excel allowed students to combine the use of recursive formulae and dragging and thus to overcome the local characteristic of recursion. As a result, most students used recursion whenever possible. In some cases, such as exponential growth, this may have been the only way available to them. The use of recursion does not allow one to find data beyond those included in the numerical table, however. We observed students extending their tables to thousands of rows to answer a question that could have been solved by using a simple position formula. Thus, one of the issues at this stage was how to “promote” position formulas, whenever they are mathematically more rewarding. An abundance of numerical data may also lead to an increased cognitive load or to lack of motivation to monitor the obtained results.

Similarly, there is an obvious equivalence between generalizing a pattern as a spreadsheet formula or as a standard algebraic expression. The difference between the two is sometimes more than syntactic however. For example, when we required students to give an algebraic expression for the multiples of 5, with \( x \) specified as representing the position of each multiple in the sequence, some students made a direct transfer from the Excel recursion formula \( B2 + 5 \) (with the variable representing the previous number in the sequence) and produced the algebraic expression \( x + 5 \), rather than \( 5x \), as expected.

In summary, the spreadsheet’s ability to produce large quantities of data by simple “dragging” of formulas provides an excellent illustration of the meaning of a variable, an algebraic expression and the pattern of a variation. On the other hand, the same ability can be abused by preferring the extension of the numerical table to the use of a more mathematically sound strategy, such as constructing and solving an equation.

**Software Transparency.** Students’ willingness to monitor solution methods and their results was increased considerably by being released from computations and
All questions refer to $2 \times 2$ arrays of numbers such as the following:

<table>
<thead>
<tr>
<th>Square 1</th>
<th>Square 2</th>
<th>Square 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 9 5 11</td>
<td>7 13 9 15</td>
<td>10 16 12 18</td>
</tr>
</tbody>
</table>

In a spreadsheet, construct a "seal design" of formulas that can produce number squares like the ones given above

- First, enter a number of your choice in the upper left corner of the square.
- Then, use the name of this upper left cell to write formulas (and not numbers.), which will produce corresponding numbers in the other three cells of the square.

Check whether your seal design produces the correct number squares.

- Enter the numbers 3, 7, and 10 in the upper left cell of your "seal" and check whether you obtain Squares 1, 2, and 3 given above.

Investigate number squares of this kind.

- Find as many interesting properties as you can.
- Justify your findings. Try to convince other people that the properties that you found are true for all the number squares of this kind.

FIG. 26.7. Excerpt from the "Seal Designs" activity.

algebraic manipulations, by being able to relate to the meanings attached to the problem situations, and by being in a situation to discuss and argue their ideas with peers. On the other hand, in some cases we detected cognitive and technical difficulties in students’ monitoring their solution processes and results. We related these difficulties not only to the cognitive load created by the abundance of numerical data but also, and more importantly, to the spreadsheets’ lack of transparency, that is, the “disappearance” of the formulas that produced the numbers. The Seal Designs activity illustrates this issue (see Fig. 26.7).

The Seal Designs activity required the generalization of a $2 \times 2$ array of numbers, which can be characterized by the expressions $x$, $x + 6$, $x + 2$ and $x + 8$. Then the students were asked to discover “interesting” patterns, common to all number squares of this kind. Two students, R and Y, used correct formulas to produce the number arrays. They also found that the difference of the products of the two diagonals always equals 12. When they were asked to justify their claim [i.e., to show that $(x + 2) \cdot (x + 8)$ exceeded $x \cdot (x + 8)$ by 12], however, they were distracted by the numbers shown on the spreadsheet for a particular array (7, 13, 9, 15) and attempted to compare $(x + 2) \cdot 13$ and $x \cdot 15$—a mixture of remembered formulas and numbers taken from the specific square.

**Documentation of Computer Work.** The energy and motivation invested in computer work frequently led to incomplete documentation or a complete lack of any paper record of the computer work and its results.
In addition, we were led to consider the question of the extent of instructions. Many curriculum developers attempt to allow the students freedom in choosing their own solution path, but at the same time, they feel the need to include some guidance in the design of the task, which will help the students to complete the task successfully and to achieve its mathematical agenda. In spreadsheet-based activities, we had to decide on each occasion whether to recommend a certain structure for the numerical table (for example, its headings), to hint at or “give away” the required formulas (especially at the beginning of the course), or to specify a sequence of spreadsheet operations needed to achieve a certain representation.

Expansion

Here, we will consider new emphases introduced as a result of the findings in the previous stage of development: hypotheses, graphical representations, and generalization.

**Hypotheses.** We found that, in any process of inquiry, hypotheses were crucial to creating meaningful learning situations and student involvement. Because students did not usually hypothesize spontaneously, we introduced specific requirements to predict at two stages. First, as part of getting acquainted with the problem, students were asked to predict some quantitative aspects of the outcome before using the computer to obtain a table or make any other systematic attempt to solve the problem. Second, students were asked to predict some qualitative aspects (usually a rough sketch) of a graph after they obtained a numerical table but before they used the computer to produce the corresponding graph. The Savings activity (see Fig. 26.8) illustrates this issue. It is based on weekly doubling of an initially small sum of money, that is, on exponential growth.

Students were required to make predictions at two stages. First, the students were asked to compare the (exponential) savings of Efrat with the (linear) savings of other children, analyzed in a previous activity. They also had to estimate the amount of her savings by the end of 1 year. These predictions were made before using spreadsheets. At the second stage, the students were required to use their numerical data on Efrat to hypothesize and sketch the shape of the corresponding graph.

We found that predictions considerably increase students’ willingness to monitor the output produced by the computer and to analyze their solution if the outcome did not correspond to their prediction.

**Graphical Representations.** Because of the influence of traditional beginning algebra, we tried to avoid graphical representations at the initial stage of development. Our findings from the first stage of development showed, however, that the use of graphical representations is a vital need in the developed activities, that there were no particular technical difficulties in students’ handling of spreadsheet graphs, and that the construction of graphs constituted a natural use of Excel’s abilities. As a result, we introduced graphs systematically, as one of four possible representations of

<table>
<thead>
<tr>
<th>Efrat’s savings grow as follows:</th>
</tr>
</thead>
<tbody>
<tr>
<td>At the end of the first week, she had 2 agorot (that is, 0.02 shekel).</td>
</tr>
<tr>
<td>Each week, Efrat’s savings grow by the amount that she has already saved up to then.</td>
</tr>
</tbody>
</table>

FIG. 26.8. The ‘Savings’ problem situation.
data, models, or solutions (verbal, numerical, algebraic, and graphical). Metacognitive discussions of advantages and disadvantages of these representations were also conducted, both orally and in writing (for example, as journal items).

**Discussing Generalization Methods.** As mentioned above, we were quite aware of the importance of generalizations as one of the central processes in learning algebra. We were less aware, however, of students’ tendency (especially when working with spreadsheets) to generalize recursively, rather than using the independent variable. In most cases, the second method seemed preferable to us because the inclusion of the independent variable shows more clearly the underlying pattern. Therefore, right from the beginning of the course, both tasks and classroom discussions started to raise the issue of various ways of generalizing. Students were required to describe verbally or algebraically their generalizations in one or more specified method. For example, students were asked to construct on a spreadsheet a sequence, by using the numbers in the column of the position index or to express in words the weekly balance of a person’s savings by using the number of weeks (and not the previous balance) as variable.

We also found that the method of generalization was frequently influenced by the following characteristics of the task:

1. **Nature of variation:** At the stage of beginning algebra, linear models are employed most frequently. As mentioned before, spreadsheets tend to encourage recursive generalizations of linear relations. On the other hand, the same recursive methods allow students to analyze many nonlinear models that otherwise could not be approached at this stage of learning algebra (see the Savings activity).

2. **Question Design:** We found that presenting the first consecutive numbers, quantities, or shapes of a sequence or a variation attracts recursion, whereas the presentation of one or two nonconsecutive representatives of a similar variation tends to encourage generalization using of the independent variable.

3. **Style of presentation:** A visual presentation of a variation or sequence (for example, sequences of dots or cubes, arranged in growing similar constructions) gives a strong meaning to general formulas, as a reflection of the counting method employed by the solver. Therefore, this presentation tends to encourage generalization by the position number, especially if nonconsecutive representatives are used, as above.

In summary, although the spreadsheet may induce some characteristic obstacles to be avoided, we found that its language, its representational options, and its mathematical capabilities combined to offer a powerful tool to establish a connection between arithmetic and algebra.

**STATISTICS: COMPUTERIZED REPRESENTATIONS AS RHETORICAL TOOLS**

Statistics is becoming ever more pervasive. Political, social, economic, and scientific decisions are made on the basis of data. Statistical reports affecting virtually all aspects of our lives appear regularly in all the news media. Therefore, statistical literacy is becoming a major goal of the school curriculum. Gal (2000) suggested that statistical literacy is “people’s ability to interpret and critically evaluate statistical information and data-based arguments appearing in diverse media channels; and to discuss their opinions regarding such statistical information” (p. 135).

We argue that the teaching–learning of statistics offers a particularly powerful testing ground for probing fundamental questions regarding the role of computerized
environments in curriculum development, as well as the links between the relationship between syllabus and curriculum.

More than other domains in school mathematics, the syllabus in school statistics is the object of intensive debate: to decide once again which statistics should be included in the school syllabus (Lajoie, 1998), which technology is appropriate for educational purposes (Ben-Zvi, 2000), and which type of research initiatives are needed. These issues led statistics educators (Garfield, 1995; Graham, 1987; Hawkins, Jolliffe, & Glickman, 1992; Shaughnessy, Garfield, & Greer, 1996) to define exploratory data analysis (EDA) or Data Handling as the content and framework of statistical education in schools. EDA is the discipline of organizing, describing, representing, and analyzing data, with a heavy reliance on visual displays as analytical tools and, in many cases, technology for making sense of data. EDA activities are often schematized by the following ideas: looking at the data (preliminary analysis), looking between the data (comparisons), looking beyond the data (informal inference), and looking behind the data (context; Curcio, 1989; Shaughnessy et al., 1996).

Following the recommendations to introduce stochastic (statistics and probability) concepts for all students from early stages (e.g., NCTM, 1989), new EDA instructional materials for elementary and secondary schools have been developed in many countries. In these curricula, there is growing emphasis on graphical approaches, on students gathering their own data and carrying out investigations, on misuses and distortions, and on probability simulations to generate data. We describe here a junior high school EDA course for Grade 7 (age 13) developed as part of the CompuMath project.

In Israel, the official junior high school mathematics syllabus assigns 15 hours in Grade 7 to cover basic statistics topics and an additional 10 hours in Grade 8 to introduce basic concepts of probability. The CompuMath EDA course was an extension of the previous curriculum development cycle, sharing with it basic approaches and the goal to design and create a learning environment in which students are engaged in meaningful mathematics. But at the same time, it intended to make full use of the power of up-to-date technological tools to redesign and reshape the EDA learning environment.

Stage I: Focus on the Choice of a Computer Tool for EDA Instruction

At this stage, the curriculum development team systematically reviewed innovative statistics curricula, research literature, and technological tools. We subsequently chose the statistical software to be used in class, and (re)evaluated our educational goals and instructional strategies. We focus here on the choice of technological tool.

The types of software generally used in statistics instruction are manifold and include statistical packages, microworlds, tutorials, resources (including Internet resources), and teacher’s meta-tools (Biehler, 1993, 1997; Ben-Zvi, 2000). Statistical packages include software for computing statistics and constructing visual representations of data, often based on a spreadsheet format to enter and store data. Microworlds consist of software programs to demonstrate statistical concepts and methods, including interactive experiments, exploratory visualizations, and simulations. Students can conceptualize statistics by manipulating graphs, parameters, and methods. For example, they allow the investigation of the effects of changing data on graphical representation, the effects of manipulating the shape of a distribution on its numerical summaries, or the effects of changing sample size on the distribution of the mean. Prob Sim (Konold, 1995) and Sampling Distributions (delMas, Garfield, & Chance, 1998) are good examples of computer simulation microworlds.

Tutorials include programs developed to teach or tutor students on specific statistical skills or to test their knowledge of these skills. The tutorial program is designed
to take over parts of the role of the teacher and textbook by supplying demonstrations and explanations, setting tasks for the students, analyzing and evaluating student responses, and providing feedback. The tutorials are often too dominant to leave enough room for students to construct knowledge autonomously. Examples include ActivStats (Currall, Young, & Bowman, 1997), and ConStatS (Brewer, 1999). Resources consist of various resources to support teaching statistics, including Internet resources. The development of the World Wide Web has produced unprecedented global means for teachers to easily share their ideas on ways to improve the teaching of statistics (Lock, 1998). Teachers’ meta-tools create an interface that enables teachers to adopt software to their specific audience and educational goals. The categories listed above are not necessarily distinct, and in many cases specific software falls into more than one category.

We considered the above possible types against the CompuMath criteria for choosing technological tools (generality, mathematization, and communicative power) and opted for a statistical package and various resources (including Internet resources) for the EDA course. The specific educationally modified statistical package we chose at the predesign stage was Stats! (LOGAL Software, Inc.). Stats! is a data-analysis program intended for middle and high school students in introductory statistics courses, stressing the analysis of real data using EDA techniques. Students start with unordered data, typing both quantitative and qualitative data into any cell in a spreadsheet-like data table. Next, they can use the classification tool to order their data and use tally sheets to display data by count or relative frequency. Graphic representations include pie charts, pictograms, bar charts, scatterplots, and accumulated frequency graphs. The representations can be enriched by also displaying the mode, mean, median, quartile, and boxplots. Students can manipulate the data interactively directly on the graphic representations and compare two variables or populations on one display.

We hypothesized that such a manipulative power would foster mathematization and that the potential of Stats! in simultaneous displays of representation gave it communicative power. Finally, we chose Stats! because it was simple and suitable for all students and did not demand adjustment for classroom activity. Also, it was still under development, and the members of the curriculum development team were invited to function as advisors to the software programmers to include what we considered as educationally desirable procedures and functions.

The predesign stage ended when we finished planning the main topics (“big ideas”) of the curriculum (described in the next section); studied the features, advantages, and limitations of the chosen software; and searched for suitable investigative situations (including real data). We were then ready to write the first version of some activities.

### Stage II: The Initial Design

The second stage consisted mainly of a first design of activities and their implementation in a few Grade 7 classrooms, accompanied by classroom research and student interviews, one of the aims of which was to learn about their statistical intuitive conceptions and prior knowledge. These interviews showed a surprisingly broad knowledge of basic statistical concepts, such as averages and charts and their applications.

One major concern at this stage was the degree of openness of the activities. Our initial inclination, when planning a “virtual activity,” was to engage students in the investigation of data in a given context and then to give them the freedom to choose research questions, tools, and strategies to analyze the data. Thus, the “virtual activity” was very open, instructions being minimal. The actual design of activities was however realized as an ongoing compromise by trying to find the appropriate blend between open and closed.
Furthermore, we experimentally added to the EDA course an extended activity, a final project, in which each pair of students identified a problem and the questions they wished to investigate, suggest hypotheses, design the investigation, collect and analyze data, interpret the results, draw conclusions, and present their main results to the class. Some of the topics students chose to investigate were superstitions among students, attendance at football games, student ability and the use of the Internet, students’ birth month, formal education of students’ parents and grandparents, and road accidents in Israel. At this stage, we had little knowledge about the implementation of projects in class, for instance, how and when to introduce the project to students, how to guide their work, or how to assess it.

To improve the learning materials and software, we observed and videotaped classrooms and students’ work, interviewed students before and after the experimental implementation, read and analyzed student notebooks and final projects, and observed teachers in inservice workshops. All of these activities helped us to redesign the learning materials to make them more attuned to student interests, motivation, capacities, and interactions and to calibrate the appropriate uses of technology (Ben-Zvi & Friedlander, 1997). For example, we changed investigation contexts, gave more attention to the “entry point” of the investigations, and improved the quality of the databases. Furthermore, we faced two major problems in this stage: (a) shaping the relations between the classroom activities and the final project and (b) the change of software. They are presented in turn.

The Relationship Between the Classroom Activities and the Final Project.

One option in the program is to structure the components linearly (i.e., to locate the final project after the classroom activities), based on the theoretical assumption that students have to acquire the necessary statistical skills, tools, and concepts, before they are able to begin on a large-scale final project of their own. A different option is to intertwine work on the final project with the classroom activities, assuming that the mutual effects are beneficial to learning. The hypothesis here is that the construction of statistical understanding is a complex nonlinear process that benefits from the combination of the semistructured classroom activities and the self-propelled and open-ended final project. For this option, one has to choose carefully a starting point for the project work.

Specific classroom circumstances and the teacher’s preferences also influenced the exact starting point of the final project. As expected, the classroom activities supplied some of the basic statistical approaches, concepts, and skills needed for the project design. It became apparent, however, that the early introduction of the final project into the course motivated the students to take responsibility for their work and methods of inquiry and gave them a sense of relevance, enthusiasm, and ownership. Students evaluated and applied new concepts and methods that were introduced in the classroom activities not only in the given context of the different activities, but also in their own projects. Thus, the project work gave them an added opportunity to experiment with the new concepts and methods and often raised new statistical issues to explore, which were not originally part of the curriculum. The early start also provided more time for the project work. For example, some students spent several weeks exploring and choosing an interesting and rich investigation topic. On the other hand, the early start of the final project caused it to dominate students’ interest to a certain extent, which required special attention and flexibility on the teacher’s part.

One of the most striking effects of the cross-fertilization between the project and the classroom activities is that the various representations of data, which were introduced in the activities as didactic means to convey statistical ideas to students, turned into means of expression by which the students presented their points of view or tried to convince opponents, mainly during work on the projects. The students thus
realized that the representations could serve rhetorical functions. The curriculum development team in the third stage took advantage of this crucial point (described later).

**Changing the Software.** During the experimental implementation of the curriculum, we were forced to reexamine the type of software we used. Although the software helped students to develop expertise in using data to solve real problems, it was limited in many respects and uncommon in schools. It had been created specifically for use in schools and consisted of a limited number of statistical procedures and representations. It was not general or mature enough as a software. Therefore, we replaced Stats! with a spreadsheet package (Excel). Our main reasons for this step were as follows:

First, spreadsheets are common and familiar. Excel in particular is now recognized as a fundamental part of computer literacy (Hunt, 1995). They are used in many areas of everyday life, as well as in other domains of mathematics curricula, and are available in many school computer labs. Hence, learning statistics with a spreadsheet helps to reinforce the idea that this skill is connected to the real world. Moreover, prior and in parallel to the learning of the EDA course, CompuMath students study algebra with Excel as described previously.

Second, Spreadsheets provide direct access that allows students to view and explore data in different forms, investigate different models that may fit the data, manipulate a line to fit a scatter plot, and so forth.

Third, spreadsheets are flexible and dynamic allowing students to experiment with and alter displays of data. For instance, they can change, delete, or add data entries in a table and consider the graphical effect of the change or manipulate directly data points on the graph and observe the effects on a line of fit. Furthermore, they are adaptable; namely, they provide students and teachers with control over the content and style of the output.

Unlike other topics described above, this stage with its field experiments and research, resulted in more than the design of a sequence of isolated activities. Most of the EDA activities were planned for 4 to 6 lessons and the EDA course eventually consisted mainly of these extended activities and the final project.

**Stage III: Research**

Three directions of expansion were discussed in Stage III. Here, we focus on the relations between the constitution of a continuum of activities and learning trajectories, which is the object of intensive research (Ben-Zvi, 1999; Ben-Zvi & Arcavi, 1998, 2001). We present an example of the relation between research on learning and curriculum development (Men’s 100-Meter Race) and an example of research on students’ rhetorical use of representations (the Work Dispute).

**Men’s 100-Meter Race: Constructing Meaning for Trends.** This study arose from a classroom activity. Students were presented with a spreadsheet table of the Olympic 100-meter records, the years in which they occurred (from 1896, the first modern Olympiad, to 1996), the athletes’ names and country, and so forth. Their first task was to work in pairs to describe the data graphically and verbally and to use the spreadsheet to produce a graph and discern trends. Our observations in several experimental classes showed that the students were able to engage quickly and with relative ease with the task. They were able to read the table of results, compare the records of consecutive Olympiads, consider the issue of outliers, sort the data, consider various graphs (some inappropriate for the given data), and create a time plot with a spreadsheet.
One frequent difficulty drew our attention: Students found it difficult to see trends in the given data (before graphing), to discern a pattern from the graph, and to report on it. Our interpretation was that at this stage, cognitive issues relating to the connections between learning different topics appeared. Specifically, they found difficulty with the relation between the EDA course and the algebra course, which is also based on the use of Excel; the deterministic nature of algebraic formulas interfered with the nondeterministic and disorganized nature of statistical data.

When we redesigned the activity, we included manipulation of scatter graphs and a serious engagement with the notion of trend. The students were asked to use the spreadsheet to manipulate data graphs (i.e., to change scales, delete an outlier, and connect points by lines, and to consider the effect of these changes on the shape of the graph. The objective was to prepare for a design task in which they were asked to design a graph to support claims, such as (a) over the years, the times of the Olympic 100-meter race have improved considerably; (b) over the years, the changes in the times of the Olympic 100-meter race were insignificant; and (c) between 1948 and 1956, the times of the Olympic 100-meter race worsened considerably.

When manipulating data in one representation with immediate feedback in another, the computer provided the means to push the activity to a conceptual level, just as in geometry, algebra, and functions (see Schwarz & Dreyfus, 1995).

Our observations indicate that the students constructed meanings by making connections between the investigation context, the data, and the graph. The computer assisted them in switching their discourse between the context, graph, and data and thus helped them to construct meanings. Because students were found to be sensitive to this use of representations, we made many of our initial activities more rhetorical. For example, we modified the Olympic records activity as follows:

Two sports journalists argue about record times in the 100 meter race. One of them claims that there seems to be no limit to human ability to improve the record. The other argues that at some time there will be a record, which will never be broken. To support their positions, both journalists use graphs.

Similarly, students were given a second database of the performances of Olympic women winners of the 100-meter race. The students were to draw a graph supporting the statement of a “feminist activist” according to which “women will sometime overcome men” and another graph supporting the claim of a “male chauvinist” according to which “women will never run as fast as men.”

This change in approach is characterized by the intention to change the status of representations from descriptive entities to entities that are to be judiciously constructed to attain a goal. The manipulation of data representations can yield graphs supporting any of the four arguments. Thus, in the third stage, activities were designed to engage students in statistical literacy. These activities resemble activities on functions and geometry in which the students learn to discern between representatives (the material displays) and the meaning of these representatives. To some extent, however, these activities were still of a prescriptive character. In the final example, the design of the activity leads students to use the technological tools to fulfill their own goals in an argumentative activity.

The Work Dispute. This activity concerns workers at a printing company. The workers are in a wage dispute with the management, who have agreed to an increase of equaling 10% of the company’s total wage expenditure. The dispute is about how this increase is to be divided among the employees. The students are given the present salary list of the 100 employees and an instruction booklet to guide them in their work. They are also provided with information about the national average and minimum
salaries, Internet sites to look for data on salaries, newspaper articles about work disputes and strikes, and a reading list of background material. In the first part of the activity, groups of students are required to take a position in the dispute and to clarify their arguments. Then, using the computer, they describe the distribution of salaries and use appropriate measures (median, mean, mode, and range) to support their position. They learn about the effects of grouping data and the different uses of statistical measures in arguing their case. In the third part, their task is to suggest changes to the salary structure that satisfy the 10% constraint. They produce their proposal to solve the dispute and design representations to support their position and refute opposing arguments. Finally the class meets for a general debate and votes for the winning proposal. The time spent on this activity is about seven class periods.

This activity led students to manipulate representatives and data for rhetorical use. They not only displayed averages and distributions of data but designed the distribution of salaries to attain a desirable goal. This constitutes a jump comparable to that between learning to speak a language and speaking to learn. Taking a stand also made students check their methods, arguments, and conclusions with extreme care. Criticism and counterarguments by peers and teacher were a natural part of the activity. When the results of their work were not in line with their position, students were forced to persevere and search for more evidence and convincing arguments. Finally, after much refining, the groups formulated their proposal.

In sum, we claim that statistics learning is a particularly powerful testing ground for probing fundamental questions regarding the role of computerized environments in curriculum development, as well as the links between syllabus and curriculum. In the EDA course, students realized that representations and representatives could serve rhetorical functions, similar to their function in the work of statisticians, who select data, procedures, tools, and representations that support their perspective. Our team took advantage of this crucial point to extend the scope of the course beyond the learning of statistical concepts to involve students in “doing” statistics in a realistic context through a set of semistructured activities and an autonomous final project in a computerized environment.

EPILOGUE

In this chapter, we have attempted to draw a picture of the compound activity of curriculum development for computer-rich learning environments in mathematics. The account was organized along a few main dimensions of the curriculum development activity: goals; participants; the potential of the technological tools, contents, and approaches; the design of the materials to be used in the classrooms; research as an integral part of curriculum development; and implementation on a large scale. We tried to show how these dimensions change dynamically during the three main stages of a curriculum’s development: predesign considerations, initial design of isolated activities, and expansion.

The dynamically developing nature of these dimensions reaches beyond the stages of development and accompanies the project team throughout the life of the curriculum; we now discuss some of the future-oriented dilemmas the decision makers and the project team face.

First, technological tools develop at an exceedingly quick rate. A curriculum such as CompuMath needs to take such development into account at two levels. At the level of more powerful and user-friendly new versions of the same software packages, the adaptation needs to be done to allow schools to use all existing versions of the software. At a more profound level, new, educationally powerful types of software may be
expected to appear. This is particularly true for the algebra course. Although we believe we have chosen the most appropriate software available today for functions, statistics, and geometry courses, we are not yet fully satisfied with our approach to beginning algebra.

Seconds, changes in the research dimension of the project emanate from two directions: issues that arise from the actual development and implementation of the curriculum in classrooms may lead to research questions, and developments in the scientific discipline of mathematics education may entice us to investigate particular issues within our curriculum. In our approach, research is a necessary and integral part of curriculum development. It enables the development team to redesign a virtual learning activity into one for real students and classrooms in such a way that the intended change will happen. Such research follows the learners and investigates their learning in their own environment. As such it has sociocultural characteristics. It also fits well with both our curriculum-based interest in the role of collaborative problem solving with computers and the current trend toward a sociocultural outlook in the field of mathematics education. Ten years ago, cognitive issues had a much larger weight in our curriculum-based research. And in the near future, the question of combining the cognitive and the sociocultural approaches is likely to take on added importance—specifically, the question of where in shared knowledge the individual knowledge is hidden (Hershkowitz, 1999).

Third, although the above dimensions are more or less in the hands of the project team, others demonstrate the power of decision makers outside the project team on the future development and realization of the project. For example, in an ideal environment the students can be autonomous and use technological tools whenever they feel the need for it. For this purpose, computers need to be available in each classroom on a permanent basis and in each student’s home. In today’s typical CompuMath environment, computers are in a lab, and students may use the lab about once per week. The resources and organization to overcome these limitations of the learning environment depend on powers beyond the curriculum team.

Fourth, as discussed in Stage III, implementation on a large scale is a vast task. Problems arise because teachers, students, and the teaching–learning processes are in a sense “unknown”: They have a much lower degree of interaction with the team members than the Stage II participants. It seems reasonable to create an electronic system for communication with teachers and students. In the CompuMath project, we piloted this idea in one mid-sized town with some success, but the bulk of work on such a “curriculum maintenance system” is waiting to be carried out and requires enormous resources.

Finally, the most serious compromise that the CompuMath developers had to make was to teach the contents of the given mathematics syllabus, which was formulated three decades ago and does not take the potential of computerized tools into consideration. In Stage I, we discussed the gap between syllabus and curriculum and how it was bridged by creating suitable standards for mathematics learning and teaching and careful design. A large number of mathematical problem situations were created and connected into a web of meanings, of which the syllabus constitutes but the barest outline. Moreover, in some instances, the syllabus was more of a liability than an asset. The time has come to reconsider the contents of the syllabus itself. For instance, the mere existence of computer algebra systems raises questions concerning the role of algebraic manipulations such as solving equations: To what extent do students have to reach mastery in solving linear or quadratic equations when they have a tool at their disposal that solves compound equations including the linear and the quadratic ones? What kind of mathematical knowledge and insight may students gain using such a tool? In short, the time has come to reconsider the syllabus in the light of the technological tools’ potential in mathematics learning.
Curriculum development projects are not limited in time and scope. Any project is necessarily based on previous projects as well as on conditions imposed from the outside, and a project’s success can eventually be measured only by the influence it exerts beyond the immediate classrooms in which it is implemented and beyond the few years during which it is taught.

REFERENCES


CHAPTER 27

The Influence of Technological Advances on Students’ Mathematics Learning

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1. INTRODUCTION

1.1. The Impact of Technology

Given the impact that computer technology has had in a brief period of time, it is likely that new technology will continue to be incorporated quickly within school practice. Thus, it becomes more urgent to identify the crucial points around which to organize the use of computers and the new technologies related to them. We need to understand how and why new technologies influence, and will continue to influence, mathematics education.

Human history has been punctuated by technological innovations that directed its course, sometimes marking a drastic change or “revolution.” The construction and the use of artifacts\(^1\)—in particular, complex artifacts—is characteristic of human activities, but even more characteristic is their contribution at the cognitive level (Norman, 1993). The passage from the sphere of practice to that of intellect, and vice versa, may be considered one of basic evolution and progress; Norman’s examples discussed on this perspective are illuminating. Certainly language, both oral and written, has a central place among artifacts and their use. The passage from oral to written language is an example of a technological revolution. At first glance, writing can be considered simply a way to improve oral expression; what is said may be recorded. Once written, it can be read and can be “said” again and again. Considering writing as just a simulation of oral expression is limited and misleading, however; writing transformed the way

\(^1\)There are many terms, such as tools and instruments, that refer to artifacts and were conceived to describe a specific use or goal. Because one aim of this paper is to clarify aspects related to the use of artifacts, the general term artifact will be used.
of human beings think2: “Writing creates the difference, not only in the expression of thought, but also and primarily in how thoughts are perceived” (J. Goody, 1989, p. 266; translated by the author).

As far as mathematics is concerned, the classic studies carried out by Nunes, Schlimann, and Carraher (1993) analyze and confirm the difference between written and oral language, highlighting how it is possible to relate the deep differences between formal and informal mathematics to this distinction. They refer to this as the distinction between “street” and “school” mathematics. It is also important to investigate what changes may occur because computers, and all recent technology derived from them, become resources to strengthen our way of thinking. In particular, we are interested here in the impact on the field of education of the use of new technologies. Perhaps there is something in the technology related to the computer that makes it a very peculiar artefact, which makes it similar to a basic technology like writing.

Effective computer-based learning environments, compared with other types of learning materials, present a unique feature—an intrinsically cognitive character, and in particular, as Balacheff and Kaput clearly pointed out, the fact that “The interaction between a learner and a computer is based on a symbolic interpretation and computation of the learner input, and the feedback of the environment is provided in the proper register allowing its reading as a mathematical phenomenon” (Balacheff & Kaput, 1996, p. 470).

This has created great expectations in the field of mathematics education, but despite the deep penetration of technology into everyday life, schools maintain a certain hostility toward it. I hope the following discussion will highlight some explanations of this phenomenon.

The evolution of technology has stimulated deep and vast research (Artigue, 1998; Balacheff & Kaput, 1996; Kaput, 1991, 1992), seeking answers to urgent questions such as, “How do users’ interaction with computer affect, shape . . . cognitive development related to math education? How do particular features of a computer environment function cognitively? How are they related to school practice? . . . Mathematics educators need answers to such questions in order to fashion learning environments and situations that exploit those environments” (Kaput, 1991, p. 55).

The use of technological devices in teaching and learning has been treated by many researchers and from many perspectives, but the issue is still complex. It is impossible to arrive at a definite answer to the questions above; different perspectives highlight different aspects and potentials that cannot be unified. What I attempt to do in this chapter is to discuss and make explicit some general theoretical principles regarding the presence of computers in school practice and, in particular, its integration into school practice. For a complementary discussion on the design of computer environment, see Bottino and Chiappini’s chapter in this volume.

2 TECHNOLOGY AT SCHOOL

2.1. A Catalyst to Transform Sociomathematical Norms

Not long ago, Anna Sfard and Uri Leron (1996) discussed the impact of computer programming on learning mathematics. Questioning “the tacit assumption that there is a direct relationship between the complexity of the problem and the students’ ability to cope with it,” the authors described how the presence of the computer might change

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2 Actually, the use of a computer and, generally speaking, working in a computational environment presupposes the use of writing. “Written words” appear on the screen and constitute the main communication medium. As we await further advances, our main modes of communication with a computer are “reading” and “writing.”
the standard way we conceive the difficulty of a problem; in so doing the computer shows its potential in upsetting and transforming the norms of school practice. The authors proposed the following meta-problem:

Which of the following problems would be easier for students to solve?

P1 Given three points (2,3), (−1,4) and (0,1) in the plane, find the centre and the radius of the circle passing through them.

P2 Write a computer program that accepts any three points in the plane (given by coordinates) and returns the centre and the radius of the circle through them. (Sfard & Leron, 1996, p. 189)

The two problems are similar, but a careful analysis shows that they are of a quite different nature. The difference is related to the computer, but it is important to see exactly how the computer is involved. With regard to P1, the problem is that of identifying a particular circle, finding the center and the radius, in the Cartesian plane; the solver must elaborate on the given data to calculate the coordinates and the measure of the radius. In the case of P2, the problem is concerned with writing a program for the computer to obtain the center and the radius for any three given points; that means that the procedure itself, which was in P1 a process to perform, in this case must be identified as a product and exhibited as a written solution (represented in a programming language).

Students' performances surprised the authors, who tried to find an explanation. In fact, against their hypothesis that students would find P1 easier than P2, they were more successful in the solution of P2 that P1. In the latter task, the problem must be solved at a level of generality that the former does not require. The presence of the computer seems to be the causes this shift. Because programming is a key element in the interaction with a computer, it seems natural that "different environments—the math classroom an the computer lab—gave each of the two problems its unique identity and character" (Sfard & Leron, p. 191).

This provides a good example of how computers can transform the nature of a problem. We can foresee that the presence of new technology transforms the relationship between problems and knowledge and that this change will occur in at least two respects: the type of problems that can be proposed to pupils and the solution processes that can be used. The available resources change, and consequently the processes used to reach a result change as well. Certainly this will be the case for technology that has a direct relationship with mathematical knowledge (e.g., symbolic manipulators such as DERIVE), but it will also be true of software that has no direct relation to mathematical knowledge but nevertheless incorporates it (e.g., professional software such as EXCEL or AUTOCAD).

As various authors have noted, many researchers, have studied the effects of computer environments, and especially of programming, on intellectual processes involved in problem solving and concept formation, but the cognitive-oriented research, carried out for many years and principally centered on the learner, must be complemented by investigation of the effect that activities in computer environments may have on the mathematical classroom as a whole. Enlarging the perspective makes it reasonable to ask oneself how the different facets of technology affect the culture of the classroom with respect to teaching and learning mathematics. It is possible to consider the computer as a "catalyst of a cultural revolution—as a bearer of a new socio-mathematical norms" (Sfard & Leron, 1996, p. 193). It was Papert (1982) who first pointed out this aspect and more recently noted the following:

Information technologies, from television to computer and all their combinations, open unprecedented opportunities for action in improving the quality of the learning environment, by which I mean the whole set of conditions that contribute to shaping learning in work, in school, and in play. (Papert, 1992, p. IX)
Papert—and many math educators agree with him—emphasizes one aspect that is interesting, but probably not crucial. Interacting with a computer offers many opportunities for meaningful activities that involve ways of thinking usually recognized as typical of mathematics but often neglected in school practice. For instance, the separation between planning and executing is difficult to achieve during traditional school activities but emerges naturally in programming activities with a computer.

Although it is possible to recognize the potential of a computer in the classroom, in recent years, much research has shown that the presence of a computer does not always produce what is expected. A deeper analysis is needed to better understand both the potential and the limitations of computers. The following is a discussion of several theoretical approaches to this analysis.

3. THE CONSTRUCTIVIST APPROACH

3.1. A New Experimental Realism

The role of concrete representation, images, or models of given situations has often been emphasized, so that thinking processes are described as essentially consisting of transformation and manipulation of physical or cognitive mental models (Dörfler, 1991, 1993). In this sense, computers (and computer software) provide ways of experiencing mathematics models that we simply did not dream of 30 years ago. The effect is so strong as to lead one to speak of a new experiential mathematical realism. Abstract and formal conceptions become increasingly accessible when reconsidered with a computer providing a concrete impression via direct manipulation of mathematical objects and relations. The extension and the versatility of these new methods has changed the traditional relationship between cognitive processes and representations (Dörfler, 1993). From this perspective, Kaput (1991, 1992) provided an illuminating analysis focusing on the functioning of notation systems and the medium in which they are instantiated and discussing the potential of new electronic media (Kaput, 1992, p. 522). The impact based on the reification of mathematical objects and relations can drive deeper changes in the curriculum, challenging held assumptions about “what mathematics is learnable and by which students” (Balacheff & Kaput, 1996, p. 469). Generally speaking, a dramatic transformation has occurred concerning the classic distinction between abstract and concrete, which is historically rooted in Western culture. As Hoyles and Noss pointed out, a radical change of perspective must be accomplished: “Abstraction must be thought not so much as a step upwards, but rather as an intertwining of theories, experiences and previously disconnected fragments of knowledge including the mathematical” (Noss & Hoyles, 1996, p. 44).

Quoting Wilensky (1991), the authors held that concreteness is not a property of an object, but rather a property of a person’s relationship to it; in other words, the focus is shifted from knowledge to the interaction between knowledge and the individual. This opinion is consistent with the words of Ilyenkov; in a completely different psychologic reference frame, this author focused on the notion of consciousness:

If consciousness has perceived an individual thing as such, without grasping the whole concrete chain of interconnections within which the thing actually exists, that means it has perceived the thing in an extremely abstract way despite the fact that it has perceived it in direct concrete sensual observation, in all the fullness of its sensory tangible image.

On the contrary, when consciousness has perceived a thing in its interconnections with all the other, just as individual things, facts, phenomena, if it has grasped the individual through its universal interconnections, then it has for the first time perceived it concretely, even if a notion of it was formed not through direct contemplation, touching or smelling but rather through speech from other individuals and is consequently devoid of immediately sensual features. (Ilyenkov as quoted by Engestrom, 1987, p. 241)
In this sense, mathematical concepts are “concrete abstractions” because they reflect and reconstruct the systemic and interconnected nature of real objects.3

Revising the relationship between abstraction and concreteness leads to the reconsideration of the processes of construction of meanings. In fact, the interplay between abstract and concrete is a source of meaning in a continuous reshaping in which abstract and concrete activities are mutually constitutive. The main consequence of this shift is that “from this perspective flows the educational corollary that it might be possible to design educational environments in which the process of abstracting becomes part of the lived-in culture of experience” (Noss & Hoyles, 1996, p. 47).

Different and often contrasting positions are summarized in the previous discussion (see, for instance, Dubinsky, 1991; Sfard & Linchevski, 1994). From another perspective, the “object metaphor” itself is questionable (Confrey & Costa, 1996). Setting aside this controversy, in a computer environment, the user acts and interacts in a virtual reality where critical coactors are “objectlike” elements that can be referred to as computational objects.

The choice of medium, in the variety of evolving technical devices, is fundamental; a careful analysis highlights often critical differences. (A discussion on this point can be found in Bottino & Chiappini, this volume.) The complexity in comparing different environments is well known and illustrated by several examples; for instance, regarding geometry, see the discussion concerning the comparison between Logo and a dynamic geometry software such as Cabri (Balacheff & Sutherland, 1994; Laborde, 1993; Noss & Hoyles, 1996). A rough distinction related to the evolution of computers is that between programming languages and software environments. Human interaction with computers has become more direct and increasingly similar to interaction between human beings. Artificial codes of communication that characterized the early generations of computers have now disappeared or been restricted to the community of “experts.” It is common opinion that increasingly sophisticated technologies are destined to replace today’s products.

The discussion on the role of programming (Noss & Hoyles, 1996; see also Healy, 2000) suggests the need for reflection on the benefits and limitations of such a specific type of communication between the user and the machine. Further investigation is needed, taking into account the evolution of the programming languages and their manifold relationship to mathematical knowledge. Perhaps, as far as the field of education is concerned, standard parameters of technological progress must be revised or at least reconsidered critically. For this reason, I turn now to a more general perspective and consider the pedagogical paradigms that can inspire particular educational projects.

3.2. The Constructivist Approach and the Problem of Meanings

According to the constructivist hypothesis, based on the idea of knowledge as an “adaptive function” (von Glasersfeld, 1991, p. XIV), learning results from a process of active adaptation of the learner to his or her environment, rather than a passive reception of information or instruction. As soon as the computer becomes part of the

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3 As Fischbein (1987) wrote about intuitive meanings, “The respective concepts and operations produce, in fact, for the individual’s intellect apparatus, the internal consistency, the empirical reliability, the practical “manipulability” which characterises real, concrete objects. . . . The mental ‘objects’ (concepts, operations, statements) must get a kind of intrinsic consistency and direct evidence similar to those of real, external, material objects and events, if the reasoning process is to be genuinely productive activity.” (p. 21)
environment, a question immediately arises: How will this new element influence the process, and, what specifically will be the effect of the pupils’ interaction with the computer?

Seymour Papert (1980) first introduced the idea of a “microworld” into the field of education. This seminal idea has since become crucial and has seen a great development. “Learning environments,” where pupils are expected to construct their mathematical knowledge, have flourished. The main hypothesis behind the idea of a microworld is the potential of stimulating a genuine problem-solving activity in which pupils can experiment mathematical ideas; without any explicit formal presentation, pupils can elaborate on mathematical ideas in coping with a problem and looking for a solution. In other words (Hoyles, 1991, p. 152), the computer can be considered a powerful tool within an informal learning environment (emphasis is mine); it is seen not only as powerful resource to accomplish a task, but, as observed above, one that is able to transform the task itself and at the same time transform the relationship of the user to the underlying knowledge. Thus, according to this perspective, although the crucial issue concerns the role of the computer, the user (the pupil), is the central element, and the research studies concerning both the design and the experimentation of microworlds are focused on the pupil’s learning.

A microworld (Hoyles, 1993) provides an environment for solving problems in which pupils can experience the constraints of the underlying mathematical system and, in so doing, construct their own mathematical system. A characterization of a microworld and its use in education can be synthesized as follows:

A microworld consists of the following interrelated essential features:

i) a set of primitive objects, elementary operations on these objects, and rules expressing the ways the operations can be performed and associated—which is the usual structure of a formal system in the mathematical sense.

ii) a domain of phenomenology that relates objects and actions on the underlying objects to phenomena at the “surface of the screen.” This domain of phenomenology determines the type of feedback the microworld produces as a consequence of user actions and decisions. (Balacheff & Sutherland, 1994)

Thus, a piece of mathematical knowledge is incorporated into a piece of software. Working with the solution of a problem within the microworld environment, acting and interacting with the computer, the user constructs his or her own knowledge. In particular, a mathematical concept emerges from the set of problems to which it provides a means of solution. The constructivist paradigm of reflective abstraction generated in the problem-solving activity is consistent with the theory of didactic situations (Brousseau, 1997) and the Vergnaud’s theory of conceptual field (Vergnaud, 1990), as is the key role attributed to representation systems by many authors (Noss & Hoyles, 1996; Kaput, 1992). In fact, according to its definition, representation systems are one of the components of a conceptual field.

Mathematical microworlds provide dynamic semantics for a formal system, thereby allowing the learner to explore simultaneously the structure of the accessible objects, their relations, and the representations that makes them accessible (Balacheff & Kaput, 1996, p. 471). The mutual constitutive relationship between software features (elements or commands) and knowledge is expressed well by the notion of ECO (evocative computational object), coined “to emphasize the resonance of the ‘object’ with the knowledge domain” (Hoyles, 1993, p. 10). A microworld offers great learning potential; its very nature, characterized by the availability of “computational objects” and the interactive processes occurring within it, plays a basic role in the users’ construction of meanings. For that reason, a microworld may have a teaching role that is qualitatively different from traditional explicit teaching.
3.3. Difficulties Within the Constructivist Approach

In summary, there are two complementary aspects. Mathematical ideas, as incorporated in interactive systems, have a "concreteness" that makes them manipulable. At the same time, meanings may emerge from abstraction processes, based on interaction with those systems. A microworld cannot be considered an environment that provides an autonomous construction of meaning, and (Hoyles insisted on this fact) the notion of a microworld has to be enlarged to include the whole teaching-learning system, in particular, it must include the teacher. A microworld does not have a teaching role unless it is included in a teaching situation in a relevant way. "How a microworld is used by students is also crucially influenced by the teacher, who has the responsibility of organizing the classroom setting in which the learning takes place" (Balacheff & Sutherland, 1994).

Where problem-solving situations are concerned, pupils may be able to reach a solution, which may be considered the result of both abstraction and generalization, but such a solution, and the processes related to it, are strictly related to the specific problem and the specific environment within which the problem is posed. In particular, verbal descriptions may contain mathematical terms, and their use can be mathematically consistent; nevertheless, they have relative meanings, they refer to situated abstractions (Hoyles, 1993; Noss & Hoyles, 1992, p. 125).

At the school level, the consequences of this phenomenon can be evaluated differently, and different reactions often depend on cultural differences. Let us take as an example the conclusion drawn by Hoyles and Healy (1997) on the results obtained in an experiment carried out with 12-year-old pupils concerning the topic of reflection symmetry. The crucial point is synthesized by the following key question, which the authors formulated commenting on the answers given by the pupils: "Clearly the invariants of reflection made explicit in the turtles' world are expressed rather differently from the "normal" set of axioms. Does this matter?" (Hoyles & Healy, p. 55). As the authors said, if the goal is that of making pupils appreciate a particular set of axioms, certainly the answer must be yes; but if the goal is more modest, for example, "to encourage students to build any formal description of that geometrical transformation" (Hoyles & Healy, 1997), consistency with standard axiomatization may be irrelevant. Actually, although the broadest sense of the word geometry is accepted, the main question remains: How is this approach to geometry related to "official geometry?"

The theory of didactical situations (Brousseau, 1997) recognizes the tension between socially constructed knowledge and students' own knowledge construction, stressing the role of the teacher in organizing the interaction between the subject and the "milieu." The way for the teacher to succeed in this task is to "construct a good situation" that allows the expected interaction and the emergence of the expected knowledge. Nevertheless the functioning of the "game" is definitely confined in the relationship between the subject and the "milieu," and this game may escape the control of the teacher.

3.4. From Microworlds to Macroworlds

Other difficulties are highlighted by studies concerning complex computer-based environments, such as computer algebra systems (CAS). In the last 10 years, technological developments has produced a number of software systems characterized by a "friendly" interface and a high computational, even symbolic computational, power.

4As discussed by Noss and Hoyles (1992), the term situated is consistent with the notion introduced by Lave (1988).
The need for programming languages has nearly disappeared, and interaction has become more direct. ECOs have become less evocative and more directly related to mathematical object, but what happens in a direct interaction with such virtual mathematical objects? Many studies in this field show that the practice of “touch and see” (Yerushalmy, 1997) presents many difficulties and may generate learning obstacles. Discussing an example in the case of CAS, Hillel pointed out that students need new skills, or what he calls “a certain art to graphical window shopping” (Hillel, 1993, p. 40). On the other hand, as Dreyfus (1993) said that acting in such artificial mathematical worlds requires more than pure perception; interpreting screen images may require mathematical knowledge that, in principle, should emerge from the computer-based activity itself. The case of multiple representations of a function in a graphic calculator is perhaps the best example, but similar phenomena were also observed in other environments (Laborde, 1993; Sutherland, 1993).

Generally speaking, the development of students’ conceptions as a result of their interaction with a learning environment raises a semiotic issue, related to interpretation of the phenomena observed on the screen. The problem becomes even more complicated if one takes into account possible discrepancies between mathematical knowledge and its computational transposition in a computer (Balacheff & Sutherland, 1994). As Balacheff (1997) noted, “because of its hardware characteristics and software idiosyncrasy, the computer introduces a new semiotic of mathematics” (p. 113).

The discussion about what can be learned as a result of the interaction with a computer becomes more and more complex. Nevertheless, the crucial point is the following: Is it possible to coordinate two sometimes-conflicting, tendencies: the autonomy of the student to construct his or her own knowledge versus the authority of mathematical knowledge as a cultural domain of knowledge?

4. THE INSTRUMENTAL APPROACH

4.1. Artifacts and Instruments

In addition to enthusiastic reports of experimental results concerning mathematical activities in a microworld and of successful use of graphic calculators or computers, the literature has described many cases of failure; often not all of what researchers expect actually happens. It may be that analysis of the results of a teaching experiment shows limits and difficulties that the paradigm of the constructivist approach cannot describe or completely explain. This was the case of a teaching experiment (Lagrange, 1999, 2000) concerning the use of DERIVE and carried out in a high school class: “students’ reactions and reflections did not have the meaning that the teacher expected” (Lagrange, 1999, p. 194).

To better clarify the effect of technical devices on learning processes, the analysis of their functioning must be enlarged and refined. Besides the study of experimental situations in which few pupils and relatively simple and controlled microworlds are involved, it is necessary to consider situations in which computers or graphic calculators are integrated in regular school practice. Moreover, it is useful to consider a different perspective, the so-called instrumental approach described by Verillon and Rabardel (Verillon & Rabardel, 1995; Rabardel, 1995). The use of technical devices has a double interpretation. On one hand, an object has been constructed according a specific knowledge that assures the accomplishment of specific goals; on the other hand, there is a user who makes his or her own use of the object. In other words, there is an artifact, that is, the particular object with its intrinsic characteristics, designed and realized for the purpose of accomplishing a particular task, and there also is an instrument, that is, the artifact and the modalities of its use, as elaborated by a
particular user. According to Rabardel (1995), the notion of an instrument refers to the subject and concerns the mental counterpart of a well-adapted use of a particular artifact.

From the perspective of the subject, an instrument is the unity between an object (an artifact such as a technical device) and the organization of possible actions, the utilization schemes that constitute a structured set of invariants, corresponding to classes of possible operations. Such schemes function as organizers of the activity of the user. According to this definition, an instrument is an internal construction, the development of which is a long-term process; this means that at different moments, different instruments are concerned, although the same artifact is actually used. Consider a simple and well-known artifact, the compass. Different utilization schemes may be associated with it and contribute to the construction of different instruments, according to the fact that it is used to draw circles, to report or compare segments, and to solve geometric construction problems.

As different and coordinated utilization schemes are successively elaborated, the relationship between user and artifact evolves. This process is called instrumental genesis. In principle, it is not assured that the development of the utilization schemes is consistent with the original purpose for which the object was designed, and it can develop in a completely different direction. The introduction of an instrumental approach makes it possible to analyze and interpret the difficulties encountered by pupils in dealing with a complex artifact such as a graphic calculator. On one hand, the complexity of the process of instrumental genesis may explain discrepancies between teachers’ expectations and students’ performances. On the other hand, instrumental genesis may be studied from an educational perspective. Consider a graphic calculator such as a TI92 or TI89. Potentials and limitations related to utilization schemes, as elaborated by the pupils in classroom activities, can be effective in didactic terms as long as attention is paid to the process of instrumental genesis and to the meanings that may emerge (Lagrange, 1999, p. 196). A careful analysis of the utilization schemes becomes fundamental to design tasks and activities involving complex calculators; because of the complexity and the potential of new technology, most traditional school activities should be reconsidered.

4.2. The Process of Instrumental Genesis

The distinction between the terms artifact and instrument clarifies the fact that an instrument does not exist in itself, but only in respect to its use by an individual. Moreover, an instrument is not a permanent but rather an evolving structure. The process of instrumental genesis is complex and should not be under evaluated. An interesting example, discussed by Guin and Trouche (1999), concerns an experiment carried out with two groups of 50 students. One group had a graphic calculator available, and the other did not. The problem was to calculate the following:

$$\lim_{x \to +\infty} \ln x + 10 \sin x$$

All the answers given by the pupils without the calculator were correct, whereas only 10% of the answers given by the pupils with the calculator were correct. Most of the students who could use the calculator made a mistake when extrapolating what they saw on the screen. These results can be interpreted in terms of a failure in coordinating utilization schemes with the development of mathematical meanings.

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5These machines have entered into school practice; besides the common functions of pocket calculators, they usually provide a spreadsheet, a symbolic manipulator, and graphic functions. The TI92 also provides a version of Cabri II.
Tasks and solution processes within a technological environment must take into account the genesis of the instrument, that is, the accomplishment of the process of instrumental genesis, as well the didactic objective requiring that the use of the calculator acquires a meaning in terms of mathematical knowledge. In the case of the previous task, in addition to being able to use the keys to obtain a result on the screen, students must be able to interpret that result correctly and productively. As Lagrange said, “the calculator acts as a mediator for the action of students . . . meeting new potentialities and constraints the students have to elaborate utilisation schemes, potentially rich in mathematics meanings” (Lagrange, 1999, p. 200).

The analysis of utilization schemes and the evolution of instrumental genesis may highlight interesting aspects useful to explain why students so often do not behave as was expected. The ideal constructivist approach may be usefully integrated by an instrumental approach but still does not solve the crucial problem: bridging the possible and sometimes inevitable gap between mathematical meanings and computer phenomenology.

4.3. Moving From the Instrumental Approach

Consider a complex artifact, such as a computer or a graphic calculator. As the previous discussion clarifies, different instruments are related to novices and to experts; in particular, quite different instruments may be in play when a mathematician (or a math teacher) and a pupil use the same artifact. A first educational aim is to foster the evolution of instrumental genesis for the pupils; but any evolution of the utilization schemes occurs in practice and makes them dependent on the specific tasks and problems for which they were used; meanings emerging in these activities may remain situated abstraction. Sometimes meanings may be strictly limited to the sphere of practice and related to a specific practical use; pupils learn to correctly respond to specific tasks, but the meaning involved is not the expected mathematical meaning, although it can share some aspects of it.

Consider a standard task such as summing decimal numbers with a pocket calculator. After few days of practice, it is possible to obtain a correct performance: pupils learn how to arrive at correct answers using a pocket calculator. The genesis of the instrument took place, and the artifact and its efficient utilization schemes are settled in rapport to the given task, but what meaning does the pupil assign to his or her activity?

The artifact, although incorporating mathematical knowledge and integrated by appropriate utilization schemes, might not function in generating mathematical meanings; its user might not access the meaning incorporated in the artifact. This is a common phenomenon; the process of constructing meanings is not related directly or simply to practice. For example, consider the notion of “base” and its relation to the positional notation system. Although most pupils (and adults) are familiar with counting in the decimal system (i.e., they use the notation system), they do not necessarily understand the concept of base. Now consider the example of a compass. The meaning of a circle—a geometric figure—is incorporated in the compass, which also can accomplish the graphic representation of circles. The passage from the use of the compass to trace round shapes to the conception of the circle as “the locus of the points equidistant for the center” is not immediate, however.

Expertise may add an additional obstacle in the passage from the genesis of the instrument to the process of the construction of meaning. In technical drawings, people use only small arcs instead of whole circles to save time and to create clean drawings. This may lead pupils to forget that circles are involved, if they ever knew it!

In conclusion, from the didactic point of view, mathematical meaning, incorporated in an artifact, might be accessible through the process of the instrument genesis, but
this cannot be taken for granted. A more accurate analysis of the relationship between instruments and meaning is required, and I attempt to do this in the following section.

5. MEDIATION

5.1. Mediation and Communication

Mediation is a common term in the literature related to computers and education. Most of the time, the term is used in an undefined way, referring to a vague potential for fostering the relationship between pupils and mathematical knowledge. Only a few authors directly discuss the idea of mediation; among others, Noss and Hoyles explicitly deal with the mediation function. They do so from the perspective of communication: The mediation function of a computer is related to the possibility of creating a channel of communication between the teacher and the pupil based on a shared language.

A computer language opens the communication between the user and the machine, but at the same time, it opens a channel of communication between the user and the teacher. In other words, the relationship between the pupil and the teacher may be transformed by the introduction of the computer, making communication between them more efficient and reciprocal. Although it is not possible to reduce the teaching-learning process to a communication process—and this is not what Noss and Hoyles do—it is important to stress a computer’s potential to create a dialogue with both the basic elements of the educational system: the pupil and the teacher: “Not the transmission of A’s understanding to B, but an arena in which A and B’s understandings can be externalised; not a means of displaying A’s knowledge for B to see, but a setting in which the emerging knowledge of both can be expressed, changed and explored” (Noss & Hoyles, 1996, p. 6).

These authors went beyond the model of pure transmission to consider the possibility of sharing a language that is neither a natural nor a mathematical language, but one that shapes a shared environment where communication is possible and meaning can be expressed. Taking the perspective of meaning as an alternative to a perspective of knowledge can stimulate reflection on epistemological and educational issues related to mathematical meaning that pupils can express through interaction with the computer.

5.2. Instruments of Semiotic Mediation

The possibility of an instrumental approach to mathematical learning, as discussed above, is rich and fruitful; in particular, the instrumental perspective clarified the limitations of a constructivist approach. Nevertheless, the relationship between instrument and meanings remains to be clarified. As far as a particular artifact is concerned, why and how does the instrumental genesis contribute to the construction of meaning? How does the use of a particular artifact make it possible to pursue educational goals consistent with the curriculum?

As historic and epistemological analysis confirms, the development of mathematical knowledge is based on a productive dialectics between theory and practice. A key element of this dialectical relationship between theory and practice is represented by artifacts.6 Take, for example, the beginning of geometry, in Euclid’s Elements artifacts

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6Artifacts conceived to be used for a specific purpose to accomplish a specific goal are usually referred to as technical artifacts; they incorporate a theoretical knowledge to assure the correct functioning of the artifact. Technology is the discipline concerning the productive relation between theory and practice.
such as the ruler and the compass played a special role. On one hand, they are theoretical products of the continuous effort to rationalize the perception and production of shapes; on the other hand, they are physical objects of the world to be modeled, and for this objective they are related to well-defined utilization schemes. Theories and practices may have developed independently for some time, but they have always nurtured each other; their dialectic relationship was constantly reconstructed with continuous shifts of meaning from one field to the other (Bartolini Bussi & Mariotti, 1999a). As commonly used in the studies about the practice and development of technologies (Simondon, 1968), artifacts have a twofold function: they are externally oriented, aimed at accomplishing an action, yet they also are internally oriented, aimed at controlling the action. This distinction can be elaborated starting from the seminal work of Vygotskij (1930/1978), who introduced the theoretical construct of semiotic mediation. Vygotskij distinguishes between the function of mediation of technical tools and that of psychological tools (or signs or tools of semiotic mediation). Both are part of the cultural heritage of mankind; they were produced and used by human beings, evolving over the centuries, but maintaining their functions. Although clearly distinguishable, sings and tools are assumed by Vygotskij (1930/1978, p. 53) to be in the same category of mediators. “The basic analogy between sign and tools rests on the mediating function that characterises each of them. They may, therefore, from the psychological perspective, be subsumed under the same category. . . . of indirect (mediated) activity” (p. 54).

As for their function, the difference between sign and tool elements rests on “the different way that they orient the human behaviour.” (Vygotskij, p. 54). A tool’s function is externally oriented; its purpose is to serve as the conductor of human activity aimed at mastering nature. A sign’s function is internally oriented; it is a means of internal activity aimed at mastering the self.

The use of the term psychological tools, which refers to signs as internally oriented, is based on the analogy between tools and signs, but also on the relationship that links specific tools and their externally oriented use to their internal counterpart. According to Vygotskij, the mastering of nature and the mastering of self are strictly linked, “just as man’s alteration of nature alters man’s own nature” (1930/1978, p. 55).

The process of internalization as described by Vygotskij may transform tools into psychological tools when an internally oriented tool becomes a “psychological tool” and shapes new meanings. In this sense, a tool may function as a semiotic mediator. Both technical and psychological tools are an integral part of social activity. Vygotskij noted that in addition to language, there are examples of psychological tools in mathematics as well, such as various systems for counting. What makes a counting system a tool for semiotic mediation is that it has been produced and employed to evaluate quantity but at the same time functions in the solution of problems to organize and control behavior.

Current literature concerning the Vygotskian notion of semiotic mediation does not make a distinction between the artifact itself and the mental construction resulting from the use of such an artifact according to particular use schemes. The generic notion of internalization actually refers to a complex process; the instrumental genesis, as previously described, is part of this process but does not cover it completely. According to the Vygotskian theory, the main difference consists of the orientation of the artifact’s use; that is, whether it is externally or internally oriented.

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7I will temporally use the term tools as it is used in the current English translations of Vygotskij. Tool is a general term, the use of which does not take into account the difference between artifact and instrument that were previously introduced. In the following, the relationship with the other terms, such as artifact and instrument, will be elaborated.
Let us consider the development of the notation system commonly used for numbers. Our notation system has its origin in the abacus, which is an artifact that keeps track of the process of counting. The evolution of utilization schemes transform the relationship between the object (artifact) and the user, and a process of instrumentation takes place. Subsequently, a shift occurs from a gesture medium in which physical movements are related to an external apparatus, a graphic medium—that is, the notation system. Some aspects of the original gesture medium are preserved in the permanent signs written on paper. For example, spatial aspects of the abacus are maintained in the notation. But, although maintaining some aspects of the external origin, the new signs become part of a system with independent rules. The graphic medium thus gains autonomy; without any reference to the abacus, it can be used in the solution of new problems, but it also can be used in the planning phase, shaping the strategy to solve an arithmetic problem. In short, the external form of social activity mediated by the abacus and by the derived signs has been transformed into an internal form, related to the solution of arithmetic problems, contributing to the construction of arithmetic operations. This example shows how the role of an instrument (an artifact related to its utilization schemes) can be interpreted in terms of semiotic mediation (i.e., meanings construction).

As in the case of the abacus, a fine-grained analysis of the origin and development of instruments may provide insight into the dialectic relationship between practice and theory in the construction of mathematical knowledge, as well as interesting educational implications. (Bartolini Bussi & Mariotti, 1999a, 1999b) The key point is the distinction between the use of an artifact in an external orientation and the use of the artifact in an internal orientation, together with the process of the transition from the external to the internal orientation. According to this distinction, the artifact and its utilization schemes result into an instrument that can function mentally and, in so doing, may shape new meanings. It becomes what Vygotskij would have called a psychological tool.

An illuminating description of such a process of internalization (Vygotskij, 1978) in the case of the compass and the circle is given by Bartolini Bussi and colleagues: “The geometric compass, embodied by the metal tool stored in every school-case, is no more a material object: it has become a mental object, whose use may be substituted or evoked by a body gesture (rotating hands and arms)” (Bartolini Bussi, Boni, & Ferri, in press). In this case, the process is described as it was accomplished within a primary school class, during a long-term teaching experiment, rather than through a historical reconstruction.

5.3. The Process of Semiotic Mediation: The Role of the Teacher

When a discussion focuses on an artifact, it is important to consider the relationship between how it was conceived and constructed and how it is used, between the knowledge incorporated in it and the utilization schemes of the user. An educational perspective requires that one also consider the relationship between meanings that emerge from the use of the instrument and meanings that are culturally recognizable as mathematics.

Previous analysis highlights the potential in terms of meaning of a technological artifact (computer, microworld, graphic calculator, etc.) when it is introduced in school practice. At the same time, the instability of the processes of meanings construction is related to the use of an artifact. The reference frame of Vygotskij offers the possibility of overcoming this impasse.

The analysis of the functioning of the artifact must be further elaborated. On one hand, in the process of instrumental genesis, the artifact becomes an instrument.
The subject develops potential utilization schemes, and then the artifact, become an instrument, may support the construction of meanings related to those utilization schemes. On the other hand, the artifact, acting as a mediator between learners and a teacher, may be used by the teacher to exploit communication strategies aimed at guiding the evolution of meanings within the class community. In other words, the artifact may function as a semiotic mediator.

The process of semiotic mediation develops on different levels, although it is centered on the use of a particular artifact:

- The pupil uses the artifact, according to certain utilization schemes, to accomplish the goal of the task. In so doing, the artifact may function as a semiotic mediator; that is, meanings emerge from subject’s involvement in the activity.
- The teacher uses the artifact according to specific utilization schemes related to an educational motive. In this case, as explained in the following examples, the utilization schemes may consist of the particular communication strategies centered on the artifact.

In the dialectics between these two levels, the construction of meanings occurs as the product of a process of internalization guided by the teacher. Thus, the artifact is exploited by a double use in which it functions as a semiotic mediator. The learner uses the artifact in actions aimed at accomplishing a certain activity, and meanings emerge from this activity; the teacher uses the artifact to direct the learner in the construction of meanings that are mathematically consistent. A computer can intervene in this activity, but it is used in different ways, according to the subjects involved in the activity. It can be an artifact, used according to utilization schemes. In this case, meanings, if there are any, may emerge, but the mathematical meaning incorporated in the artifact may remain inaccessible to the user. The computer can also be used as an instrument of semiotic mediation. In this case, the teacher uses it to accomplish communication strategies aimed at developing a specific meaning related to the mathematics content, which constitutes the motive of the teaching and learning activity.

The mathematical meaning related to the knowledge incorporated in the artifact becomes accessible to the learner by its use, but teacher’s guidance fosters the construction of meaning as long as specific activities are organized, the motive of which is the evolution and construction of meanings that are mathematically recognizable and acceptable. According to that perspective, the following hypothesis can be stated: Meanings are rooted in the phenomenological experience (actions of the user and feedback of the environment, of which the artifact is a component), but their evolution is achieved by means of social construction in the classroom, under the guidance of the teacher.

The following sections will be devoted to illustrating this theoretical hypothesis in the case of a particular artifact, the software Cabri-Géomètre (Baulac, Bellemain, & Laborde, 1988), and a particular mathematical meaning, that of proof.

6. CONSTRUCTION OF MEANINGS

6.1. Semiotic Mediation in the Cabri Environment

A long-term experiment concerning geometric constructions in two environments (ruler/compass and Cabri) was carried out in recent years. This research focused on the problem of introducing students to theoretical thinking. According to our didactic problem, the main motive of classroom activities proposed to students is the evolution of the idea of proof, and this is realized by means of the evolution of the idea of geometric construction within the field of experience of geometrical constructions in the
The term *field of experience* is used after Boero, Dapueto, Ferrari, Ferrero, Garuti, Lemut, Paranti, and Scali (1995) to refer to “the system of three evolutive components (external context; student internal context; teacher internal context), referred to a sector of human culture which the teacher and students can recognise and consider as unitary and homogeneous” (p. 153). The evolution of the field of experience is realized over time through the social practices of the classroom. In this experiment, in addition to spontaneous forms of interaction, there are specific forms of controlled and planned classroom verbal interaction (Bartolini Bussi, 1998) realized by means of “mathematical discussion”: “a polyphony of articulated voices on a mathematical object, that is one of the objects—motives of the teaching—learning activity” (Bartolini Bussi, 1996).

The “polyphony of voices” in this case concerns the dialogue between the voice of practice (i.e., a practical conception of graphic construction) and the voice of theory (i.e., a theoretical conception of geometric construction). Through the dialogue between these voices the ability to create and read configurations (to recognize Gestalts), as it is continuously practiced in the production of drawings in both environments, has to be enriched with theoretical control. On one hand, the concrete production of a drawing on a sheet of paper is a practical activity; its correctness is controlled by empirical verification. On the other hand, geometric constructions have a theoretical meaning that supercedes the apparent practical objective. Every geometric construction is based on a theorem that guarantees theoretical control of the procedure by which it has been realized. As experimental evidence shows, theoretical control is not spontaneously achieved but can result from the activities that pupils perform within the chosen field of experience (Mariotti, 1996).

The nature of the particular environment—here, the Cabri environment—in which geometric constructions are realized may foster the shift from practical to theoretical meaning of construction. Nevertheless, the context itself is not sufficient, and the intervention of the teacher becomes determinant. We shall limit ourselves to describe some elements of the external context to present an analysis of the process of semiotic mediation that can be realized in the classroom activities.

### 6.1.1. The External Context

The external context is determined by the concrete objects of an activity (paper and pencil; a computer with Cabri software; signs, e.g., gestures, figures, texts, dialogues). According to the main goal of this discussion, we focus on the “objects” offered by the Cabri environment, although they must be considered in a dialectic relationship with all the other objects available. The Cabri environment offers the following objects:

- Primitive commands and macros, which create the geometric relationship characterizing geometric figures (these are the external signs of the basic elements that constitute the theory); and
- The dragging function, which starts as a perceptual control tool to check the correctness of the construction and then becomes the external sign of the theoretical control.

In the frame of the geometric construction activities, Cabri primitives, macros, and the dragging function constitute the external signs on which the evolution of pupils’ internal context is based; such evolution concerns the development of the geometrical theory and the development of the meaning of theory itself. These objects may become instruments of semiotic mediation (Vygotskij, 1978) in as much as they can be used...
by the teacher in the concrete realization of classroom activity and according to the motive of introducing pupils to theoretical thinking.

6.2. The Evolution of the Meaning of Geometrical Construction

In the Cabri environment, the construction activity (i.e., drawing figures through the available commands of the menu) is integrated with the dragging function. Actually, several possible constructions can be realized on the screen; when dragged, some constructions maintain their geometric properties, and others do not (Laborde, 1993; Laborde & Strässer, 1990). In the field of experience of Cabri constructions, the meaning of geometric construction emerges both from the activities of construction and the activity of mathematical discussions related to them. In the social practice of mathematical discussion, the way pupils make sense of the activity of construction, in rapport to the “domain of phenomenology” (Balacheff & Sutherland, 1994) and the type of feedback it allows, is elaborated and developed under the guidance of the teacher.

The key point here is that a Cabri figure has to be related to the procedure that produced it, so that what must be validated is the correctness of the construction; it is not the product of a procedure that must be validated, but the procedure itself. The distinction between drawings and geometric constructions is related to the general distinction between a “drawing” and a “figure” (Arsac, 1989; Laborde, 1993; Laborde & Capponi, 1994; Mariotti, 1996), but in this case the discussion is limited to the process of semiotic mediation based on the use of the instruments offered by Cabri. The distinction between drawing and geometric construction results as a consequence of the internalization of the control “by dragging.”

The following analysis of a mathematical discussion aims to show the complexity of the process and to point out how, within the Cabri environment, a teacher can find a specific instrument of semiotic mediation, contributing to the development of the meaning of geometric construction.

6.2.1. The Mediation of the “History Command”

The case discussed here involved one of the experimental classes of the Cabri project (a 9th-grade class at a scientific high school [liceo scientifico]); 19 out of 23 pupils in the class participated to this activity (Mariotti, in press; Mariotti & Bartolini Bussi, 1998). This is the first activity and constitutes the very beginning of the long-term experimentation. The first part of the activity took place in the computer room, where pupils sat in pairs at the machines. Pupils had general expertise with the computer, but they had never used Cabri; after a short acquaintance with the Cabri environment (they were let to freely explore the software for about half an hour), the following task was presented: “Construct a segment on the screen. Construct a square that has the segment as one of its sides.”

Pupils were asked to create a figure on the screen and to write down a description of both the procedure and their reasoning. The term construct is expected to be ambiguous, but it was selected intentionally: The comparison between different interpretations of the task will fuel the following discussion. Pupils obtain their solutions in different ways; some of them refer to geometric properties, others refer to perceptual control, but most refer to both. Dragging will transform the figures obtained differently.

The teacher opened the discussion by suggesting that students analyze the solution given by Group 1 (Giovanni and Fabio). The solution was obtained by drawing four consecutive segments and arranging them into a square until the image is perceptually satisfactory. To judge whether the solution is correct everybody agreed that control must be exerted on the particular drawing and, according to the well-known
properties of a square, pupils suggested measuring sides and angles. They debated the ways of checking the correctness, but the only elements that arose were the use of measure and the precision related to it.

The following day, when the discussion was resumed, the teacher presented the drawing produced by Group 1 and dragged one of the points (Fig. 27.1). The figure is altered, and everybody agrees that it is no longer a square.

At this point, another solution (Group 3, Dario and Mario) is proposed by the teacher.

21 Teacher: Well, I’d like to know your opinions about Dario and Mario’s construction.
22 Marco: They made a circle and then two perpendicular lines.
23 Teacher: Do you know how they started?
24 Michele: We can use the history command.
25 Teacher: Let’s do it. They took a segment, then they...[...the construction step by step follows]. They drew a line perpendicular to the segment, then the circle... in your opinion, what is it for? What is its use?

SILENCE

Is there logic in doing so, or did they do it just because they felt like drawing a perpendicular line, a circle, Alex, tell us.

26 Alex: The measure of the segment is equal to the measure reported by the circle on the perpendicular line.
27 Teacher: You mean that the circle is used to assure two equal consecutive segments, the first one and that on the perpendicular...
28 Class: is used to obtain an angle of 90°.
29 Teacher: I know that the square has an angle of 90° and four equal sides or three equal angles,... then let’s see if it is true... let’s go on. Intersection between line and circle. They (Dario and Mario) determined the intersection point between the line and the circle... why did they need that point?
30 Chiara: The intersection point between the line and the segment...
31 Teacher: And what should you draw from there?
32 Chiara: A segment, perpendicular to the line.
33 Teacher: What else??
34 Chorus: Parallel to the segment...
35 Teacher: Let’s see what they did.
The first reaction to the teacher’s questions came from Marco. He attempted to describe the procedure; facing difficulty reconstructing it, he suggested the use of the history command. The teacher took his suggestion and executed the command on the “master computer.” While the computer executed the first steps of the construction (Fig. 27.2) and the corresponding elements were drawn on the screen, the teacher described what was done. At one point, she interrupted the procedure and asked pupils to detect the “motivations” that caused their classmates to use these commands. The teacher’s question, asking for an interpretation of the commands used (we call it the “interpretation game” (23)), aims to provoke the first shift from the procedure to a justification of the procedure itself.

The silence that follows the teacher’s first question shows the difficulty of that shift: Moving from action to its motivation is not immediate. The analysis of the discussion (for details see Mariotti & Bartolini Bussi, 1998) reveals that the teacher’s communicative strategies were directed by a main motive, that is, shifting the control from the description of the procedure to the motivation of the procedure. For instance, when Alex (26) expressed the relationship between two of the segments according to the series of commands previously executed, the teacher (27) reformulated his statement in terms of motivations: “You mean that the circle is used to assure two equal consecutive segments.” The class appropriated the teacher’s expression and continued discussing motivation. A natural evolution of the interpretation game is what we have called the “prediction game”: by stopping the execution of the procedure, the pupils are asked to predict the next step, determine its motivation, and then compare it with the step recorded in the history. Different solutions are compared, always negotiating the acceptance of a Cabri figure as the correct solution of a construction task. What is most interesting is the fact that, together with the acceptance of a solution in terms of the dragging test, a new relationship to drawing is achieved: It is possible to explain the correctness of a construction by controlling the “logic” of the procedure.

6.2.2. The Role of the Teacher

During the discussion, the use of the history command played a crucial role in changing pupils’ relation to drawing, that is in changing the meaning of “construction” from the drawing produced on the screen to the procedure that realized it. As clearly shown, however, the teacher also plays a fundamental role. Using the specific objects offered by the external context, specific strategies are made available to the
teacher to augment standard strategies and manage discussions. Through the mediation of the history command, the teacher may put into practice the interpretation and prediction games. The interpretation game is introduced with questions such as, Why did the authors choose this operation? What is the use for? This focuses on what could have been the intention or goal of the person who made the construction. The prediction game is introduced with questions such as, How would you go on from this point? What could have been the following step in this construction? This induces pupils to enter into the construction procedure and offers the opportunity to discuss different possibilities.

The previous example shows the functioning of a particular element of the software in the process of meaning construction. In this case, the history command is an instrument of semiotic mediation and has a double function. Pupils use it to show the sequence of the construction steps, and the teacher uses it to mediate the meaning of construction as a procedure and introduce pupils to the idea of justifying the correctness of a construction. The two games are possible because the software makes available a decontextualized copy of the construction procedure (i.e., its sign).

In the following, the history command remains a basic element in the activities of a theorem’s production. The availability of an external sign, referring to the procedure of construction in its temporal sequence, very often contributes to the production of a description and a correct justification of a construction.

In the dynamic of the Cabri figures, the relationships between the geometric properties are expressed globally, so that the pupils may miss some hidden relationships of logic dependence (see the protocol of Lorenzo that follows). The dragging invariants represent perceptually all the geometric properties at the same time. The history command reintroduces the temporal dimension and allows one to grasp the construction process as it develops; the temporal sequence of the construction’s steps represents the counterpart of the logic hierarchy between the geometric properties of a figure. In so doing, it supports the control of the logic relationships between the properties involved.

6.3. The Theoretical Meaning of a Construction: An Example

The analysis of pupils’ protocols shows the slow evolution of the meaning of construction. At first, a construction is conceived as a concrete process to reach a drawing, which is justified by the acceptability of the final product; a construction is then conceived as a theoretical procedure, which is justified by a theorem. Here is an example. The following task was presented to the pupils: “Construct the bisector of an angle. Describe and geometrically justify your solution.”

This is one of the first construction problems that was proposed to the pupils. Analogous to the Euclidean axioms, in the Cabri construction menu, the commands are reduced to “intersection of two objects,” “compass,” and “report of angle.” From a theoretical point of view, this menu provides the three criteria of congruence for triangles. These criteria are already part of students’ theoretical system and they can refer to them to justify their solution.

Lorenzo (9th Grade)

I consider the triangles ABD and ACD. They have side AD in common, and side AB of the first is equal to side AC of the second. In fact, if I take a circle with its center in A and point B, it passes through both B and C. Thus, the sides AB and AC are equal because they can be considered rays of a circle. If I also put the point of the compass in D with the ray DC, the circle passes through both C and B. Thus, the sides BD and DC are equal for the same reason. I discovered that the triangles ABD and ADC have equal the sides; for this reason the two triangles are congruent for the third criterion of congruence. If the
two triangles are equal, the rule is that equal sides are opposite to equal angles. Thus, the angles 1 and 2, which are opposite to equal sides BD and DC, are equal.

In this case, the difficulty in selecting the correct hypotheses appeared. Such a difficulty is made clear by the fact that in the first step the equality of two of the sides is correctly derived form the construction, while the equality of the other sides is obtained by considering the circle with center D and ray DB, which does not pertain to the original construction. Actually, the construction D was accomplished by the intersection of two circles, center C and ray CB and center B and ray BC, and, as a consequence, the circle with center D that passes through C will pass through B as well.

In the sketch drawn for this protocol (Fig. 27.3), all three circles are present at one time (center B/ray BC, center C/ray CB, center D/ray BD), but the sequence of the operations used in the construction is not preserved in the drawing. Not even in the dynamic of the Cabri figure can the correct order of the construction be reestablished; when the figure is moved, the mutual relationships among the three circles are preserved, but the necessary order in the construction disappears. In this case, the use of the history command, showing the construction step by step, can help overcome the obstacle. The sequence of the construction steps reconstructs the order in the procedure, and thus it is possible to detect the correct relationship and keep the logical control of the geometric figure.

7. CONSTRUCTION OF A THEORY: THE ROLE OF THE MICROWORLD

7.1. Using Commands and Using Axioms

With regard to proof, there are two interconnected areas of difficulty. On one hand, the idea of validation must be introduced; on the other hand, the rules for validation must be stated, and the acceptance of validation depends on the acceptance of these rules. According to the basic aim of our research—introducing pupils to theoretical thinking
and, in particular, to geometry theory—we decided to build a dialectic relationship between Cabri constructions and geometric theorems. Starting from the Cabri environment, pupils should have been guided to enter into the geometric system. The key to access was the link between the logic of Cabri, expressed by its commands, and the geometry theory expressed by its axioms and theorems.

From the theoretical point of view, the complete Cabri menu corresponds to the whole system of Euclidean Geometry; in other words, the geometrical primitives of Cabri condense axioms and theorems expressing the relationships between the main concepts and properties. For instance, consider the perpendicular line command. This command incorporates the piece of theory concerning the definition of perpendicularity and the theorem validating the existence of a perpendicular line. Thus, the focus on the underlining system may be too difficult to manage because the corresponding geometric system, may be too complex to grasp all at once. In fact, because of the richness of "geometrical tools," it is difficult to state what is given (axioms) and what must be proved (theorems). Generally speaking, the richness of the environment might emphasize the ambiguity of intuitive facts and theorems, so that it can even constitute an obstacle to grasping the meaning of proof.

Rather than giving pupils a ready-made Cabri menu, corresponding to the complete Euclidean theory, we decided to let the class construct its own Cabri menu, step by step. Taking advantage of the flexibility of the software environment, the microworld was adapted to follow the evolution of the theory. At the beginning, an empty menu was presented and the choice of commands discussed, according to specific statements selected as axioms. Then, in the sequence of the activities, the other elements of the microworld were added, according to new constructions and in parallel with corresponding theorems. In this way, the geometrical system is slowly built up, its complexity gradually increasing. The aim is that of reaching levels of complexity that pupils can manage while making them participate (Leont’ev, 1976/1964) in the construction of a menu and its corresponding axiomatization.

It is impossible to discuss here the details of the protocols’ analysis. Nonetheless, the following examples aim to give an idea of the evolution of meanings described above, from the Cabri command to the theoretical statements (axioms and theorems). The descriptions of the procedure change, improving in clarity through an increasing mastery of correct terms. At the same time, the argumentation approach to theorems, that is, the justifications the pupils provide, assume the form of a statement and a proof, referring to the given theory. Let us consider the first example.

**G. and C. (9th Grade) 1C Liceo Scientifico**

Create the midpoint of a segment.

I create a segment through two points. I make three other points on the screen and construct an angle with them. With (the command) “report of angle,” I carry this angle on the edges of the segment, and I create the intersection of these two rays. Using (the command) “circle (center, point),” with the center on the edges of the segment and point on the intersection of the rays, I create two equal circles. Joining the two intersections, I find the midpoint O.

I did that because by creating the equal angles on this segment, I created an isosceles triangle. Using the equal sides of this triangle as radios of two new circles, I can construct two equal circles on the edges of the segment.

This protocol does not contain a drawing; the pupil was probably referring to the screen image. In Fig. 27.4, the construction has been reproduced step by step to illustrate the procedure described by the pupils.
This is a good example of a first step in the evolution of the meaning of theoretical justification. The description of the procedure is still mixed with its justification, showing the difficulty of separating the operational aspect (creation of the drawing) and the theoretical aspect (identifying the geometric relationships) drawn from the figure according to its construction. For this same reason, however, it is possible to observe the transition from the use of command and the theoretical validation.

As the protocol clearly shows, in the process of internalization the external signs—the Cabri commands—are transformed into internal tools related to theoretical control. According to a classic axiomatization (Heath, 1956, p. 229), the particular command “report of angle” corresponds to one of the axioms introduced in the theory. Such an unusual construction, quite different from those found in the textbooks, provides a strong support to our interpretation.

Similar examples can be found (Mariotti, 1998; Mariotti, 2001) and provide evidence of the fact that the construction problem has achieved theoretical meaning, while at the same time, the commands of Cabri have been transformed into external signs of theoretical control that correspond to axioms (and theorems).

It is interesting to note how the double status of the visual perception in Cabri is achieved. Dragging is both a legitimate tool of control (for example, it can be used to verify the robustness of the construction), and an “illegal” means of proof (i.e., it is not acceptable to say: “it is true because I see it”!).

8. INTERNALIZATION OF DRAGGING AS THEORETICAL CONTROL

Within the Cabri environment the dragging function plays a crucial role in the solution of open-ended problems, when pupils must formulate and validate a conjecture. They create a Cabri figure according to the given hypothesis and explore its geometric properties by dragging and observing perceptive invariants. However, the interpretation of perceptive invariants in terms of a geometric conjecture requires the interpretation of the dragging control in terms of theoretical control. One must be aware that the hierarchy of a construction realizes a logical relationship relating given properties to properties perceived as invariants in the dragging mode. In other words, the use of Cabri in the generation of conjecture is based on the internalization of the
dragging function as a logic control, capable of transforming perceptual data into a conditional relationship between hypothesis and thesis. As Laborde and Laborde noted, “the changes in the solving process brought by the dynamic possibilities of Cabri come from an active and reasoning visualisation, from what we call an interactive process between inductive and deductive reasoning” (Laborde & Laborde, 1991, p. 185).

Being aware of the fact that the dragging process may reveal the relationship between geometric properties embedded in the Cabri figure directs a way to transform and observe the screen image. As Laborde and Laborde pointed out, describing the solution process, “This way of solving the problem is strongly related to the fact that the solver does not work on the drawing but really on the figure,” (1991, p. 185). At the same time, that consciousness is indispensable to exploit some of the facilities offered by the software, such as the “locus of points” or “point on object.”

Evidence of the process of internalization of the theoretical control can be shown by the way pupils construct and transform the image on the screen, during exploration activities. First, students construct the figure to be explored, taking into account that the properties given in the hypotheses are achieved through the corresponding commands of Cabri. A conjecture may then emerge from the exploration of the figure by dragging, but its validation is sought within geometric theory, that is, in principle, a conjecture asks for a proof. The following example aims to analyze this process of internalization.

8.1. A Particular Case: The Rectangle Problem

Consider the following activity, proposed to the pupils during a session in the computer laboratory. The pupils sat in pairs at the computer and were told to accomplish the following task: “Draw a parallelogram, drag the figure in order to make one of its angles right. Write your observations.”

Different behaviors could be observed in this activity, all of them highlighting different aspects related to the process of internalization relating the control by dragging to the theoretical control on the geometrical figure.

Almost all the pupils constructed a parallelogram and describe the construction correctly; some of them provide a drawing reproducing the image on the screen, completed by labels referring to the construction accomplished (for instance: “line by two points,” “parallel line,” see Fig. 27.5). This shows the pupils’ preoccupation with reporting on paper the complexity of the Cabri figure, constituted by the image and controlled by the properties/commands used in the construction. One can interpret these elements as traces of the intention to create a parallelogram that corresponds to the hypotheses for producing conjectures.

After the construction of the parallelogram, the pupils follow the instruction given by the task and consider the new hypothesis (“make an angle right”). They then transform the figure by dragging, adjusting the perpendicularity of the sides “by eye,” and the possible conjecture becomes immediately evident: “it is a rectangle!” Actually, the difficulty of this task is in the immediacy of the conclusion expressing, in a contracted way, the fact that if one of the angles is right, all the other angles of the parallelogram are right. All the conjectures were correct, but errors occurred in the proof.

The exploration strategies used to accomplish the solution reveal the presence of particular “signs” used by the pupils as a support in the solution process. Such signs can be interpreted as means of external control on the logic operations required to relate the hypothesis to the thesis and produce the logic deduction that constitutes the proof. Such signs are generated within the software environment and derive their semantics from those of the software. In other words, their interpretation is consistent
with the system of meanings that emerged in the practice, up to that moment, but clearly refer to theoretical control. Let us analyze two of the observable signs.

8.1.1. “Mark an Angle” and the Corresponding Sign on the Drawing

Before dragging the figure, some of the pupils used the command “mark an angle.” That command, without which it is impossible to obtain the measure of an angle, is used as a means to control the hypothesis of “one (only one) right angle,” under which the exploration must be carried out and the conjecture must be formulated. The drawings, which the pupils included in their written reports, reproduce a sequence of snapshots, describing the different phases of the solution as they developed while interacting with the software (see Figs. 27.5 and 27.6).
The images in the figures show the angles marked successively. At first, the angle that will become right, then the angles that are assumed to be right. The software distinguishes between the sign for “mark an angle” and the sign for “right angle.” The same distinction is reproduced in the drawing (Figs. 27.6A and 27.6B).

8.1.2. The “Perpendicular Line”

Other pupils used a different type of sign. They construct “a perpendicular line,” drawn from one of the vertices of the parallelogram to one of its side. The exploration of the figure is then realized by dragging a free vertex and making one of the oblique sides coincide with the “perpendicular line.” In the protocols, besides the description of the exploration, pupils attempt to express the dynamics of the Cabri figure. For example, pupils drew arrows connecting the vertex and the moving side to the “perpendicular line” (see Fig. 27.7).

Being aware that properties obtained by adjusting the figure “by eye” do not grant the validity of the derived properties, leads pupils to look for a control in an element obtained “by construction.” Strategy “by eye” and “by construction” definitely entered into practice, and pupils used specific elements of the software (external signs) with the aim of keeping the control of the two different meanings.

The problem of the rectangle, in its simplicity, focuses on a basic aspect of the process of exploration in the Cabri environment: the need to maintain control of the figure in terms of given properties (hypothesis) and derived properties (thesis). It is possible to observe the need of a support in the process of exploration, and the semantics drawn from the interaction with the software allows the pupils to generate external signs to support theoretical control that is not completely internalized (Bartolini Bussi, Boni, Ferri, & Garuti, 1999).

Often the phenomena observable on the screen hide the asymmetry of the conditional relationship between properties. All the properties hold at the same time and there is no distinction between the hypothesis and the thesis.

In principle, during the exploration of a figure by dragging, it is possible to keep the theoretical control needed to express the conjecture in a conditional form but in fact that control can be very difficult to achieve. It requires one to relate the basic points (the elements variable by dragging) to the observed properties and express that in a conjecture, that is, it requires relation between hypothesis and thesis.

As the previous examples show, construction activities in the Cabri environment can provide a field of experience where the meaning of geometric construction is expected to evolve, introducing pupils to the theoretical world of geometry. The process of internalization, based on the social practice of the classroom, transforms the commands available in the Cabri menu (external signs) into internal psychological tools, which organize and direct pupils’ geometric thinking, in producing both conjectures.
and proofs. Appropriation of theoretical control related to the dragging function is not easily grasped, however. The process of internalization may be slow, and its difficulty should not be overlooked.

9. CONCLUSIONS: TEACHERS AND DIDACTIC ISSUES

The entry of computers into schools has been slow, and their integration in school practice even slower. "No more than 15% of the teachers include graphic calculators in their teaching, in spite of the fact that all students have a graphic calculator in scientific classroom. Teachers appear to resist the integration of new technologies even at an elementary level" (Guin & Trouche, 1999). Obvious differences between countries notwithstanding, there is a general complaint that the great expectations of 20 years ago have not been realized (Balacheff & Kaput, 1996, p. 470). The educational system functions at its own pace, characterized by a resistance to change, and this characteristic may explain the difficulty and the failures that occur not so much in introducing but rather in integrating new technologies into the school. This is a general tendency of the educational system, but in the case of computers, the explanation can be taken further. The previous discussion should have clarified how and why the great potential of new technologies cannot be exploited immediately. There are at least two levels of difficulty. Computers, different from other technology because of the effect they have on mathematics itself, require a radical change of perspective and a profound change in the curricula. At the same time, they demand a change in school practice. If a graphic calculator is available, traditional tasks, such as "calculate the limit," may lose their meaning. According to a traditional approach, tasks concerning the calculation of limits aim to construct schemes of reasoning, allowing a rapid and efficient evaluation of a function in the boundaries of its critical points. With a graphic calculator, this kind of task is reduced to the simple motor activity of pushing on a key. As a consequence, if such tasks and their motivation must be preserved, there is no other possibility for the teachers than reject the computer.

Integrating computer technologies into school practice requires a radical change of objectives and activities. From teachers’ point of view, this is a very expensive request: “Teachers want to be convinced that the internal efficiency of the educational system will be increased by such change in teaching means” (Artigue, 1998, p. 122). Artigue (p. 126) explains the difficulty of escaping the vicious circle of minimizing cost and difficulty of integration and overestimating the benefits for mathematics teaching and learning. Taking an instrumental perspective and exploiting the process of semiotic mediation that artifacts make possible permits consideration of new technology in a broader problematic context. Computer technology, like “older” technology, has the potential that comes from being products of human culture, incorporating knowledge and expertise (Bartolini Bussi et al., 1999; Bartolini Bussi & Mariotti, 1999). Nonetheless, as previous analysis shows, the teacher still plays a central role, complex and delicate at the same time. This role supercedes that of selecting a good problem.

Conscious use of available technology for semiotic mediation requires an attentive and careful planning of the activities, taking into account the double use of the artifact in play. A deep knowledge about the artifact and the process of instrumental genesis is necessary. This requires more than simple familiarity with the use of the artifact itself. It is also necessary to analyze the artifact and its potential to organize and carry out classroom activities according to this function in terms of semiotic mediation. Of course, teachers cannot do this by themselves.

All this represents a great challenge in the field of math education research, to provide teachers the knowledge and the support they need. Further studies and investigations focusing on new technology may profit from comparison with the results from studies of “old” technology.
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REFERENCES

CHAPTER 28

Flux in School Algebra: Curricular Change, Graphing Technology, and Research on Student Learning and Teacher Knowledge

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INTRODUCTION

School algebra is a complicated curricular arena to describe, one that is undergoing change. Yet our capacity to track curricular change, to identify ways in which technology supports such change, to study the knowledge teachers use in implementing such change, and to understand what students learn as a result of such change all hinge on descriptions of this curricular arena.

With respect to school algebra, at one and the same time, there seem to be shared assumptions and strong disagreements. On one hand, leaving aside many issues of how classrooms are organized, in terms of the content to be studied, school algebra seems to be well defined and understood. Even in international settings, many mathematics education papers and talks assume that school algebra is a term that refers to shared curricular notions and to particular kinds of student experiences. The shared image seems to be that introductory school algebra coursework consists of experiences designed to help students learn to factor and multiply some range of polynomial expressions and to solve linear and quadratic equations. Of course, there are concerns about some ways of reaching such goals; some experiences designed to reach these goals are described as involving students in meaningless manipulation.
On the other hand, mathematics educators do not seem to be in broad agreement about how one might conceptualize this curricular area (e.g., see Lee, 1996, for results of interviews with a range of mathematics educators). In particular, technological innovations seem to have fractured whatever agreements may have existed with regard to school algebra. As mathematics educators try to envision “algebra in a technological world” (in the words of Heid, Choate, Sheets, & Zbiek, 1995), they are drawn in different directions. Some, working in directions first suggested by Klein at the beginning of the 20th century (Klein, 1945) and championed by others subsequently (e.g., Fehr, 1951; Hamley, 1934; van Barneveld & Krabbendam, 1982), are particularly drawn to visions that involve making “functions” an important component of school algebra. They argue that such changes might enable desirable changes in the nature of classroom interaction. Others would argue that whatever the potential benefits of such a direction, it is not representative of algebra (see, e.g., Lacampagne, Blair, & Kaput, 1995; Pimm, 1995). Perhaps one way to integrate these seemingly different phenomena is to suggest that with regard to school algebra, there are important differences in the ways in which educators imagine students might achieve a set of competencies, in which there is some agreement with respect to these competencies.

Further complicating matters, mathematicians, steeped in the dramatic successes of this past century’s use of algebraic methods, often find it confusing to understand what mathematics educators mean by school algebra and how it relates to current foci of the research community. For example, when interpreting discussions of school algebra in a piece titled “Algebra with Integrity and Reality,” Bass (1998) emphasized that as someone who has “lived professionally with the subject we [mathematicians] call algebra, I am not an expert on what algebra means and looks like in the school curricula” (p. 9). Nonetheless, in referring to debates about school algebra among mathematics educators, he suggested that the discussion often “seems to envisage broad areas of mathematics, such as functions or modeling, and to identify within them certain algebraic ideas and techniques” (p. 9). In our reading, this comment suggests that to Bass, the links between these broad areas and his conception of algebra are tenuous.

This difficulty describing school algebra as a curricular area poses challenges to curriculum developers, teachers, and researchers of student learning, teacher knowledge, and the impact of technology on teaching and learning, in addition to mathematicians trying to understand what mathematics educators mean by algebra. For those developing instructional materials, there is the challenge of articulating one’s approach amongst a set of choices. So far, most new materials make only rough contrasts between “traditional” or “standard” approaches and materials taking different approaches (e.g., Huntley, Rasmussen, Villarubi, Sangtong, & Fey, 2000), many of them technologically based (e.g., Fey & Heid, 1999; Heid et al., 1995). For teachers, this challenge takes on a different form. In choosing curricular materials, both textual materials and technological support, teachers are choosing across a variety of approaches to school algebra. Important questions center around the ways in which particular pieces of software, for example, do or do not support or work in concert with particular textual materials.

In addition, for researchers of student learning, teacher knowledge, and the impact of technology on teaching and learning, the presence of different approaches to school algebra raises a host of questions. Are phenomena, such as the didactical cut (Filloy & Rojano, 1989; Herscovics & Linchevski, 1994), approach independent? Or are they, as suggested by Kieran, Boileau, and Garancon (1996), related to how students learn to solve linear equations and to the cognitive tools they have to tackle such problems? Is the “deep structure” of word problems approach independent, or is there an interaction between approaches to school algebra and categorization of word problem tasks (Yerushalmy & Gilead, 1999)? As a result, there is the important challenge of
describing the type of approach to algebra in which a particular study is carried out and in which phenomena of learning are identified.

Similarly, what is a conceptual or relational understanding of school algebra? Are such understandings approach independent as well? Is there an essential core or set of understandings that teachers should master and students develop? Does one think that an important component of teachers’ knowledge is their capacity to distinguish different approaches to the same subject? Is that an important component of having a “subject matter understanding for teaching” (in the sense of Usiskin, 2000)? Is such an understanding important for students?

In terms of understanding the impact of technology on teaching and learning, it seems important to have a more nuanced understanding of the role of technology in supporting the teaching and learning of algebra. One step might be to ask how particular pieces of technology support particular curricular approaches. Perhaps it even makes sense to look inside pieces of technology and examine components of software, not just complete packages.

The situations of curriculum developers, teachers, and researchers of student learning, teacher knowledge, and the impact of technology all suggest that it is important to develop more nuanced descriptions of school algebra curricula. Such ways of speaking will enhance our capacity to describe new curricula, possibilities of new technological innovations for supporting such curricula, transitions that students are asked to make in the course of studying in a particular approach to school algebra, and challenges that teachers face as they aim to support such learning. In this chapter, we work toward this goal. We begin by outlining distinctions in the cognitive literature on students’ learning of algebra. We believe that these distinctions can support ways of speaking about school algebra that will move the field forward. To illustrate the potential of these distinctions, we use them in four ways. First, we use these distinctions to describe a U.S. text that might be taken as a standard or traditional approach to school algebra (the sort of approach critiqued in Fey, 1989; Thorpe, 1989). This description details what we would consider a standard, or traditional, approach to school algebra. Second, we use these distinctions to analyze ways in which technology, specifically spreadsheets and graphing technology, support particular perspectives on school algebra. Third, we use these distinctions to examine how teachers think about curricular approaches to school algebra. In particular, we examine a tension experienced by a teacher integrating the use of graphing calculators into a preexisting curricular approach. Finally, we use the distinctions to raise issues related to research on student learning. In particular, we examine how a class of students sought to use what they had learned over a 3-year period to make the transition from the solving of equations in one variable to the solving of system of equations in two variables. We hope that the attempt to use these distinctions in such a variety of settings will illustrate their utility and suggest the importance for many research endeavors of developing more nuanced ways to describe school algebra.

**DISTINCTIONS FOR DESCRIBING SCHOOL ALGEBRA**

Over the last 15 years, research from a cognitive perspective has identified tasks that cause learners of school algebra difficulty in systematic ways (reviews of this literature include Kieran, 1992; Bednarz, Kieran, & Lee, 1996; Wagner & Kieran, 1989; Leinhardt, Zaslavsky, & Stein, 1990; Wenger, 1987). In this section, we outline distinctions from that literature that seem potentially useful for describing nuances in school algebra curricula. Although the labels we use are sometimes mathematical labels, in our view these distinctions are psychological rather than logical. With all of these distinctions, there is a fuzziness of the kind that mathematical definitions are often meant to dispel.
Rather than to try to define such fuzziness away, we intend instead to emphasize that employment of these distinctions will always be a matter of interpretation.

**Different Usages of the Same Representation**

**Systems: Letters and Graphs**

Particular studies in the research literature have focused on the equal sign (e.g., Herscovics & Kieran, 1980; Kieran, 1989), literal symbols (e.g., Schoenfeld & Arcavi, 1988; Sleeman, 1984; Usiskin, 1988; Wagner, 1981; Kuchemann, 1978), and graphing (e.g., Bell & Janvier, 1981; Goldenberg, 1988; Monk & Nemirovsky, 1994; Nemirovsky, 1994; for a book length treatment, see Romberg, Fennema, & Carpenter, 1993). These studies suggest distinctions that attempt to capture nuances and differences among learners’ interpretations of representational systems.

For example, what is an equation? Although textbooks define the term, as we will see shortly, there are important differences between the strings of symbols that are often labeled as equations, differences that research suggests are experientially real to students (Usiskin, 1988, for example, suggested that equations can have different “feels” to them). Let’s focus on an equation in two variables (or perhaps unknowns), say $5x + y = 5$. This sort of equation has a particular flavor. It fits well with one textbook’s (Dolciani & Wooten, 1970/1973) definition of an equation as “a pattern for the different statements—some true, some false—which you obtain by replacing each variable by the names for the different values of the variable” (p. 44). If one replaces $x$ with 1 and $y$ with 0, one has a true statement. If one replaces $x$ with 1 and $y$ with 1, one has a false statement. When one graphs an equation like this one on the Cartesian plane, the plane is made up of points. The graph consists of those points for which the equation is a true statement. For the remaining values in the plane, the statement is false. Creating the graph is a matter of identifying members of the solution set.

Although in the textbook we quoted the word *variable* is employed, there is a subtlety here. On one hand, when focusing on solutions to equations, to the statements that are true rather than false, the $x$s and $y$s in an equation in two “variables” seem less fixed than the $x$s in an equation in one “variable.” If one is given a value for $x$ in $5x + y = 5$, one then has an equation in one variable that can be solved. One can solve for the related value for $y$, or for $x$ if one is given $y$. On the other hand, compared with the independent variable in a function of one variable (e.g., $f(x) = 5 - 5x$), $x$ and $y$ in $5x + y = 5$ have a different flavor, as does the equal sign. When the focus is solely on the values of the letters for which the equation is true, $x$ and $y$ might be thought of as of yet unknown numbers, rather than quantities that vary or indeterminate objects (Bell, 1995; Schoenfeld & Arcavi, 1988; Usiskin, 1988). The $x$s and $y$s that make the statement true, taken in pairs, constitute members of the solution set; whereas in a function of one variable with an explicit correspondence rule, values of the independent variable are substituted into a computation that determines the dependent variable. Thus, in the function of one variable truth somehow does not seem directly relevant (although of course one can force it to be).

In the context of solving $5x + y = 5$, the equal sign between two sides of the equation can be thought of in two ways. On one hand, the equal sign can be thought of as what Matz (1982) called equality as constraint—the equality of two expressions constrains the values that a letter can take. A second approach takes for granted the condition of truth. It suggests that when the statement is true, the equal sign indicates that two different expressions, $5x + y$ and 5, are names for the same number; they can be used interchangeably (Kieran, 1981, called this use of the equal sign equivalence relations).

By way of contrast, there are similar-looking equations in school algebra that have quite a different “feeling” to them (following Usiskin, 1988), for example, $y = 5 - 5x$. From one perspective, nothing has changed. We still have an equation. The solution
set to this equation is identical to the solution set to $5x + y = 5$. These are arguably different representations of the same cognitive construct (as is suggested in Crowley & Tall, 1999).

There are subtle changes to the equation and its constituents, however. Whereas $5x + y = 5$ was an implicit function—if one substituted a value for $x$, only one $y$ value would result—$y = 5 - 5x$ is explicitly a function. This form of the equation indicates that given any $x$ value, one does not need to solve for $y$, one can merely compute it. In this sense, the value of $y$ can be said to depend on the value of $x$. To capture this sense of the equation, one might use the notation $y = f(x) = 5 - 5x$, where $y$ is a function of $x$.

This notation helps capture the notion that the equation has changed. This equation no longer fits the definition “a pattern for the different statements—some true, some false—which one obtains by replacing each variable by the names for the different values of the variable” (as suggested in Dolciani & Wooten, 1973, p. 44) as nicely. Instead this “equation” feels like a correspondence rule which “assigns to each number in the domain of the variable $x$ another number, the value of $y$” (p. 146). Similarly, the letters and equal sign in the equation have a different feel (as discussed in Mason, 1989). In an attempt to capture this difference, one might now refer to $x$ and $y$ as variables rather than unknown numbers (Bell, 1995; Schoenfeld & Arcavi, 1988; Usiskin, 1988). The form of this equation allows one to keep track of how $y$ varies as $x$ varies. The equal sign no longer constrains the values of the variables. It can be thought of instead as assigning a label to the outcome of a computational process (discussed in Freudenthal, 1973; Matz, 1982). When one graphs this function on the Cartesian plane, although the graph appears to be identical to that of the previous equation, the nature of the axes and of the plane have changed. $x$ is now the independent variable and $y$ the dependent. For any value of $x$, there can only be one value of $y$; the whole plane is not active in the way it previously was. As mentioned earlier, truth seems less relevant. The points on the graph are a set, but what creates them seems less a statement that is true or false than a defining rule. As a result, the domain structures one’s view of the function; one can create the graph by systematically substituting the values of $x$ into the function and computing the $y$ values.

**Describing Functions in School Algebra**

The notion of function is a complicated one, the mathematics of which has developed in complex ways historically (e.g., Kleiner, 1989; Malik, 1980; Markovits, Eylon, & Bruckheimer, 1986). As its role in mathematics has expanded, so have calls for its use in structuring curricula (for an examination of some of this history in a U.S. context, see Cooney & Wilson, 1993). Thus, during many parts of this century, sections or a chapter on functions have been a part of introductory algebra texts. In such sections, the concept of a function is introduced as a set of ordered pairs and is primarily contrasted with the notion of relation. Related studies of the concept of function have focused on students’ and teachers’ capacity to define functions and to distinguish functions from other relations (e.g., Even, 1993; Haimes, 1996; Vinner & Dreyfus, 1989; Wilson, 1994).

Recent technologically supported approaches to introductory algebra often emphasize the use of multiple representations of functions and thus seem to articulate a different curricular role for functions in school algebra. Thus, as various representations of functions have become a part of how mathematics educators conceptualize the algebra curriculum, new layers of complexity are added to the task of developing terminology to describe school algebra. Researchers have developed a set of distinctions around the notion of function that are similar to the ones introduced earlier about letters and graphs, distinctions that once again require interpretation in their use.

For example, in describing curricula, some researchers have focused on the sorts of translations between representation systems of function that curricula ask students
to master (Janvier, 1987). Other researchers (Cooney & Wilson, 1993; Hershkovitz & Schwarz, 1997; Leinhardt, Zaslavsky, Stein, 1990) have employed the notions of concept image and concept definition (Vinner, 1991) as way to think about how curricula approach the concept of function and how teachers implement such curricula. Finally, still others focus on ways in which functions can be conceptualized both as process and as object (Dubinsky & Harel, 1992; Sfard, 1992). In such a view, accomplished performance seems to require flexibility in taking different interpretations at different stages of problem-solving processes.

Although each of these sets of distinctions seems potentially useful and revealing, we spend more time here on another distinction in the literature. Nemirovsky (1996) and Monk & Nemirovsky (1994) talked about pointwise and variational approaches to function, whereas Smith and Confrey (1994) discussed covariational and correspondence views. Building on their work, we borrow the terms explicit and recursive, usually used to discuss sequences, for rules written to describe functions from both covariational and correspondence views. This terminology allows us to distinguish the sorts of rules one might write to describe a function’s behavior based on a table of values. The two terms do not simply describe views of functions but also pose different ways of thinking about, for example, the role of the equal sign.

Mathematically, there are two ways to describe a sequence—when one view is chosen as the definition of a sequence, the other view highlights properties of this process and vice versa. For example, if the definition of the sequence 1, 4, 9, 16, . . . is based on the explicit expression \( f(n) = n^2 \), then a property of this sequence is that it has a pattern of differences that forms a linear sequence (3, 5, 7, 9, . . . in this example). The symbolic rule that describes this property is \( f(n+1) - f(n) = 2n + 1 \). This expression can be obtained easily by substituting and simplifying in the explicit expression \( [f(n+1) - f(n)] = (n+1)^2 - n^2 \). This description is a recursive description, and although it describes the same sequence as \( f(n) = n^2 \) (assuming \( f(1) \) is known), it is a very different rule. This recursive rule uses a constraint equal sign, as opposed to the “assignment” equal sign in \( f(n) = n^2 \). Furthermore, this rule has two types of variables, \( n \) and \( f(n) \). It is an equation on functions—a complicated one at that. The “solution” to this sort of equation is a function that has a property for any value of \( n \). If, instead, the defining description of a sequence is that it is formed by adding terms of a linear sequence (for example: 1, 1 + 3, 1 + 3 + 5, 1 + 3 + 5 + 7 . . .), then its explicit description will be quadratic (in the case of the example given above each term corresponds to the square of its place in the sequence). However, creating this explicit description is not easy. In most cases, creating a recursive description of a pattern by using explicit expressions for \( f(n+1) \) and \( f(n) \) is easier than constructing an explicit description of a pattern from a recursive one (which requires integration in the continuous cases or a sum of sequences). This analysis of sequences, which can be defined as functions on integers, can serve as a basis for moving on to analyze continuous functions. To symbolize that transition, we change notations to \( \Delta f \) and \( \Delta x \). These symbols, and the notion of difference equations, are usually connected to precalculus or calculus courses and are not usually part of algebra curricula.

As with the other distinctions described above, however, our interest is didactical, not mathematical. As Nemirovsky (1996) said, when used in a didactical sense, this distinction is one that is “made for analytic purposes. The two aspects are profoundly related.” (p. 296). Indeed, we argue that these two views are complementary and that even from quite early on fluency in moving from points to explicit expressions involves thinking with both explicit and recursive views. Each view provides different information about a function, and using both views together is what ultimately provides a satisfactory understanding of a function and its properties.

How do these two views of function appear in school algebra? These two views appear from the moment students are asked to move from data (in the sense of modeling
changing processes) or points or terms in a given sequence (in the sense of “guess my rule”) to an explicit expression of a function. Unless an expression is quite simple (e.g., a constant or of the form \( f(x) = mx \)), to choose an appropriate explicit expression, students work from what is at hand, from the recursive properties of the data or terms in the sequence. Stacey and MacGregor (2000a) identify this phenomena as typical with the students they studied in Australia. Thus, algebraic modeling starts by attending to how a process changes. As described by Confrey and Smith (1995) and others, students often describe a process by using the “difference” between two consecutive events (or terms in a sequence). As they suggest, differences in that sense do not mean just additive differences that then lead to rate of change as traditionally defined in calculus, but rather describe any local behavior such as multiplicative difference as well. Using the term recursive helps us describe a process of thinking about patterns and functions by taking the view of variations either multiplicative or additive. To choose an appropriate expression, however, students cannot simply identify the recursive properties of a function. Unless they know the relevant explicit description of the function, they will not have generated a model that is computationally efficient.

**DESCRIBING APPROACHES IN AN ERA OF CHANGE: A “STANDARD” U.S. APPROACH TO SCHOOL ALGEBRA**

In attempting to describe school algebra curricula, particularly as technology has come to influence curricula, educators have begun to use the kinds of distinctions we have just reviewed to outline differences between curricula. In particular, a curricula’s view of variables has been used to describe how newer approaches differ from what is often referred to as traditional or standard practice (e.g., Huntley et al., 2000; Kieran, 1997). Even though we have done this ourselves (e.g., Chazan, 1999; Yerushalmy & Gilead, 1999), we find such descriptions too rough and seek more nuanced possibilities.

In what follows, we try to characterize what we think of as a standard, or traditional, U.S. approach to school algebra, one that represents the practice against which curricular change might be identified. Although any text or set of curriculum materials is unique, we use one text to represent what we would argue is a set of practices that is common across a large number of U.S. texts. To use the distinctions we have introduced to describe this text, we suggest that there are a number of important questions to consider. There are questions of order. For example, what sorts of views of equal sign, letters, and graphs appear first in the curriculum? When do new views appear? How long does the curriculum stay with a particular view? How much does the curriculum emphasize particular views? Then there are questions of consistency and multiplicity. How many views of equal sign, letters, and graphs are entertained at a given time? How explicit is a curriculum about the presence of multiple views?

When letters are treated as variables, the equal sign as assignment, and the Cartesian plane as a space for graphing functions, then a major pedagogical question is how do curricula introduce students to functions? Should students first meet recursive or explicit views of functions? Thinking recursively is at least as intuitive and natural a way to describe the patterns. In many cases, it eliminates the “guessing” involved in finding the explicit form. And there may be ways in which technology supports thinking recursively with new representational tools. Symbolically, however, it is a difficult task to generate an explicit expression from a recursive one. On the other hand, arriving to an explicit definition from a set of points or a table is a difficult task, but once one has an explicit rule, it is easier to deduce this rule’s recursive properties. Because the two views should be acquired, then how to start and what to emphasize and for how long are crucial to the description of an approach to school algebra. Similarly, regardless of the view of letters, the equal sign, and the Cartesian coordinate
To illustrate how we might employ these questions and the distinctions described earlier, we provide examples from our reading of Dolciani and Wooten (1970/1973). Even though this text is 30 years old, in terms of its mathematical approach to introductory algebra, in our reading, it is quite similar to approaches taken in many contemporary U.S. texts as well. We also chose it for the connection of its authors to the revamping of school algebra in the United States that came out of the School Mathematics Study Group (SMSG; for discussion of this reform, see Cooney & Wilson, 1993).

In what we call a standard approach to algebra (some might call such an approach “structural” in connecting such approaches with SMSG), students meet letters before they meet graphs or tables. Indeed, tables are only emphasized at the point where students are introduced to equations in two variables and learn to construct graphs from points. Thus, for example, in Dolciani and Wooten (1970/1973), initially letters are called “variables,” a variable is defined as “a symbol which may represent any of the members of a specified set called the domain or replacement set of the variable” (p. 32). Expressions are patterns for carrying out a computation. And, an equation, as indicated earlier, is “a sentence about numbers” (p. 24) or “a pattern for the different statements—some true, some false—which you obtain by replacing each variable by the names for the different values of the variable” (p. 44).

In the second chapter, students learn to substitute values into expressions to evaluate the expression and into equations to determine whether a particular value leads to a true or false statement. In the third chapter, they are introduced to the axioms that determine the structure of algebra and that justify simplifying and multiplying expressions. It is here that the nature of students’ activity changes. Expressions are rewritten without being evaluated. In the next chapter, equations are transformed according to a set of axioms that justify the equivalence of the resulting equations. Solving equations now becomes a search for the initially unknown numbers that will create true statements. The values that create false statements are no longer of interest. Letters, although still called variables, do not take on a range of values but wait to have their identities revealed and become known. Thus, even though the authors continue to call the letters they use in equations variables, based on the literature described above, we would describe the activity of the students as working with unknown numbers.

Once students have met letters as unknowns, other than three sections on inequalities in chapter 5 where solution sets have an infinite number of members, the text stays with that point of view for a considerable length of time, until chapter 10. In chapter 10, “Functions, Relations, and Graphs,” letters as variables return. In addition, tables of values, the assignment view of the equal sign, and different uses of the Cartesian coordinate system all appear for the first time. Functions are redefined as a special kind of relation, and an arrow notation for functions reappears (both after fleeting mention in chapter 4), but examining a function in a recursive or covariational way is not introduced.

Even though multiple views of equations are present, in chapter 10 the text is not explicit about movement between different points of view. For example, the chapter moves back and forth between functions and equations. This is possible because functions are conceptualized as a special type of relation, relations that have a special characteristic, even though the nature of the equal sign is different in each case.

To illustrate, the first four sections of this chapter are about “functions and relations.” The first section focuses on “functions described by tables.” The third section introduces relations and defines functions as a special type of relation. This first part of the chapter ends with a consideration of open sentences of the form $Ax + By = C$ and specifies a method for “solving” linear equations and inequalities in two variables by isolating the variable, and thus rewriting a relation as a function.
(This is a different meaning for solving than was used earlier in the text. Earlier solving meant finding a particular initially unknown value for the single variable). Students are then asked to plot the points in the solution sets to these functions to create a graph for the original relation.

The second part of the chapter is called “linear equations and functions.” This part begins with a section titled, “The graph of a linear equation in two variables” (where the equations are still in $Ax + By = C$ form). It then moves to a section titled, “The slope-intercept form of a linear equation” (where equations are in the form $y = mx + b$, and the equal sign can be considered as assignment). The notion is that students should come to see that the graphs of these two types of relations are both lines in the Cartesian coordinate system. The third part of the chapter moves on to “Other functions and equations.” As long as one keeps in mind that a function is a particular type of relation and thus may be expressed by open sentence in two variables where one variable is isolated, then the exposition in the text is clear. But the text moves between function and equation quite quickly and does not address differences between the use of the equal sign or the Cartesian coordinate system when working with functions and relations.

Having introduced equations in two variables, the next chapter in the text returns to the theme of solving, this time solving systems of two equations in two variables. Students are taught three solution methods, a graphical one that uses the Cartesian coordinate system to represent the solution sets of equations in two letters, a substitution method (for the types of transitions between views that such a method necessitates, see Chazan & Yerushalmy, in press), and a method involving linear combinations. Chapter 12 enlarges the number system to include irrational numbers, and then this enlarged number system is used in chapter 13 to solve quadratic equations and inequalities. The final chapter of the text begins an introduction to geometry and trigonometry.

Thus, it is fair to suggest that this text emphasizes an unknown point of view of letters over a variable point of view, an equal sign as constraint or equivalence relation view over equal sign as assignment, strings of letters over tables and graphs, explicit descriptions of function over recursive ones, and more (what Kieran, 1997, might describe this as an emphasis on algebra without functions). Those emphases don’t capture the whole story, however. Detailed examination also described the order in which these views were introduced, how new views were added to existing ones, and that the text provided little explicit scaffolding for students to negotiate changes in point of view.

UNDERSTANDING THE INSTRUCTIONAL AFFORDANCES OF SPREADSHEETS AND GRAPHING CALCULATORS IN AN ERA OF CHANGE

Elsewhere we have argued (Yerushalmy, 1999) that technological tools are not neutral with respect to the different conceptions of function and of algebraic symbol representation systems outlined earlier. As a result, it does not seem fruitful to discuss the capacity of technological tools to support algebra instruction in general. Instead, it seems more useful to describe how particular technological tools, or perhaps even specific parts of a particular tool, have the capacity to contribute to particular curricular approaches. Doing so, once again, involves us with the distinctions outlined on pp. 727–731 of this chapter.

To raise questions about relationships between technology and curricular approaches to school algebra, we discuss in this section spreadsheets and graphing calculators, two technological tools that are touted for algebra instruction. Neither spreadsheets (e.g., Excel) nor graphing calculators (or software with similar capacities) were created initially with the teaching of school algebra in mind. Instead, they were designed to support educated users doing mathematics, users already familiar with
the multiplicity of meanings of the representational systems used in school algebra. Spreadsheets and graphing calculators are both tools that support users in making observations about relations between quantities, given either with algebraic symbols or developed from the user’s own mathematical ideas. Both tools support multiple representations of functions (Heid et al., 1995). Both tools aim to reduce the cognitive load of interaction with some aspects of mathematical symbolism. Both tools value learning from examples in various linked representations (graphs and tables).

In our view, however, these two kinds of technological tools are different with respect to some of the distinctions outlined earlier. As a result, they can support different curricula and the same curriculum in different ways. We think it is important to analyze relationships between technological tools and curricular approaches before integrating tools not explicitly designed for educational purposes into teaching practice (for a similar analysis, see Philipp, Martin, & Richgels, 1993.)

With these issues in mind, we start the section by outlining how these two tools support beginning algebra students asked to solve an algebra word problem that involves an equation in one variable. Stepping back from this description, we end the section by raising questions about assessing the strengths and weaknesses of these tools in supporting approaches to school algebra.

**Solving a Word Problem Involving a Linear Equation in One Variable With Spreadsheets and Graphing Calculators**

Filloy, Rojano and Rubio (2000), Sutherland and Balacheff (1999), and Rojano (1996) described the solution process with a spreadsheet of a word problem. The problem asks students to compute the length and width of a rectangular field, if its perimeter 102 meters and its length is double its width. As Rojano (1996) suggested in her conclusions, students who are beginners at algebra do not necessarily use “algebraic methods” to solve problems of this kind. Quite often, they use reverse arithmetic operations, “undoing,” or using what Rojano describes as a “whole/part” method (they compute the measure of each “part” where the whole is made in this problem of six parts). For students who have not yet had any formal algebra studies, however, all three research studies suggest that spreadsheets can support an “algebraic” strategy. By this they mean that students use spreadsheet “formulas” to write an explicit rule for the perimeter of the rectangle based on a particular cell representing the width and specified by its location. As Sutherland and Balacheff (1999) indicated, the problem “was crafted to provoke pupils to use a spreadsheet cell to represent an unknown quantity and to build relationships with reference to the unknown” (p. 11). In the given problem, the notion is that students will begin with a cell containing a value for the width and build cells in other columns, eventually building a cell with information on the perimeter as a function of the value in the original “width” cell. Students will then vary the values of the width by typing values into the original cell or by incrementing its values down a column. Regardless, the task is to search for values of a width cell that generate a perimeter of 102 meters.

The correct width value is unknown. In that sense, students using the spreadsheet in this way are working analytically, as opposed to synthetically, by reasoning with an unknown to make it known. Yet this sort of work is different from traditional algebraic methods to solve for an unknown. In fact, in the spreadsheet table, there is no direct representation of the unknown or of the equation itself. One brings this to the tool. Perhaps the closest to a representation of the equation that one can get with a spreadsheet is making a column that is always 102 meters, and then creating another column to compare with the 102-meter column to compute the perimeter based on an initial width value. The formula of the comparison column would be something of the following form: = cell in the computed perimeter column = cell in the always-102-meters column. That “formula” (with its two equal signs) would create outputs
of either true or false. Other ways would include creating another column that would be the difference between the computed perimeter and 102. The user would then look for a value of 0 in this column. Or the user could create a column that would be the ratio of the computed perimeter and 102. The user would look for a 1 in this column. Regardless of the lengths to which the user goes to identify the solution set of the equation, it is clear that these solution strategies are not based on operating on both sides of an equal sign in sanctioned ways. Instead, they are based on successive computations while observing input and output relationships, a systematic “trial & error” method.

Describing the types of conceptions present in this type of spreadsheet work is complicated. When students are working with symbols representing locations in the spreadsheet table, these symbols are neither unknowns nor variables. They represent particular locations and in that sense seem too particular to be variables, although of course the values in cells to which they refer can change; the cells to which they refer either do or do not have values; when they do, it seems strange to call them unknowns. Yet somehow the reasoning involved with the columns suggests notions of variables (the cells in a column seem something like the points along an axis representing a quantity), and the task that was posed asked students to find an unknown. Copying rules down the spreadsheet’s columns makes use of the tool’s capacity to carry out recursive operations, yet the formula students must develop to compute one column from another is an explicit function rule on these variables.

To complete our view of this sort of solving with a spreadsheet, it is worthwhile thinking about how students would approach the task if the perimeter of the field were supposed to be 100 (rather than 102). In many descriptions of students’ work with spreadsheets (including the one in Rojano, 1996), the input is increased by a unit step, thus creating a sequence of integer inputs. This might be supported by an important resource for students provided by the tool. It might be that a tendency to view the entries in a column as terms of a sequence is supported because each row is labeled by an ordinal number. This makes us wonder whether a noninteger solution would make it more difficult for students to find a solution for the given perimeter.\(^1\)

Rojano and Sutherland’s study of this problem was part of a longer sequence of learning with spreadsheets that we cannot address here. Instead, we would like to conjecture about posing this task to beginning algebra learners with a graphing calculator. Using capacities of the calculator, a user could insert ordered pairs created from the problem description—such as (10,60), (20,120), (30,180), (40,240)—and use a regression line to find an expression that might fit this “data.” Perhaps as described in Hershkovitz and Kieran (2001) algebra beginners would choose this option in specific circumstances.

However, beginning students who are successful in solving the task would most likely start by typing in a function rule (e.g., \(y_1 = 2x + 4x\)). Such a rule describes all possible perimeters as a function of the width, \(x\). Students might then graph this function and search for an \(x\) for which the value of the function is 102, perhaps by tracing the graph and generating values. It is possible, although less likely, that students would start with an analysis of tables of values of the function generated automatically from the expression. Still another option is to enter two functions (rather than just the one that computes the perimeter from given widths): \(y_1 = 6x\) and \(y_2 = 102\). The \(x\) value of the intersection point of the two functions’ graphs would indicate the \(x\) value that solves the width value.

Stepping back from the particular problem, the graphing calculator here supports a view of \(x\) as variable, even though the problem wants a particular \(x\) value as the

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\(^1\)Given known difficulties in having students accept non-integer values as solutions, this question is a distressing question.
solution to the equation. As with the spreadsheet, when using a graphing calculator, there is no explicit representation of the equation, but rather expressions for two functions. Whatever functions one graphs, it is necessary to formulate an algebraic equation without the support of the tool. The tool's support is in freeing the student from the need to manipulate the equation to find a solution.

Some concerns must be considered with beginning learners, however. Researchers, such as Stacey and MacGregor (2000b) and Herscovics and Linchevski (1994), have reported that beginning algebra students tend to avoid algebraic expressions in their problem solving. In such cases, it seems unlikely that the graphing calculator, with its need for an explicit rule as a starting point, would support beginning learners of algebra.

Based on the way in which spreadsheets seem to support beginning learners, what if one wanted to be able to use graphs with a spreadsheet, just like in the graphing calculator? To make use of the graphing capacities of a spreadsheet, one must choose appropriate input and output columns (there are usually more than two present because students insert intermediate computation columns on their way to the final column). Then one must choose from among a series of graphic representations, if one wants a continuous graph that represents dependency. It is most likely that a teacher would have to make these choices for the students.

**Using These Tools in School Algebra: Some Considerations**

The cursory discussion of the solving of a linear equation given above suggests that there is much in common between graphing calculators and spreadsheets when it comes to the solving of equations. To the degree that these tools support “algebraic thinking,” it is by virtue of the writing of expressions and then by supporting students in finding unknown values, rather than finding solutions by operations on both sides. This example also suggests differences between the support that each tool provides.

Spreadsheets offer syntax for writing recursive and explicit rules without forcing issues of generality. One clicks on cells or writes the cell’s location into a formula for another cell. Such formulas can be generalized to relationships between other cells without the user writing a general rule. These rules can be either explicit correspondence rules for a function or recursive rules for generating a sequence. The capacity of spreadsheets to support the writing of recursive rules, in particular, enables users to model phenomenon that might be understood recursively but that are difficult to describe with explicit correspondence rules, such as exponential growth or diffusion. The graphical representation then supports analysis of the generated data and perhaps even the generation of explicit correspondence rules to describe a relationship between columns. These capacities can be important contributions to an approach to school algebra.

Graphing calculators primarily treat $x$ as an independent variable, $y$ as a dependent variable, the equal sign as indication of assignment, and the Cartesian plane as a space for graphing functions. More recently, some models have begun to support the use of letters as parameters, as well as other uses of the Cartesian plane. The links between explicit function rules and graphs and tables that graphing calculators offer powerful visual feedback on symbolic manipulations, both manipulations involving a single expression and those involving equations in one variable.

Perhaps as a result of the differences between graphing calculators and spreadsheets, these tools seem to be used quite differently by school algebra curricula.

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2The menu terms and icons used to choose graph type are sometimes difficult to interpret and are not necessarily in line with the appearance of Cartesian graphs in algebra textbooks.
Examination of technologically supported school algebra curricula suggests that these tools are used differently depending on a curriculum’s approach to algebra. If curricular focus is on developing understanding of linked expressions, graphs, and tables to expect, explain and conjecture about symbolic manipulations and about qualities of expressions, about the nature of solutions and equations, then graphing calculators are frequently used (Heid et al., 1995). When the focus is on using mathematics as a powerful means of modeling and using mathematics to solve problems across a range of subjects outside of mathematics, then spreadsheets are often used.

Of course, many technologically supported approaches to school algebra try to reach both of these goals. The question of commonalities and differences between the tools then becomes an important one. Attention to the distinctions outlined earlier suggests that if one intends to make the best use of these two sorts of tools, then the order of introduction of the tools is not merely a matter of technical facility with each tool. There are important pedagogical and conceptual nuances with regard to the functionality of the tool that, together with the curriculum, suggest particular courses of action. When considering the timing of the introduction of these tools into a curriculum, it is useful to use the distinctions introduced on pp. 727–731 to consider what aspects of the tool will be familiar and unfamiliar to students.

First, spreadsheets seem to be consistent in their use of letters as coordinates for specifying columns in a table. Does the spreadsheet’s use of letter and number pairs as coordinates in specifying locations support students in learning to use letters as variables, unknowns, and parameters? What kinds of tasks with the tool might help students learn these perspectives on letters, learn to distinguish them one from another, and learn to employ these views flexibly? Does it matter what students have learned before they use spreadsheets? Are there conflicts that arise if students are used to viewing letters as unknowns that do not arise when students are used to viewing letters as variables or vice versa? Unlike its use of letters, the spreadsheet supports both the equal sign as assignment and as constraint. Similarly, the spreadsheet syntax for writing rules can be used to write both explicit and recursive rules; one cannot distinguish the nature of a rule by its syntax alone. And finally, spreadsheets support a number of different uses of the Cartesian plane. How might the tool be used to learn to distinguish these different views of the equal sign, these different kinds of rules, and these different uses of the Cartesian plane, as well as to learn to use each of them and to employ them flexibly as needed? Again, does it matter what sorts of experiences students have prior to their use of spreadsheets? These seem to be important questions for research.

Second, with respect to graphing calculators, can this tool be used by learners who have just begun to use letters, the equal sign, tables, and the Cartesian plane, or is the tool only for more advanced users? If these tools are to be used by learners new to these representations, does the tool support learning of the representations? Or can activities be constructed around the tool that would support such learning? Does it make sense for someone who has not learned about graphs of functions to use this tool? With respect to students who have already begun their study, does it matter what view of letters, for example, students have met before they use the calculator? For example, is the calculator easier to use for students who have not yet met letters as unknown? Are there important considerations related to the sequence of experiences a learner has with this tool?

These also seem like important questions for consideration as we try to carry out research on the capacities of technology for supporting instruction in algebra. It seems likely that tools will differ in their capacity to support different approaches to school algebra, and once again, it seems important to have nuanced ways of describing different approaches to school algebra and differences in the affordances provided by technological tools.
As reformers envision curricular changes as well as changes to the nature of classroom interaction (e.g., National Council of Teachers of Mathematics [NCTM], 1991, 2000), researchers have asked what teachers need to know to teach in such ways (e.g., Ball, 1992). As a result of the nature of some of the changes proposed for the algebra curriculum, this general question takes on a unique twist in algebra. As teachers consider using technology with their algebra students, they make many decisions. What technology should I use? How does it support what I want to do with my students, what my text already does? What sorts of tensions might arise as a result from the use of this technology? Responding to these sorts of questions requires that teachers use their understandings of the approaches that they wish to take with respect to school algebra or that their text materials are taking. As a result, the increasing availability of technology and of curricula taking new approaches to school algebra have spawned a raft of studies of teacher thinking and knowledge, particularly focused on both preservice and inservice teachers’ thinking about functions (e.g., Huntley et al., 2000, Chazan, 1999; Even, 1993; Haimes, 1996; Lloyd & Wilson, 1998; Williams, 1998; Wilson, 1994).

There are larger questions to explore here as well. Do teachers believe that there are alternative approaches to school algebra? If there are alternative approaches, how do they characterize differences in approach? What terminology do teachers have for describing differences? In their view, what does technology offer? Does it support alternative approaches or simply offer novel solution strategies? Do they feel that there are tensions in their instruction as they try to carry out technologically supported curricular change? If so, how do they conceptualize these tensions? Do teachers feel that such tensions can—or should—be addressed explicitly with students?

In this section, we illustrate how the distinctions we have introduced might be used to understand how teachers conceptualize approaches to school algebra. We use our analysis of a standard U.S. approach to school algebra and of graphing calculators as background to the examination of one intern teacher (see Chazan, Larriva, & Sandow, 1999, for further details). This preservice teacher was in a unique position. Like many who were preparing to teach in the late 1990s, this future teacher did not use graphing calculators during her high school years. Her apprenticeship of observation was in a standard approach to school algebra as depicted earlier. When she arrived at the university, graphing calculators were an important part of the landscape, particularly in the lower division courses. As a teaching assistant from her sophomore year on, she became versed in the graphing calculator. Her observations about the use of graphing technology with her students are a result of her efforts to integrate such technology into the instruction offered in a high school using a textbook similar to the one analyzed above.

We interviewed this teacher about her conceptions of school algebra and about her classroom instruction. During the section of the interview on classroom instruction, she identified some benefits of the use of graphing techniques with her students, but she also raised a number of concerns. She described tensions that arose for her as she introduced graphical techniques for solving equations into a standard approach to school algebra. In describing these tensions, she initially came face to face with different uses of the Cartesian coordinate system in graphing. Subsequently, these differences in use led her to feel that her instruction did not differentiate sufficiently between solving an equation in one variable and solving a system of equations in
two variables. We suggest that the sort of thinking in which she was engaged is an important kind of pedagogical reasoning that involves a kind of understanding of mathematics that is important in teaching (for a more elaborate argument, see Yerushalmy, Leikin, & Chazan, 2001).

When describing her concerns about graphical techniques, this preservice teacher indicated that she became aware of a conflict between different aspects of her teaching. When asked about defining what an equation is and solving an equation, she said:

that’s how I define an equation, is that it has an equal sign in it. . . .To solve an equation means you find the unknown value that makes both sides of your equation equal, that’s what solving an equation is . . . You’re looking for the unknown value . . . that makes both sides of the equation true. (Chazan et al., 1999, p. 197)

But when she taught students to graph expressions as functions, either when they were present singly or when she taught students to graphs the expressions on both sides of an equation, she was using a different view of the letter; it was a variable rather than an unknown:

I guess it’s thinking about a variable in a really different way . . . x is an unknown, but when you’re graphing, you’re thinking about y and x, all possible solutions, kind of different concepts. I wonder if that makes it confusing at all for students? (p. 197).

What she did not address is whether she discussed this tension about the nature of letters in algebra with her students explicitly.

There was another tension as well that was more difficult for her to articulate clearly. This tension is not unique to her practice. In our view, it results from attempts to integrate the graphing of equations conceptualized as comparisons of functions into the standard curriculum without adjusting the curriculum’s approach to school algebra. One way to articulate this tension is to say that the graphing calculator in its standard mode graphs functions of one variable in the Cartesian plane, not equations, whereas most standard texts use the Cartesian plane for equations (of all kinds) in two variables. (Of course, one can reduce this tension by suggesting that the functions in one variable that one can graph with a graphing calculator can be regarded as equations in two variables.)

To illustrate, let’s start with a system of equations, say $6x + 8y = 650$ and $x + y = 100$. When graphing such a system with most calculators, one has to solve for $y$; one cannot graph these equations as they are, but one can graph $y = (650 - 6x)/8$ and $y = 100 - x$. However, the text would present graphs for $6x + 8y = 650$ and $x + y = 100$ (even if, as in the Dolciani and Wooten text referred to previously, students were taught initially to solve for $y$ when graphing such equations). These would appear to be the same lines that are generated on the calculator (but as we suggested earlier, they are not; the nature of the coordinate systems in these two cases differ). Regardless of whether one is working with the text or using the graphing calculator, the solution to the system of equations is related to the intersection point of these two lines; numerically, it consists of the values of both the $x$ and $y$ coordinates of the intersection point. These coordinates are the values of $x$ and $y$ that will make both equations simultaneously true.

By contrast, for the text, an equation in one variable, like $3x + 7 = 2(x + 5) + x - 1$, is never graphed; graphical techniques are irrelevant. Yet this teacher was introducing just such a graphical technique to her students. To graph such an equation with a graphing calculator, she was suggesting that each expression be graphed separately ($y = 3x + 7$ and $y = 2(x + 5) + x - 1$). Given that for this preservice teacher, however, an equation is anything with an equal sign, when entering these two strings into the calculator, she now had two equations and a coordinate system with two lines
graphed on it. From her perspective, it looked like “I have a system of equations.” This “system” seemed indistinguishable from the graph of a “true” system of equations, but in this case the solution to the equation is the x-coordinate of the intersection point of the two lines. This number is the number that will make the equation a true statement. In her experience, this made it difficult to explain to students what a solution for a system of linear equations in two variables, as opposed to an equation in one variable, is:

[With an equation in one variable, as opposed to a system], what is really my solution? If you’re having students graph, then why is y not part of your solution? Well, because you’re looking for your unknown x. You have to think back to the original problem. (Chazan et al., 1999, p. 2–196)

Somehow, simply telling them to pay attention to what sort of problem it was did not seem sufficient to her. Why is the solution to $3x + 7 = 2(x + 5) + x - 1$ different from the solution to $y = 3x + 7$ and $y = 2(x + 5) + x - 1$? Why do the “and” between the two equations in a system and the “=” between the two expressions in an equation indicate that there is a difference in the nature of the solution set to be sought?

For us, these questions are solid questions for which the distinctions we have introduced are useful. Beyond the explicit presence of two variables in one kind of problem and one variable in another, the Cartesian coordinate system is being used differently when graphing solution sets in two dimensions and when graphing functions of one variable. When graphing the solution set to $6x + 8y = 650$ on the Cartesian plane, for each point on the plane the question true or false can be answered. Similarly, when a system of equations is graphed on such a plane, again the question of true or false can be addressed for each point. But, when graphing two functions to find inputs for which they two produce the same output, the question of true or false can only be asked once for each value of the domain.

These questions indicate this preservice teacher’s sensitivity to the dilemmas of instruction that come as teachers try to chart a course through this terrain; others might perhaps have let this issue slide. These questions indicate the kinds of complexity that can arise as teachers and curriculum developers try to change and incorporate curricular ideas and technological innovations that support a range of approaches to school algebra. But, at the same time, they suggest to us that techniques, as we argued before in relation to software, are not neutral. Techniques presuppose the sorts of conceptions that the distinctions in the literature aim to capture. Thus, a part of the tension was that this teacher was trying to introduce a technique that the approach she was taking did not support well. In fact, during the interview, she subsequently described her approach to school algebra as “that [pointing to her definition of equation given up above] with a little of this [pointing to a depiction of graphing both sides of an equation].”

**RESEARCH ON STUDENT LEARNING IN AN ERA OF CHANGE: TRANSITIONS BETWEEN CONCEPTIONS IN GENERALIZING FROM EQUATIONS IN ONE VARIABLE TO EQUATIONS IN TWO**

We have used the distinctions developed early in the chapter to describe a standard U.S. approach to school algebra and to describe the capacities relevant to school algebra that spreadsheets and graphing calculators afford. We then used these distinctions to consider the types of knowledge a teacher brings to bear in making pedagogical decisions. In this section, we move to issues of research on student learning.
As mathematics educators debate what school algebra in a technological age might be, what is the role of research in influencing curricular decisions? During such a time of instability, it seems that research can play an important role, but the challenges for researchers are many. How do researchers study a subject matter that is unstable and changing (or one that is changing as rapidly and dramatically as school algebra)? What are the implications of curricular change for research on student learning? Studies of alternative approaches to school algebra reveal new dimensions to students’ understandings and misunderstandings (e.g., Confrey & Smith, 1995; Kieran et al., 1996; Moschkovich, Schoenfeld, & Arcavi, 1993; Yerushalmy, 1997a, 1997b, 1997c, 1997d). As curricular approaches change, what is essential and how does one identify this? How does one create tasks that allow for research across different approaches that can illuminate the impact of curricular changes? How does one conceptualize results across different approaches to school algebra? Is it useful to examine the nature of particular points of transition for students, categorizing them as either “epistemological obstacles,” in the sense of being difficult points without qualification, or “didactical obstacles,” obstacles for one particular approach but not for others? Finally, if changes in technology can have an impact on what is done in school algebra, how does a research agenda keep pace with innovations in technology without being driven solely by the evolution in technology?

To examine these issues of research on student learning raised by technologically supported curricular change in a concrete manner, we present an examination of a class working in a particular approach to school algebra. Using the distinctions developed in the second section, we begin by describing the approach to school algebra taught to these students. We then describe the work of three groups from this 9th-grade class on a task involving a system of equations that asked them to move from their knowledge of equations in one variable to equations in two variables. As students began to think about systems of equations in two variables, the teacher focused their attention on representing a single equation in two variables. This subtask challenged them to change their perspectives on the role of $y$ in an equation and to reexamine tables of values and the Cartesian plane as tools for solving equations. As we present their work, we use some of the same distinctions again to explore why this task was so challenging for them; we try to understand how the task asked them to make a transition from the perspectives they had learned already to new perspectives. The question that our examination raises is the nature and magnitude of the transition that this class was making. Do students in this class experience the transition from equations of one variable to equations of two variables differently as a result of the approach to school algebra that was used in their class?

Some Notes on Visual Math

The learning we describe here took place in a 9th-grade class in ISRAEL. Students in this class were in their 3rd year of a functions-based, technology-intensive algebra curriculum (Visual Math, 1994/1995). In this “function-based” (in the sense of Chazan & Yerushalmy, in press) curricular sequence, the dominant conception of letters in introductory algebra is as representations of quantities that vary, either independently ($x$) or dependently ($y$ or $f(x)$) and that taken together define a Cartesian plane. The

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3We have described the dominance of a variable conception of letters in this curriculum, and how the conception of unknowns flows out of the view of variable. In this curriculum, students also experience other perspectives on letters. For example, they also use literal symbols as parameters to describe the behavior of families of functions. Standard examples of such activities include the notion of changing one coefficient in a quadratic function to understand the role of that coefficient (e.g., Heid et al., 1995, p. 1).
explicit rule for calculating the dependent variable from the independent one can be graphed in the plane. Solving equations in one variable is conceptualized as a particular kind of comparison of two functions. With such an approach (as described in the fourth section of this chapter, understanding the instructional affordances of spreadsheets), there are graphical, nonalgebraic, means of keeping track of equivalent equations in one variable (see Fig. 28.1). This representation allows students to explore questions of equivalence as they learn to manipulate equations algebraically (studies of this representation include Yerushalmy & Gafni, 1992; Yerushalmy & Gilead, 1997). If the solution of an equation is the perpendicular projection of the intersection point down onto the independent axis, then the number of intersection points in a new equation must be the same as the number of intersection points in the original equation, and these intersection points must come in pairs that share the same x-coordinate.

In general, an important goal of this curriculum is to help students develop strong algebra skills and to learn to do a variety of standard algebraic manipulations, including the solving of systems of equations in two variables. But the curriculum is aimed at helping students learn to do such manipulations with an understanding of the graphical and tabular meanings of these manipulations, as well as a sense of the purposes for which such manipulations are useful.4 Such proficiency involves moving across the various views of symbols, graphs, and functions that we described early in this chapter. Thus, an important goal for instruction aimed at producing people proficient in algebra is to help students learn to shift their point of view. This goal is an ambitious one.

Footnote:
4Examination of student learning with the curriculum include Yerushalmy (2001), Yerushalmy (1997b), and Yerushalmy and Shternberg (2001).
To illustrate the ambitiousness of this goal and how it plays out with respect to the transition from equations in one variable to equations in two, we return briefly to the question raised by the preservice teacher in the previous section. Why do the “and” between two equations in a system and the “=” between the two expressions in an equation indicate that there is a difference in the nature of the solution set to be sought? In the context of the Visual Math curriculum, how do students who have learned to solve equations in one variable by noting the x-coordinates of the intersection points of the graphs of expressions learn to understand how solving systems of equations in two variables is both similar and different?

The set of strategic decisions made with respect to this question by the developers of the Visual Math curriculum involves a complex evolution in students’ understandings. If an equation in one variable is a comparison of two functions each of one variable, then an equation in two variables is a comparison of two functions of two variables. The same way the expressions that form an equation in one variable can be graphed in a two-dimensional plane, the expressions that form an equation in two variables can be graphed in a three-dimensional space. This space allows for the coordinated representation of the shared domain of the equation’s two functions of two variables, as well as their outputs.

The same way that the solution set of an equation in one variable can be indicated on a one-dimensional number line, the solution set of an equation in two variables can be indicated on a two-dimensional plane that represents the shared domain of its functions of two variables. In this context, there is an important difference between the “=” in $f(x) = g(x)$ and the “and” linking two equations in a system, even though functions in one variable and the solution sets to equations in two variables can both be represented on the Cartesian plane. The “=” indicates a comparison of two functions. A judgement of true or false is made for each element in their shared domain. By contrast, with the system of equations, the “and” links equations of two variables, comparisons of two functions of two variables that have already generated a set of true and false judgements for each element of their domain (the whole of the Cartesian plane). The “and” indicates the desire to find members of the shared domain for which the judgements with respect to each equation both yield “true.” Of course, this story is complicated by two factors. First, $3x + 7 = 2(x + 5) + x - 1$ and the system $y = 3x + 7$ and $y = 2(x + 5) + x - 1$ seem quite similar. Second, even if one views these as tasks with different solution sets, nonetheless students also need to appreciate that both tasks involve the instruction “solve” and generate a “solution set” as an answer.

Moving to Two Variables: Changing Perspective on Y

The lion’s share of the early parts of the Visual Math algebra curriculum focuses on functions of one variable and equations of one variable. The teacher of the class we examine wanted to assess what his students had learned about algebra in one variable. He wanted to see whether students had internalized differences between function and equation of one variable, an understanding of operations taken on each, and a disposition to use tabular and graphical representations as tools for obtaining feedback on symbolic manipulations. He wondered whether his students could solve a task involving systems of equations in two variables without being taught a method for solving such systems? Would his students understand the task? Could his students work with their understandings of functions, equations, and solution sets in one variable in order to come to grips with functions, equations, and solution sets in two variables? Could they generalize their graphical and tabular representations to deal with the new

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5This discussion of “and” also holds for combining solutions to equations of one variable.
complexity? Did they understand the purposes, capabilities, and structure of these representations enough to modify them to fit the new circumstances? Would they be interested in discussing such issues? How would they argue about the commonalities and differences between this task and tasks they had worked on in the past?

With this in mind, he approached students with a system of two linear equations in two variables \[x + y = 2x - y,\ 2x + 1 = 3x + 3y\] and asked student to think about a solution of this system. From the beginning of his discussion with students, it became clear that his question was too ambitious. Students ignored the second equation; they were astounded by the appearance of a type of equation that they had not seen before, an equation with two variables. The teacher, attentive to their reaction, revised the question for the class to explore. He asked them to focus on the following question: "How would you describe the equation \(x + y = 2x - y\)?" We illustrate the nature of work that occurred in this class by outlining and analyzing the attempts of three groups of students to tackle this question (there were five other groups).

**Group A: Searching for Consistency While Generalizing.** One group of students decided to view both \(x\) and \(y\) as independent variables, even though that meant that \(y\) would be different from the \(y\)s with which they were used to working. They described this change in \(y\) by saying that it used to be the output variable and now was free to be an input variable. Following this discussion, in their group, they saw their task as generalizing what they had learned for a single variable to two variables. They started by observing that \(x + y\) "looked linear." To investigate this conjecture, they wanted to construct a table from which they could make a graph. They noted that with two independent variables, they would need more than just two columns in their table: "How are we going to write it? We have three numbers! How should we organize it in a table? And how are we going to graph it? For each value of \(x\) we have got infinite number of \(y\)s."

Their discussions of the issue of graphing started from the notion of dimension when drawing cubes and ended up with the notion of three-dimensional graphs. They struggled with their technical capacity to draw two-dimensional images of three-dimensional graphs. They wondered how to include values that were negative; negative directions seemed to be "covered" by traces of other dimensions. Knowing that they would need to present their ideas to the rest of the class, they then began to use a range of tools to create three-dimensional spaces. They started with their fingers. This proved awkward, and they then tried to hold their pens at 90° angles to a sheet of paper as a way to represent their idea. Finally, they developed a concrete model of wooden sticks brought from a storage nearby and attached with rubber bands (Fig. 28.2). Within the space outlined by this model, they used a sheet of paper to represent the \(x + y\) plane.

Although this group was inventive in its work, during the time allotted, they worked only on representing one function of two variables; they did not develop a strategy for comparing two functions of two variables. We wonder what they might have done with more time. Following the methods of graphical solutions they had been taught, would they view \(x + y\) and \(2x - y\) as two surfaces? Would they make an analogy between the intersection of the two planes in the space and the intersection

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6 In this section, we represent 2 days of work in this class. Our stories come from 8 days in a larger unit that culminated in an examination of the solution of systems of two equations in two variables. The data we report below comes from the report of a researcher who the teacher invited to video- and audio-tape class discussions and group work during this time period.

7 Although this choice might seem surprising for students learning in other approaches, we have seen such generalizations in previous studies of students taught this curriculum (Yerushalmy, 1997a; Yerushalmy & Bohr, 1995).
points with equations in one variable, even thought the solution set to the equation in two variables is a set with an infinite number of elements? Similarly, would they make an analogy between the projection of the intersection points down onto the independent axis and projection of the intersection in three dimensions down onto the x–y domain plane? Would they have been able to do all of this if supported by technology that graphs functions in two variables as surfaces in three dimensions (Fig. 28.3)?

Of course, all of this would not yet provide them with ways of solving the system of two equations in two variables that was originally posed. They would still need to represent the solution set of each of the two equations in the system by a line in the x–y plane and appreciate that the solution set to the system is the coordinated values $x$ and $y$ of the intersection points of the two solution sets graphed in the plane. Thus,

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8In Yerushalmy and Bohr (1995), there is some evidence to suggest that this might happen.
their generalization strategy directly builds on what they have already learned about equations of one variable. If it is done carefully, there is a direct analogy that can be built step by step. Following this analogy is quite demanding, however; it requires changing their views of tabular and graphical representations, as well as the semantics of symbols in equations and systems of equations.

**Group B: Interpreting Symbols to Stay Within the Known.** Another group developed a different way of thinking about expressions in \( x \) and \( y \). Using their knowledge of parameters to describe families of functions, they decided to think of \( x \) as an independent variable and of \( y \) as a parameter that could take on a range of values. For every value of \( y \), a new linear function in \( x \) would be generated.

We started by looking for a function's graph. The first function is \( x + y \). But if we look at specific \( y \) lets say \( y = 1 \) then \( x \) is a variable, too, so we definitely have a problem with two variables. It is a function with a constant slope 1, but it involves another variable \( y \).

To represent this way of thinking, they sketched a series of parallel lines with slope 1 to represent a series of functions in \( x \) of the form \( f(x) = x + a \) (seemingly without realizing that such a graph would fill the whole plane because \( y \) is a continuous quantity). They then moved on to represent the left-hand side of the equation by sketching a set of parallel lines with slope of 2 to represent functions of the form \( g(x) = 2x - a \) (Fig. 28.4).

With this idea, they had figured out a way to represent each side of the equation in two variables as a family of selected functions. However, this left them with the
question of where on their representation of the two sides of the equation was one to read the solution set to the equation. What would be the analogue to the $x$ value of the intersection point of two functions of one variable? Thinking about their method, it is quite complex. Taking the pair of functions of one variable generated by one value of $a$, one has an equation in one variable Fig. 28.5. The $x$ values of the intersection points are the $x$ values of members of the solution set to the equation in two variables, but even though the group labeled the axes of their plane as $x$ and $y$, the other coordinate of these points is not relevant; the $y$ values of the solution set of the equation in two variables are the values of $a$, not the output values of the functions of one variable ($f(x) = x + a$ and $g(x) = 2x - a$); see Fig. 28.6.

Their strategy also has other wrinkles. With their strategy, they would also have to make sense of a solution to an equation in two variables as a conjunction of the solution sets of infinitely many equations in one variable involving paired members of the two families of functions. Then, to fully understand the representation they used, they would need to come to grips with the dual roles that $y$ was fulfilling for them. On one hand, they used the label $y$ for the output of their functions of one variable; on the other hand, it was a parameter that they called $a$ that was incremented for each pair of functions of one variable.

In sum, what initially seemed like a promising strategy turned out to be quite complex. This group tried to figure out a way to reduce its complexity, to turn an equation of two variables into an equation of one variable and a parameter, but this turned out to be subtler than they expected. In terms of the mathematical ideas involved, this strategy seems even more complex than the generalization approach taken by the first group.
FIG. 28.5. Cartesian plane as a space of functions of one variable: For $a = 3$ the point $(6,9)$ is the intersection point of $f(x) = x + a$ and $g(x) = 2x - a$, that is, the intersection of $x + 3$ and $2x - 3$.

FIG. 28.6. Cartesian plane as a space of solutions: On the graph of $y = x/2$, the point $(6,3)$ indicates the solution to the equation $x + y = 2x - y$ when $y = 3$.

**Group C: Manipulating Symbols to Return to the Known.** Another student group also approached the question by thinking of the equation as a comparison of two functions: $f(x,y) = x + y$ and $g(x,y) = 2x - y$. But from that point, building on their work with equations in one variable, they moved to look at the difference function. By using the difference function and symbolic manipulations, they reduced
the question of finding the solution of an equation in two variables, to a question about the zeros of a single function in two variables. When is the output of \(-x + 2y\) equal to 0? In justifying their actions, members of the group seemed to be generalizing from what it meant to find zeros for an equation in one variable. In the end, they changed \(-x + 2y = 0\) to \(y = 1/2x\). In their words,

We are familiar with a linear function \(ax - b\). We often use another way to describe it [perhaps the solution of \(ax - b = 0\): \(ax = b\). Here we have \(x - 2y\) [they switched from \(-x + 2y\) at this point]. So [in the two variable case \(x - 2y = 0\)] we also have a linear function and we can draw a line through any two points generate by this dependency rule. And we have \(x = 2y\).

Thus, they moved from an equation in two variables to a question about a function of two variables, and from there with another manipulation to a recognized function of one variable.

By doing this, they seemed to be doing what Crowley and Tall (1999) called proceptual thinking and what is taught in the text we reviewed in Dolciani and Wooten (1970/1973). They were able to see the linear relation \(Ax + By = C\) and the function \(y = mx + b\) as two facets of the same relationship, a relationship that in either form can be described by the same line, but this conventional result required that these students make a number of transitions from what they were taught. The beginning of their work was like what they would have done with an equation in one variable. They wrote a difference function and were able to reduce the discussion to just one function \(-x + 2y\) instead of the two that were given.

The struggle apparent in their explanations started from this point. Although they had not been taught how to isolate one variable in a relation involving two independent variables, they manipulated \(x - 2y = 0\) to \(x = 2y\). This step seems to involve two crucial ways of thinking. First, they did not treat the equal sign here as indicating assignment; they imagined that it was the sort of equality that would allow operations on both sides. Given that context, they allowed themselves to add \(2y\) to both sides, or to move \(2y\) from one side to another, undeterred by the presence of \(x\) as well. We are not sure exactly how they justified this action; perhaps they were simply acting on an analogy with manipulations of equations in one variable. After dividing both sides by 2, they decided to change their view of the resulting object and then to treat the equal sign as assignment and \(y\) as the output variable. Thus, without being taught how to isolate a variable, they did so. What remains unclear at this stage is how they are thinking of this “function.” Are they really thinking of \(y = f(x)\), where \(y\) depends on \(x\)? If so, what is the solution set of the original equation? Or are they thinking of the whole graph of this “function” with both of its \(x\)- and \(y\)-coordinates as the solution set to the original equation? For us, this raises important questions. When should students be taught to isolate a variable? How does a curriculum distinguish between \(y = x/2\) as a relation and as a function? Or should it?

Understanding Transitions: Research on Curricula and Student Learning

Historically, the allure of modeling with algebraic symbols was that such models might lead to equations that could be solved mechanically. There was a simplicity to the solving of equations. As a result, Leibniz hoped that his work as a diplomat would become unnecessary when a universal language of calculation would allow disputes to be reduced to equations whose mathematical solutions would also be diplomatic solutions agreeable to all parties (see, e.g., Kline, 1953, especially pp. 239–240). Mathematics education research has helped us to understand how complicated the symbol systems of algebra are, how much complexity lives beneath the
surface of seemingly straightforward techniques and uses of symbols. Yet much of the power of algebra derives from the ways in which it ignores all of these subtleties. Experienced users of the algebra taught in school move comfortably between different uses of the same representational systems and a variety of mathematical objects. A key question for mathematics educators is how to help students develop such flexibility and comfort and at the same time acquire knowledge of key conceptual distinctions (Sfard & Linchevski, 1994). One way to interpret the development of different curricular approaches is that they are instantiations of particular notions about how to arrive at this goal, that curricula represent strategic decisions of developers. This perspective challenges researchers on students' learning of school algebra to identify and study such strategic decisions and their impact on student learning across different approaches to algebra.

We will close this section by exploring whether the Visual Math approach to school algebra changes the transition for students between equations in one variable and equations in two variables. We now examine two components related to this transition and speculate about the role of the curriculum in helping students learn to move flexibly between different conceptions.

Moving Flexibly Between Representations of Processes and Solutions. Thinking back what we called a "standard" approach to the school algebra curriculum, movement from equations of one variable to equations in two variables asks students to make transitions in their ways of thinking. If students have learned to solve equations in one unknown, they are used to solution sets whose elements are numbers, rather than ordered pairs. Solution sets to equations in two variables are ordered pairs, but there is another difference as well. Solution sets in two variables are less constrained. When the set of the real numbers is the replacement set for the variables, although the solution to a linear equation is often a single value, the solution to a linear equation in two variables is always an infinite number of ordered pairs. Related to this difference, although the solution sets to equations of one variable are represented on a number line, the solution sets to equations in two variables are represented in the plane.

This focus on representing solution sets is complicated by the technological tools we have for representing the processes on either side of the equal sign in an equation. If, as discussed previously, graphing of functions of one variable is introduced in a standard approach as a method for solving equations in one variable and equations in two variables are also graphed on the Cartesian plane, then there are a series of questions that arise when teachers introduce the task of solving systems of equations. There is a tension between the use of the Cartesian plane for representing solutions and for representing "processes" or covariational thinking.

An alternative is for a curriculum to focus, at least initially, on representations of process as opposed to representations of solution sets (as Visual Math does). This is feasible as long as one works with functions of one variable and equations of one variable. With equations of one variable, one can present students with representations that provide information on processes and at the same time allow students to identify solution sets. If a curriculum tries to maintain a consistent focus on process, however, the move to equations of two variables is problematic. Because of the limitations of most of our tools for representing comparisons of processes resulting from two independent variables, it is more difficult to attend in this case to both processes and solutions. As a result, there is a strong curricular need to help students appreciate the Cartesian plane as a representation of a solution set and a need for students to move flexibly between this usage and preceding usage. Of course, this need could be obviated in other ways. As technological tools with better three-dimensional representation systems become available, there may be less
pressure at this point, but such tools will not address equations in more than two variables.

How does one assess the relative difficulties of the transition from using the Cartesian plane to represent solution sets to its use for representing functions, as opposed to the transition from using the Cartesian plane to represent functions to its use in representing solution sets? And how does one interpret such an assessment against the backdrop of the value of using representations of process to help students understand equivalency between equations and equivalency of expressions? These seem like important questions for research on student learning of school algebra.

**Moving Flexibly Between Equations in Two Variables and Functions in One.**

When initially learning to graph an equation in two variables or when solving systems of two equations in two variables by substitution (see Chazan & Yerushalmy, in press, for an analysis of this method for solving this task), one begins by isolating a variable. By doing so, one has written an equation in two variables as a function in one, as was done by the students in group C. This raises a host of questions. How does doing the same operations to both sides of an equation, a seemingly mechanical manipulation, lead to such a change? How do learners come to appreciate that equations that look quite similar and may even be thought of as representing the same “relationship” (in an “ordered-pairs” [Crowley & Tall, 1999] sense) may have quite different qualities? These questions seem challenging no matter what approach one takes. For example, Crowley and Tall (1999) showed that although a curriculum with a “standard” approach such as that described previously expected students to treat $Ax + By = C$ and the related $f(x) = mx + b$ as representations of the same mathematical relationship, students did not do so. Similarly, in the context of graphs, the text we reviewed previously (Dolciani & Wooten, 1970/1973), and others like it, present the use of the Cartesian plane for functions of one variable and equations in two variables as the same. In critiquing such approaches, Bell, Berkke, and Swan (1987) pointed out difficulties that result from not distinguishing between such uses of the plane.

The challenge of moving flexibly between equations in two variables and functions in one has a different complexion for an approach in which equations are conceptualized as comparisons of two functions than for an approach in which functions are a special type of equation. For example, if equations in two variables are conceptualized as comparisons of two functions of two variables, then they are not graphed on the same Cartesian plane as functions of one variable. Such an approach seems to put a greater distance between students’ conceptions of functions of one variable and equations in two. Does this make it more difficult for students to move flexibly between these two? On the other hand, does this choice help students come to understand the difference between the solution to a system of two equations in two unknowns and the solution to an equation in one variable? Are there ways to compare such issues across different approaches to school algebra? These questions illustrate the challenges that face research on students’ learning of school algebra.

**REFLECTIONS ON FLUX IN THE SCHOOL ALGEBRA CURRICULUM**

Over the last two decades, the school algebra curriculum has become a site for innovation. Technological advances have helped curricular developers imagine that it might be possible to implement redesigned school algebra curricula. Some curriculum developers have been particularly drawn to a curriculum in which algebraic expressions are conceptualized as representations of functions. Approaches to school algebra predicated on such notions may still maintain some goals that are similar to
standard approaches. Goals may include having students learn to factor and multiply some range of polynomial expressions and to solve linear and quadratic equations. Nonetheless, such approaches may represent dramatic change, change to the order of introduction of material, the length of time students work with particular interpretations of symbols systems, and the explicitness with which the curriculum discusses these interpretations.

Changes in the school algebra curriculum pose potentially fascinating challenges to curriculum developers, software designers, teachers, students, and researchers interested in the impact of technology, teacher knowledge, and student learning. What sort of approaches to school algebra facilitate both the development of conceptual understanding and skilled performance? How can technology support curricular approaches and enhance their effectiveness? How do teachers come to understand the potential of new approaches supported by technology and use them appropriately? Then there are important research questions. In what ways does technology support a variety of approaches to school algebra? What sorts of knowledge do teachers have of the school algebra curriculum and what sorts of knowledge are necessary for faithful implementations of particular approach to the subject? And, of course, what sorts of student learning result?

It is an exciting time for research in the teaching and learning of school algebra. In this review, we have suggested that advances in all of these areas require a more nuanced language for describing differences in approaches to school algebra. We have suggested, and hopefully illustrated, that distinctions developed in the cognitive research literature on the learning of school algebra are a useful starting point.

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CHAPTER 29

Advanced Technology and Learning Environments: Their Relationships Within the Arithmetic Problem-Solving Domain

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The advent of the microcomputer in the early 1980s brought with it high expectations regarding this tool’s potential to drive change and innovation in schools. Although a number of projects have produced significant results at a research level, it is nevertheless true that these expectations appear to have remained largely unfulfilled (see Andrews, 1999; Becker, 1993; Bottino & Furinghetti, 1998; Pelgrum, 1996). Indeed, it would seem that computer use has had a limited impact on schooling throughout the world (Pelgrum & Plomp, 1993). One of the main reasons for this (disregarding factors related to hardware availability and management and to the traditional resistance of both school systems and teachers to change) is that technology has often been introduced as an addition to an existing, unchanged classroom setting (De Corte, 1996).

Often, the introduction of information and communication technologies (ICT) into education has been linked to a vision of learning as an individual process whereby knowledge emerges from the interaction between the student and the computer. This vision is borne out by the terminology frequently adopted in the literature, in which educational software applications are often referred to as learning environments, thus focusing attention on the fact that it is the software itself, through interaction with the student, that is to form the environment where learning can be developed.

In this chapter, we analyze the relationship between advanced learning technologies and learning environments that arise from a different perspective. In adopting
the term learning environment, we consider the teaching and learning situation as a whole. In other words, we are interested in analyzing teaching and learning processes that happen within activity-rich, interaction-rich, and culturally rich social environments, which the intelligent use of technology is making possible (De Figueiredo, 1999).

In this framework, ICT have an important role as artifacts mediating teaching and learning processes (as Mariotti has clearly pointed out in chapter 27), but they do not embody the entire learning environment. In the following, we briefly analyze the main aspects of evolution in educational computing research that have led to greater consideration for the learning environment as a whole. We refer to activity theory and, in particular, to the work of Cole and Engeström (1991), and we analyze the main aspects of a methodology that has been derived from this theory.

The activity theory framework offers an appropriate tool to instantiate the main relationships that characterize a learning environment. Practical examples of this instantiation are described, making reference to a project involving the design, implementation, and evaluation of an ICT-based system, the ARI-LAB system (Bottino & Chiappini, 1995). This system has been created for the development of arithmetic problem-solving capabilities with students in compulsory schooling. Hence, this project is reported here as an example of a practical application of the analysis methodology we have adopted to study the relationships between advanced technology and learning environment.

EDUCATIONAL COMPUTING RESEARCH IN MATHEMATICS: A SHIFT IN FOCUS FROM THE INDIVIDUAL TO LEARNING ENVIRONMENTS

Research on ICT-based mathematics learning and instruction has undergone a deep transformation due in part to the parallel evolution of pedagogical and cognitive science theories. A set of ideas and principles has been produced that has substantially changed orientations, at least at the research level, regarding the design and use educational software.

One of the major forces driving change has been the assumption that meanings are lost if learning is simply the transmission of information. This approach is suitably expressed in key words used by many authors to describe and frame their own work; expressions such as "learner-centered systems" and "problem-based learning" are becoming more and more frequent in the literature (see, also, Norman & Spohrer, 1996). At the heart of these researchers’ work is the idea that students learn best when engrossed in a topic and are motivated to seek out new knowledge and skills because they need them to solve the problem at hand. Hence, learning is viewed as based on an active exploration and personal construction rather than on a transmissive model.

At first, these research projects mainly focused on the design and implementation of software tools based on the new opportunities increasingly offered by technology. The attention was on individual behavior, and the objective was to design and analyze learning situations in which knowledge could emerge from interaction between the student and the computer environment.

The designs of educational software have been accompanied by in-depth experimentation with the implemented software. Analysis of this experimentation and the results achieved has helped to shed light on the fact that, by itself, technology does not lead to an educational change; that is, technology itself does not have the power to give greater meaning to the educational activity (Sinko & Lehtinen, 1999). The
pedagogical significance of a tool cannot be defined by taking into consideration only its characteristics, but rather by considering aspects that are external to the tool itself (Salomon, 1996).

Many research studies reveal that it is pointless from a pedagogical point of view to make computers available at school if the educational strategies and activities the students engage in are not suitably revised (De Corte, 1996). This observation arises from analysis of how ICT is normally used in current practice. Often, a technological tool is used for educational purposes on the assumption that somehow or other it will lead to an educational improvement simply because the tool itself is considered to be “good.” Seen in this light, technological tools are appreciated if they are rich in features or have a pleasant interface; no regard is made as to whether the tool in question is conceptually complex, whether it entails lengthy training before it can be used effectively, or how the teacher’s role or teaching methods and contents need to be redefined to accommodate its use in the classroom (Noss, 1995). This simplistic approach usually generates initial enthusiasm for a system, followed by disillusionment. The problem is that software environments are often evaluated on the basis of general, ill-defined expectations, resulting in a lack of understanding about the conditions under which the educational use of such tools might be meaningful.

In recent years, this issue has represented a major topic for discussion in the debate that researchers have been conducting in the domain of educational computing. As far as mathematics education is concerned, the work performed by Pea (1987) demonstrated that the value of a software tool for mathematics learning does not depend solely on its inner characteristics but also on the activity that is developed through its mediation in the context of use. In his work, Pea began to consider the context of use as an integral part of the design and implementation process of educational software.

Making reference to Pea (1987), we note that not only do technological artifacts influence and transform the activities performed with their mediation but also that the results of these activities can deeply influence the technology used. This is particularly true at the present time, when technological progress is constantly opening up new opportunities (for elaboration, representation, communication, etc.), the potential in the educational field of which has yet to be fully explored. Technology is a determining factor of the learning environment because of the influences it exerts on cognitive, motivational, and social aspects of the activity performed by the user with this technology. Moreover, it affects the possible interpretations of this activity.

These interpretations change over time according to the way in which technology is actually used in social practice: On one hand, this use can prefigure new functions to be included in the technology; on the other, these new functions can change the models of practice that have inspired the construction of the technology itself (Pea, 1987). Consequently, there is a dialectic relationship between technology and learning environment, one that has to be considered in its becoming. Technology continuously undergoes changes as a consequence of the needs emerging from its contexts of use and, at the same time, it changes the aims and the objectives of mathematical education because it contributes to modifying the structure of learning environments.

Technology design and use are thus being progressively considered in relation to the whole teaching and learning process and not merely with the development of specific abilities or the accomplishment of particular tasks.

For example, Bellamy (1996) reveals how technologies must be designed to support not only students’ learning activities but also teachers’ activities, because it is only by understanding and designing for the whole education situation that effective and valuable changes can be brought about in the classroom. In this way, the organizational and management aspects of technology-mediated activity are also taken
into consideration. Increasingly, technology is being studied in relation to long-term teaching and learning processes of the kind needed for the development of complex, multifaceted skills (e.g., arithmetic problem solving, conjectures and proofs in geometry, etc.).

For the development of such abilities, the student–software unit of analysis is not sufficient because it is necessary to consider the whole set of interactions established in a class over the course of time. As the matter of fact, the mediation offered by a given software to cognition is not sufficient to explain the learning aspects related to motivation, to goals formation, and to the attribution of a meaning to the whole activity that goes beyond the meaning of the single actions involved in the performance of a task.

This concern for the environmental aspects is not confined to research in educational computing but is also being expressed in areas such as social ergonomics and human computer interaction (HCI). Studies being conducted in these fields consider the environment in which a given tool is used as an integral part of the tool itself (see, for example, the works of Brown, 1986; Bevan & Macleod, 1994; Suchman, 1987; and Norman & Draper, 1986).

But this realization about the importance of the learning environment brings with it many difficult questions. What are the aspects that characterize a learning environment? If the individual is no longer to represent the unit of analysis, what is to take its place? What are the relationships between tools, individuals, and the social groups that individuals belong to?

A learning environment can be described as a composite of constituent factors: physical setting, set of agreed behaviors, consensually held expectations and understandings, particular tasks, around prespecified contents for explicitly stated goals that are guided by a person who has been given the responsibility over that setting, its participants, and activities. In other words a learning environment is first and foremost a system that consist of interrelated components that jointly affect learning in interaction with (but separately from) relevant and cultural differences. (Salomon, 1996)

Attention to the learning environment is bringing about a shift of focus in the analysis of the changes that take place in classroom practice that are due to technological innovation. The changes in the individuals’ learning are in fact a part of a larger change, that of the learning environment.

The study of how the changes in students’ learning are connected to the changes distributed over the whole learning environment as a consequence of the use of technology appears nowadays a necessity for research. How is it possible to study changes that occur in the learning environment with the introduction of ICT?

Learning environments are not presented to individuals and thus directly observable by an analyst; rather, they are constructed by individuals in the activity. Such constructions are deeply interwoven with historical achievements as well as values and norms.

As observed by Salomon (1996), few researchers have actually followed this approach until now, even though a number of them have recognized the need to do so. In actual fact, it is difficult to shift the focus from the individual to the concept of learning environments as a whole. A number of reasons might be mentioned to explain this difficulty. Disregarding those resulting from researchers’ professional background, which often lies in the field of cognitive psychology, the main obstacle appears to be the lack of a viable theory on which to base the simultaneous study of individual and environmental changes within the same conceptual framework.

Focusing the analysis on the learning environment, it is necessary to study how different aspects of a learning environment interrelate, and how these relations change over time as a consequence of the introduction of technology and all that this entails.
We think that the relationship between technology and learning environment must be studied from a historical–cultural perspective. This perspective seems appropriate for analyzing the transformations that, in the course of time, affect technology and the teaching and learning activities that are carried out with its mediation. The theoretical reference we have adopted for analyzing the relationship between advanced technology and learning environments is that of activity theory.

Activity theory is a philosophical and cross-disciplinary theory for studying different forms of human practice, such as teaching–learning practice, as development processes mediated by artifacts, in which individual and social levels are interlinked at the same time (Kuutti, 1996). Activity theory (Leont’ev, 1978; Engeström, 1987) gives us a framework, namely, terms and notions associated with those terms, that are useful for describing the interactions emerging in the learning environment as a result of ICT integration. Activity theory is concerned with the historical development of activity and the mediating role of artifacts within it.

In activity theory, an activity is a form of acting directed toward an object, and it is the object that distinguishes one activity from another. Transforming the object into an outcome motivates the existence of an activity. Activities consist of actions or chains of actions, which in turn consist of operations. If we consider activity theory as applied to the educational field, the object of an activity is the learning of a given knowledge or the development of a given ability; the outcome of this activity, the motive for which the activity is developed, is students’ acquisition of that knowledge or that ability (Bellamy, 1996). Previously we evidenced that individual learning cannot be understood without considering the learning environment in which it takes place. Using the framework of activity theory, we can state that the learning environment is constituted by the enactment of a teaching–learning activity oriented toward an object involving students, teacher, and artifacts. Studying the learning environment means studying the teaching–learning activity oriented to a didactical objective. In other words, studying the changes that learning environments undertake as consequence of the introduction of a new artifact means analyzing how activity changes and how this change is meaningful for the students and the teacher.

Cole and Engeström (1991) devised a model to formulate the complex relationships between elements in an activity (see Fig. 29.1) that is particularly appropriate to study the relationships that take place in the teaching–learning activity (see also Engeström, 1987, 1991). Their systemic model highlights three mutual relationships involved in every activity, namely, the relationship between subject and object, that between subject and community, and that between community and object. Each of these relationships is mediated by a third entity. The relationship between subject and
the object is mediated by artifacts that both enable and constrain the subject’s action. The relationship between subject and community is mediated by rules (explicit or implicit norms, conventions, and social interactions), whereas that between community and object is mediated by the division of labor (different roles characterizing labor organization). The model depicted in Fig. 29.1 also reveals that each entity mediates all the relationships described in the model.

Artifacts used in the activity mediate not only the relationship between the subject and the object but also that between subject and community and that between community and object. Moreover, mediating entities are not mutually independent but exert influence over one another. For example, the introduction of a new artifact in an activity influences both the norms regulating participant interaction in the activity and the roles that the participants can assume.

In the following, we refer to Cole and Engestrom’s model to study the complex relationships between advanced technology and learning environment in the field of the arithmetic problem solving. According to this model, it is first of all necessary to clarify the relationship that in our work has been established between the object of the activity and the ICT-based artifact used in the problem-solving activity. This issue is discussed in the next section.

AN ICT-BASED SYSTEM FOR ARITHMETIC PROBLEM SOLVING

In this section, we present the framework adopted for analyzing both the nature of knowledge involved in arithmetic problem solving and the conditions that may foster the construction of skills needed for mastering that knowledge. This framework has guided the design and classroom experimentation of the ARI-LAB system, which is briefly outlined below. We then describe how the system was tested.

The Theoretical Framework Underpinning System Design

The relationship between natural language and symbolic arithmetic writing is a complex one; it affects the way thought is organized and takes shape in problem-solving activities. Arzarello, Bazzini, and Chiappini (1994) showed that it is possible to display a double register in problem solving, even for the simplest of arithmetic problems.

On one hand, there is oral arithmetic and its uttered numbers, in which the flow of reasoning is based on natural language and on the potentialities offered by a numeration system for building a problem-solving strategy. On the other, there is written arithmetic, with its symbolic language that embodies some of the features of algebraic language (e.g., ideographic and synthesis functions); these can be drawn on in the thought process for expressing or interpreting the arithmetic solution to a problem in the form of a written number sentence.

The goal of the arithmetic school is the development of students’ capacity to master this double register in arithmetic problem solving (Carraher, Carraher, & Schliemann, 1987). Chevallard (1989) noted that, historically, arithmetic know-how is essentially an oral know-how. In particular, he noted that the signs we now improperly call arithmetic (+, −, *, :) were originally introduced into algebra in the 16th century and later adopted in arithmetic as a quick and concise way of expressing the solution process. Until that time, humanity had used natural language for solving arithmetic problems, enriching it with what might be called “a language of numbers,” that is, one enriched with the potentialities provided by a given numbering system.

Chevallard (1989) observed that the solution of an arithmetic problem can be seen as discourse, a discourse that must be uttered in a single breath because arithmetic
Problem
Six bottles of fruit juice cost 27,000 lire. How much do 14 bottles cost? How much do 17 bottles cost?
The girl’s answer:
Six bottles cost 27,000 lire. Now, 27,000 plus 27,000 makes 54,000, and that is 12 bottles. To find out how much two bottles cost, this is what I do: I divide the banknotes making up 27,000 lire into three groups and I find that they can be divided into groups of 9000 lire. So 12 bottles cost 54,000. I add 9,000 and that makes 63,000. To find out what 17 bottles cost, I work it out this way: I know that 14 bottles cost 63,000 so I add 9,000, which is two bottles, and so I get 16 bottles, and that is 72,000. But I have to get 17 bottles, so I need to divide 9,000 in two parts, and that is 4,500. So I add the 4,500 to 72,000 and I get 76,500.

FIG. 29.2.

know-how is intrinsically an oral skill. To gain a clearer idea of what Chevallard meant by oral arithmetic, the solution given in Fig. 29.2 is in the form of a discourse. The discourse conveys the validation of a thought process that is supported by the problem’s concrete situation on the basis of shared common sense. Natural language and the Italian currency system (which the child had previously used in concrete problem-solving tasks of progressively increasing difficulty) were the instruments used for the problem-solving activity. The calculation and solution processes are inextricably intertwined within the discourse.

The child solved the problem by relaying each step to the concrete situation, which allowed her to keep a constant check on the meaningfulness of her actions. In other words, the concrete situation allowed the child to validate her reasoning, or the way she has used the functionalities offered by the numeric system at hand in the concrete situation.

We note that natural language permits a good semantic control; “oral” solutions make this control transparent throughout the problem-solving activity. The same is not true of arithmetic symbolic language used for expressing or interpreting the solution to a problem by means of written number sentences.

When arithmetic signs are introduced and used in the solution of arithmetic problems, a distinction is drawn between the solution process and the way in which the process is codified into a product to be expounded through a written number sentence.

As a matter of fact, as pointed out by Carey (1991), a number sentence may directly model the actions performed in problem solution or may represent an arithmetic solution. The relationship between the oral and written registers in arithmetic is by no means always straightforward. For example, in additive and (to a lesser extent) multiplicative problems, natural language is far richer and more sensitive to the different semantics of context than is formal arithmetic language. Often, synthesis demanding the adoption of arithmetic formulas forces students to transform completely the structure of their individual oral arithmetic processes. This is seen, for example, in the transformations required for formalizing an additive completion solution strategy by means of an \( a - b = c \)-type formula.

In this regard, an important reference point is provided by Vergnaud’s studies into the classification of additive and multiplicative problems and the procedures, concepts, and representations involved in solving them (Vergnaud, 1983, 1991). Vergnaud’s work helps us to understand the evolution of students’ skills and conceptions in the process of arithmetic knowledge construction and sheds light on the types of obstacles that must be overcome to master the operational invariants (theorems in action) needed for tackling problems of a certain class. Vergnaud also provided a useful reference point for explaining the problems that may arise in the transition from oral to written arithmetic. Indeed, many researchers have drawn on his studies when examining the strategies adopted in “street math” compared with those commonly developed and learned at school (see, for example, Carraher, 1988; Schliemann, 1995).
These studies highlight that street arithmetic, based on oral competencies, and school arithmetic, based on written competencies, are not one and same. Using Vergnaud’s terminology, they show that, although the invariant underlying street arithmetic and school arithmetic are the same (i.e., they refer to the same property, that of associativity for problems of additive structure), there are differences in the way subjects represent numbers through situations and solve problems. The major difference is in the symbols used in and out of school because the different sets of situations in the two contexts define concepts of different extension.

Hence, the study of the role played by context has great implications for understanding the cognitive development that takes place within it. Context does not influence but essentially determines the kinds of knowledge constructed (Lave, 1988).

The studies of arithmetic sociogenesis conducted by Saxe (1992) show that the cognitive developmental progression in the construction of arithmetic problem-solving strategies cannot be understood in isolation from culture-specific symbolic forms and practices:

The problem arises as to which references to use for designing and running meaningful classroom activities that will lead to the development of new understanding in the arithmetic field. In this direction, studies into cognitive development in arithmetic problem solving offer general guidelines for tackling the learning problem set. These guidelines demonstrate the importance of linking school mathematics with everyday mathematics and are motivated by the need to begin from what the child already knows. Nevertheless, we note that everyday situations also pose limits that often hinder the exploration of new facets of mathematical knowledge that are not part of everyday situations but are vital to students’ mathematics education (Schliemann, 1995).

To approach this didactically crucial aspect in our research work, we have used the notion of “field of experience” developed by Boero in (Boero, Dapueto, Ferrari, Ferrero, Garuti, Parenti, & Scali, 1995). A field of experience is a sector of human culture that the teacher and student can recognize and consider as unitary and homogeneous (examples of which are the fields of experience of “purchase and sales” and “calendar”; in the long run, arithmetic, too, may become a field of experience). In studying teaching–learning problems related to a given field of experience, consideration must be given to the complex relationships that develop at school between the student’s inner context (experience, mental representations, procedures concerning the field of experience), the teacher’s inner context, and the external context (signs, objects, objective constraints specific to the field of experience). At the core of didactical practice based on fields of experience is the evolution of the student’s inner context, fostered by activities organized and guided by the teacher.

A remarkable aspect of activities referring to real world fields of experience is the possibility of developing processes of social construction of knowledge in the classroom, because students’ inner context and the teacher’s inner context may enter into immediate resonance on topics referring to common experience. All this may also bring about a favorable climate for productive discussion about mathematical strategies and objects involved in those activities, preparing the ground for discussion in the mathematical field of experience. (Boero et al., 1995)

From the didactical viewpoint, the field-of-experience notion allows us to anchor the general reference to everyday situations to the need to design learning situations in homogeneous, unitary cultural fields, ones that are meaningful for both the student and the teacher. The main goal of didactics based on fields of experience is to master systematically the field in which work is being done and to make explicit the mathematical knowledge built within the activities performed in that field.

The field-of-experience notion poses the need to analyze the potential and limits of the mathematical tools commonly used in the particular field concerned. The aim is to understand the type of mathematical knowledge brought into play by practice
in that field. With the evolution of this type of didactics, the mathematical knowledge gained will be useful in other fields, whether mathematical or not.

The field of experience helps to build the sense of mathematical concepts, procedures, and strategies needed to master the cultural aspects that characterize the field itself, fostering evolution in the students’ existing preconceptions. The student’s awareness of these tools is the reference point for subsequent evolution of activities in mathematical fields of experience, and the explication of the tools’ limits motivates the need for this evolution. The purpose of that evolution is to develop mathematical skills that are more advanced than those that can be developed solely within the field.

As part of our studies into the role that ICT can play in the development of mathematical problem-solving skills, we have designed and tested the ARI-LAB system, which is based on the framework outlined in this section. In accordance with this framework, the system was designed to provide students and teachers with the following tools:

- Microworlds that model the resources and constraints of real-world and arithmetic fields of experience by means of computational objects. The student can interact with these objects producing effects and receiving feedback that can be interpreted as mathematical phenomena (within the context of the fields of experience).
- Tools that make it possible to reify the solution process enacted within the microworlds, transforming it into an object for use in the dialogues and communication exchanges held between the teachers and students and among the students themselves. This allows reflection about the processes undergone.
- Communication support tools that foster exchange, allow comparison of processes, and results in problem solving and that permit these activities to be introduced into an educationally effective social interaction mechanism.

In the following section, we briefly outline the main characteristics of the ARI-LAB system, developed along the lines described. We then use activity theory methods to justify the changes that system use have brought about in the structure of the learning environment and to explain how these mediated the students’ learning process.

A Brief Introduction to the ARI-LAB System

ARI-LAB is a system that combines hypermedia and network communication technologies. It integrates tools of different natures to support didactics based on fields of experience in the domain of arithmetic problem solving. The commercially available version of ARI-LAB presented here (ARI-LAB, 1999) is the result of major transformations made to the initial prototype as a result of the experimentation conducted.

In ARI-LAB, two different kinds of user are expected: the student, who is to solve a given arithmetic problem, and the teacher, who is to assist the student and plan and structure the educational activity. Problem solving is carried out by exploiting the action, representation and communication possibilities made available by the different tools that have been integrated in the system. These are tools to develop the solution process (microworlds), a tool to describe and present the solution product (solution sheet), a tool allowing communication between users, and a tool to store all the actions the user performed in the microworlds and in the solution sheet while solving a problem (monitoring). An environment for designing individual learning activities is also available to the teacher.

**Microworlds.** Microworlds are mediating tools for the construction of the solution process. Within microworlds, the user can create and manipulate computational objects to develop the solution strategy for a problem. The microworlds currently
available in ARI-LAB are called *coins, calendar, abacus, number line, number building, simplified spreadsheet*, and *histogram*.

The first two microworlds have been designed to model two real-world fields of experience: “purchase and sales” and “measurement of time related to the calendar.” The other microworlds have been designed to model different aspects of the field of experience of arithmetic.

During problem solving, the microworlds allow the user to manipulate computational objects and interact with them using operational tools specific to each microworld. While interacting with these computational objects, the user receives various kinds of feedback that may foster the emergence of goals for problem solution and the construction of meaning for the strategies developed. Moreover, some microworlds make available functions for the validation of the rules used by the user in the activity. Because space limitations, an in-depth description of all the microworlds is not possible here. Some of their characteristics will be presented during discussion of examples of ARI-LAB use.

**Solution Sheet.** The solution sheet is where the solution to a problem is developed as a product to be shared with others (teachers, other students, etc). In the solution sheet, users build their solution to a problem by copying into this space the graphic representations produced in the microworlds that they consider meaningful for working toward and sharing the solution. The user employs verbal language and arithmetic symbolism to comment on the graphical representations copied and thus to explain the solution performed.

**Monitoring.** ARI-LAB features a tool that automatically records anything a pupil has done while solving a problem (every action and operation performed in the microworlds). This monitoring function makes it possible, at any time, to view the whole sequence of steps performed as a sort of movie. The tool transforms the resolution process undertaken in the microworlds into an object that can be used in the activity for different didactical aims.

**The Communication Tool.** While solving a problem with ARI-LAB, it is possible at any moment to access the communication tool, which allows the user to establish a connection with another user and share messages via a local network. The other user may be either a classmate or the teacher. If the two participants are solving the same problem, it is also possible for either of them to send the other their solution and the related monitoring. In this way, the receiver not only sees his or her classmate’s solution to the problem, but also how that solution was reached.

It is worth noting that a solution (or solutions) received from classmates cannot be copied into the solution sheet. Access is granted for the sake of analysis but to use received solutions in the solution sheet, the user has to reconstruct them through interaction with the microworlds.

**Teacher’s Environment.** The teacher is granted access to an environment that allows him or her to configure the system according to the specific needs of the students involved, for example to define and impart to the students a set of arithmetic problems to solve, to choose which microworlds should be accessible to the students, and to make available or hide the functions for validation of the counting processes. Hence, the system interface can be changed in accordance with the teacher’s options.

Figure 29.3 shows the solution sheet produced by a user tackling a given problem. On the left-hand side, there are buttons that allow access to the microworlds and the communication tool. In Fig. 29.4 some of the main interfaces of some microworlds are presented.
FIG. 29.3.

Mary received three-thousand, five hundred lire from her auntie as a birthday present. She also got five-thousand, seven hundred lire from her grandmother. With all the money she received, can Mary buy a T-shirt that costs eleven-thousand, five hundred lire?

This is the money that Mariella received from her auntie: three thousand, five hundred lire.

This is the money that Mariella received from her grandmother: five thousand, seven hundred lire.

In total Mariella received nine-thousand, two hundred lire.

FIG. 29.4.
### TABLE 29.1
Classes Involved in the Experimentation of ARI-LAB

<table>
<thead>
<tr>
<th>Type of Class</th>
<th>Duration of the Experiment</th>
<th>Frequency of Computer Sessions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary school class</td>
<td>6 months (Grade 2)</td>
<td>2 hours/week</td>
</tr>
<tr>
<td>Primary school class</td>
<td>4 years (from Grade 2 to Grade 5)</td>
<td>2 hours/week</td>
</tr>
<tr>
<td>Primary school class</td>
<td>4 months (Grade 3)</td>
<td>2 hours/week</td>
</tr>
<tr>
<td>(deaf children)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower secondary school class (only with low achievers in math)</td>
<td>4 months (Grade 6)</td>
<td>2 hours/week</td>
</tr>
<tr>
<td>Lower secondary school class (only with low achievers in math)</td>
<td>6 months (Grade 7)</td>
<td>2 hours/week</td>
</tr>
</tbody>
</table>

*Average number of pupils in each class: 12 (both girls and boys).*

For each problem, it is possible to access the entire dialogue conducted by the user and other students while solving the same problem (messaggi ricevuti [messages received] button). Moreover, at any point in the activity it is possible to select the monitoring function (“monitor”) to see the whole sequence of actions the user has performed while solving that problem. It is also possible to access the solutions to the same problem received from other users (soluzioni ricevute [solutions received]) and the related monitorings (monitor ricevuti [monitorings received]). In this way, the whole individual and social activity conducted in ARI-LAB for solving a problem is incorporated and stored in the system.

**Experimentation with the System**

Over the past 7 years, the ARI-LAB system has been experimented with widely and exhaustively. Some information on the experiments performed is briefly reported in Table 29.1. As we can see, the system has been tested in different school situations and with different kinds of students (different school levels, normal students, deaf students, students who are considered as low achievers in mathematics). All the experiments were developed in the long term. With one class, the system was used for almost the whole cycle of primary school (from the second to the fifth grade), with significant integration into the mathematics curriculum. During experiments, one computer was available for each student.

The experiments were carried out by the authors. During all experiments, the teachers of the classes involved were present and participated actively both in planning the teaching itineraries and in following the students’ work on the computer. Each experiment was accomplished in a real class situation and during normal class hours, not in an ad hoc laboratory setting.

Evaluation of the experiments had been based on analysis of the written observation protocols we took during the work with the different classes, analysis of the problem solutions devised by the students and saved in the system, analysis of the written dialogues held via the communication environment, and analysis of the monitoring of students’ solution processes automatically stored by the system.

The experiments focused on the establishment and development of both the problem-solving strategies and mathematical concepts and skills through medium-to long-term class activities centred around the fields of experience supported by the ARI-LAB microworlds. One field of experience that had particular importance
in the work of the various classes involved was that of purchase and sales. The examples presented in the next section refer to class activities performed in this field of experience.

In many countries, problems and activities related to the use of money are a common feature of primary school activity (Brenner, 1998). Even if the reference is the same field of experience, the objectives and the pedagogical approaches followed differ considerably from one country to another; for example, the degree to which activities in this field of experience are performed differs widely. One approach is episodically to set students word problems which recall situations related to buying and selling. Another possibility is to work for a considerable period of time in the field of purchase and sales, orienting the work according to the requirements of building up mathematical concepts and skills. A third possibility is to orient the work basically according to the requirements of developing knowledge concerning the field of experience itself. In our experiments, we followed the last approach, drawing on the support offered by the “coin” microworld of ARI-LAB. The approach followed was to rely on out-of-school constraints and resources and on the cognitive strategies developed in out-of-school experience to introduce (or develop) procedures, signs, and mathematical concepts suitable for solving problems of increasing difficulty, which draw their meaning and validation from the real-world field of experience (Brenner, 1998).

Different class activities were developed with the system’s support; these involved different practices such as communication, explanation, general discussion, comparison of students’ work, and so forth. The type of class activities performed varied from standard word problems to less standard activities made possible by the new opportunities that the ICT-based system offers. For example, the communication opportunities offered by the system allowed us to propose nonstandard activities for which communication and cooperation among users is meaningful for the specific learning object of the activity.

During the experimentations another real-world field of experience widely explored was that of “measurement of time with the calendar.” Through the support offered by the “calendar” microworld, activities were created to develop knowledge in day counting and day interval managing. Activities performed in this field constituted the ground to develop meanings for concepts such as that of lowest common multiple.

In accordance with the needs and curricula of the different classes involved, and thanks to the mediation offered by the different ARI-LAB environments, long to medium activities were also accomplished in different fields of experience such that of real data handling (histograms, basic statistics parameters, etc.). Because of space constraints and the interest content homogeneity, in this work, we do not provide examples of activities performed in these fields.

THE ARITHMETIC TEACHING LEARNING ENVIRONMENT MEDIATED BY ARI-LAB

In this section, we draw on Cole and Engestrom’s model to analyze the relationships that are established between advanced technology and the learning environment using some examples taken from experimentation with the ARI-LAB system.

In particular, we discuss the nature of the changes induced in the learning environment by the introduction of ARI-LAB and how this introduction modifies the didactical practice and its meaningfulness for the students and the teacher. The meaningfulness of a didactical practice refers to the emergence of goals for students during task solution and the possibility for teachers to involve students in reflection on the mathematical knowledge involved in the activity.
In the analysis reported below, we first consider the way in which the artifact mediates student action during the solution of arithmetic problems and then how the artifact mediates the appropriation of rules that characterized the field of experience in which the problem is placed and the assumption of responsibility and obligation by the students with respect to the knowledge involved in the activity.

**How ARI-LAB Mediates Student Action in Arithmetic Problem Solving**

The microworlds incorporated in the system are important tools that mediate student behaviour during problem solving. Here we use the term mediation in the sense of Vygotsky who defined semiotic mediation as the use of signs to produce effects on other subjects or on oneself (Vygotsky, 1978). The main effects of this mediation are the emergence of goals for tackling the problem at hand and the development of suitable action schemes for solving the problem.

As previously observed, during experimentation with ARI-LAB, students worked over a long period of time to tackle economic problems such as buying and selling. The action possibilities available in the coin microworld allow the user to perform all the solution strategies that might be involved in problems of additive and multiplicative structure: total–part–remainder, completion, partition, and so forth. This may take place through activities involving the grouping and ungrouping of coins in additive-type problems and repeated groupings in multiplicative-type problems. We note that these activities retrace the street mathematics strategies described by Carraher (1988).

In the following, we seek to demonstrate how microworld mediation of the students’ actions is determined by two complementary factors that help to make sense of the mathematical concepts involved in the solution strategy: mediation provided by reference to a field of experience and the possibility to express action and solution schemes through conceptual metaphors that are accessible to students and grounded in actual experience (Nunez, Eduards, & Matos, 1999).

Let’s look at this example taken from experimentation with a Year 2 primary class. It is a standard word problem set for 7-year-olds. The children had already used the ARI-LAB’s coins microworld to represent amounts of money and to solve simple problems involving the summing of different amounts.

**Problem:** “You want to buy a can of Coca-Cola costing 2,000 lire, a package of chips costing 1,200 lire, and a chocolate bar costing 700 lire. You have 3,000 lire. How much more money do you need to buy all these things?”

To solve the problem, two students, working independently, generated in the microworld working space all the coins necessary to purchase the items described in the text of the problem. They then grouped the money generated, forming a single set of coins that were then totalled (3,900 lire); the students also represented the coins they already had (3,000 lire). At this point, the students seemed stumped and, for a time, appeared unable to find a solution strategy. Subsequently, the first student cancelled the 3,000 lire from the working space and, using the mouse, separated 3,000 lire from the total amount. The crucial moment in the performed strategy occurred when the student understood that the 3,000 lire he had generated were not useful for task solution. When he physically separated the 3,000 lire from the total amount (emergent goal), the student attributed a double meaning to the coins that made up the 3,000 lire, both as symbols that contributed to the total amount and as symbols that represented the amount possessed. The second student did not cancel the 3,000 lire from the working space but began generating new 100 lire coins (emergent goal), which he placed next to the 3,000 lire until obtaining an amount equivalent to the whole sum needed for the purchases. In this case, we see how for the student, the generated coins took on a dual meaning, both as symbols representing the shortfall to be made up and symbols that helped to put together the whole amount.
Recognition of this double meaning is the result of appropriation of the coin’s cultural forms which, in the activity, are specialized as a symbolic vehicle to serve new particular cognitive functions (Saxe, 1992). In this transformation, we observe a complex developmental process in which the sense-making cognitive activities of the students and the external structure of the coins microworld are interwoven.

As already stated, the structure of the microworld refers to a field of experience with resources and constraints modeled by computational objects, namely, the coins of the Italian currency system. These objects are characterized by the following operative dimensions: generation, cancellation, exchange, and movement of coins in the microworld’s working space. These operative dimensions, incorporating the properties of the Italian currency system, make it possible to express the solution process in terms of relationships and spatial–dynamic operations involving coins and groups of coins rather than relationships and operations involving numbers by way of ordinary arithmetic symbols.

In this way, when working in the microworld, problem solving is not at all seen as the application of procedures and algorithms from written arithmetic, something that bears little significance for the student approaching problem solution and that is remote from the situation context. Instead, problem solving is regarded as development of the capacity to use the potential of the Italian currency system efficiently. This characterizes the mathematics that develops when working within the microworld-modeled field of experience and distinguishes it from the mathematics traditionally adopted in school to tackle problems of the same sort.

Our study shows how classroom practice with microworlds leads to the acquisition of mathematical tools and thinking strategies that are specific to the field of experience and that allow the pupil to think and act coherently with references to the external world. Reference to the knowledge, the solution patterns, and the linguistic expressions of the field of experience is what enables the student to exploit the microworld’s operative possibilities to build solution strategies for the problems set. In any case, the cognitive mechanisms underpinning interaction with the microworld are ordinary ones, such as those used for basic spatial relations, groupings, motion, distribution of things in space, changes, basic manipulation of objects, iterated actions, and so on.

Drawing on the theory of embodied cognition (see Nunez et al., 1999), we note that these mechanisms are grounded on image schemes that are perceptual–conceptual primitives that allow the organization of experiences involving spatial relations. We observe that in the two examples presented, the respective solution strategies developed (total–part–remainder and additive completion) are conceptual extensions of image schemas, grounded in the bodily experience of the students, such as container schema (add–take away) and source–path–goal schema (from–to).

In this framework, the solution strategies adopted prove to be visual–conceptual extensions of these image schemas. What allows these conceptual extensions to be produced is the sense provided by reference to the field of experience, reified in the physically and bodily grounded operative possibilities of the microworld. This enables the students to control their behavior during the solution process and defines the nature of the microworld’s mediation in the student’s actions.

The analysis undertaken to this point leaves open two questions of vital didactical importance. The appropriation of pertinent solution strategies for problem solution does not derive solely from the result of interaction between pupil and microworld on the basis of perceptual conceptual primitives grounded in students’ bodily experience. As previously observed, it also depends on the pupil’s mastery of the cultural aspects of the field of experience. All pupils entering primary school do not usually possess this mastery. Where this mastery is lacking, it must be constructed within the social practice enacted in class. The goal of this practice must be appropriation of the rules characterizing the field of experience in question and the student’s acceptance of specific obligations and responsibilities concerning the knowledge involved in problem
solving within that field of experience. This raises the question of understanding how
the system might help in building a learning environment that achieves the afore-
mentioned goal.

The second crucial question regards evolution of the strategies the students em-
ploy in the microworld when tackling problems related to the field of experience at
hand. In this regard, we have noted that when students are tackling, say, multiplica-
tive problems in the coins microworld, their strategies demonstrate a preservation of
meaning and understanding of proportionality relationships but prove inadequate
for dealing with numerically complex problems. In other words, the strategies in-
evitably fall back on what Vergnaud (1983) described as the “scalar approach” for
solving missing value proportionality problems by means of action schemes based on
the repeated grouping of coins. These limitations, related to the nature of the math-
ematical tools that make up the commercial transaction field of experience, are well
described by Schliemann in his studies of “street mathematics” (Schliemann, 1995).
Analysis of these limitations led to the question of how to lead students to develop
new mathematical knowledge that is wider in scope than what they develop in this
field of experience but that preserves the focus on meaning found in the field itself.
This question is linked to the transition from the field of experience of purchase and
sales to the mathematical field of experience and to the role that the teacher can assume
in this transition.

The two crucial questions listed here will be explored in the following sections.

How ARI-LAB Contributes to Change the Way in Which
the Rules Mediate Teaching–Learning Activity

Cole and Engestrom’s (1991) model shows that the relationships between the indi-
viduals involved in an activity and their learning community are mediated by rules.
In our context, the relationship between the individual and the community, that is,
between the student and classmates and between student and teacher, is embedded
in a network of activities mediated by the ARI-LAB system. The main purpose of this
is arithmetic problem solving within both real-world fields of experience and those
specific to arithmetic.

To achieve this, there are rules that mediate the relationship between the individual
and the community. Their purpose is to define what is and what is not acceptable use of
the operational tools employed in problem solving in fields of experience (real world
or arithmetic). We note that within the educational context, the rules also represent
an object of learning themselves.

In the following, we seek to demonstrate how the ARI-LAB system can represent
an important mediating tool for managing the dialectic between these dual roles that
the rules play in the educational context. The transformation of rules from being
individual–community mediators to objects of learning takes place in a network of
activities in which shifts of focus and breakdowns occur within the system mediation.

Here we are interested in breakdowns that occur when the work with the operative
instruments used in problem solution is interrupted both because a gap has emerged
between what the subject had anticipated and what he or she had actually accom-
comlished with the system and because contradictions arose among the participants
during system-mediated activity. The breakdown always represents a marker of the
contradiction in the individual–community relationship about what is considered an
acceptable use of the rules.

A focus shift is a change in the activity or in the purpose of the action that may
emerge in system’ use as a consequence (but not necessarily) of a breakdown. Through
the shift of focus, the rules cease to be a reference element mediating the operations
the student performs automatically and unconsciously and become an object of his or
her targeted actions. Analyzing and tracing the actual focus shifts and breakdowns is important to understanding of how the ARI-LAB system can contribute to the negotiation and appropriation of the socially shared rules underlying the solution of these problems. We now give some examples of this.

Example 1. Consider how the children developed the capacity to count with the coins of Italian currency for solving problems entailing the implementation of computation strategies for forming given amounts.

The coin microworld contributes to the structuring of a learning environment that allows students to explore the system of rules and conventions underlying the use of the Italian currency system. In the coin microworld, the student can generate, move, group, and cancel coins in a given working space and can change them by specifying an appropriate modality of change via specially designed interface features.

The microworld offers a representation register that enables students to perform the three cognitive activities that Duval (1995) stated are inherent to any representation: perceptive identification of the significant elementary units (the coins made available by the system interface), treatment based on the internal rules of the system that leads to the gaining of knowledge in relation to the initial representation (exchange and movement of coins in the working space), and conversion into other representation systems so as to reveal new meanings related to what is represented (conversion into the decimal system; see, Example 3 in this section).

Moreover, the coin microworld offers students a validation possibility that allows them to select a (previously generated) coin or group of coins and to hear the amount pronounced orally by means of a voice synthesizer incorporated in the system. This validation tool takes on a vital importance for the appropriation of treatment rules within the representation register of the coins microworld.

During experiments, this validation feature was exploited in tasks of different types. For example, students were assigned problems that required them to form given amounts (with or without constraints on the coins that could be used). In the text of each problem, the amounts that the student had to form were expressed in written verbal form. We noted that when carrying out these tasks the pupils counted by moving the coins into the available working space and regrouping them in a new position. At the end, the system’s validation function allowed the students to verify the work accomplished. If there was an error, a breakdown occurred between what students thought they had done and what they had actually accomplished because of the student’s misuse of a rule within the system.

This breakdown was overcome by a shift of focus in the use of the validation tool, from validating the work performed, namely, the amount formed, to validating the student’s counting process. This shift of focus turned the rules into an object of learning. In addition, it enabled the students to go back over their counting process step by step and validate it with the system’s mediation. After having generated each coin, the students predicted orally the value of the amount reached, validating their counting process via the system.

The shift of focus engages the students in this game of prediction–validation, which develops appropriation of the rules and conventions entailed in use of the Italian currency system. At the end of this activity, the teacher proposed a new activity aimed at evolving the students’ counting strategies and the control they exerted over them (hitherto mediated by the system).

Example 2. The students were set the following task, of a type that exploits the system’s communication potential. The teacher duly inhibited the possibility of validating the counting process (the ARI-LAB interface can be configured so as to inhibit the use of the voice synthesizer).
The tasks is as follows: The teacher chooses a given amount of money and tells the whole class. Each student has to generate coins to make up the given amount and then, by means of the communication environment, has to send his or her classmate a message indicating the amount obtained and the coins used to make it up (e.g., one 1,000 lire note; three 100 lire coins, etc.). The interlocutor has to check if the solution received is correct and then make up the same amount in another way. Then he or she in turn sends her or his solution to the other student as in the previous manner, and so on, until one of the two students makes an error or is unable to find a new combination of coins. The rule is that it is not possible to propose previously used representations. If one of the student’s representations is wrong (with respect to the amount given) and the interlocutor identifies the error, the latter wins the game.

Playing the game with the support offered by the system, students explored different ways in which an amount can be formed and had to coordinate the representation in two ways: using coins and in written language so as to communicate with the partner. The game developed through the written dialogue between the two interlocutors.

The game structure fosters the need for the students to evaluate their own personal strategies as well as those of their interlocutors. This changes the way in which verification of the counting strategies is carried out. Validation of counting procedure correctness is no longer performed by the teacher or by the system but in a natural way by the student because this action is functional to the development of the game. The desire to win the game (as opposed to simply carrying out the teacher’s instructions) led to the production in some cases of increasingly complex solutions.

The system automatically stores dialogues between students for later analysis, together with the work that each student has carried out in the coin microworld (monitoring function). For example, let us consider an excerpt (translated into English) of a real-time dialogue mediated by the computer between two children tackling the task at hand (forming the amount of 2,350 lire):

**ANNA-M:** Two 1,000 lire banknotes, three 100 lire coins, and one 50 lire coin

**MARIO-R:** That’s correct

**MARIO-R:** Two 1,000 lire banknotes, a 200 lire coin, a 100 lire coin, and a 50 lire coin

**ANNA-M:** You did it correctly.

**MARIO-R:** Two 1,000 lire banknotes, seven 50 lire coins.

**MARIO-R:** Four 500 lire coins, a 200 lire coin, a 100 lire coin, and a 50 lire coin.

**ANNA-M:** Ten 100 lire coins, Five 200 lire coins, three 100 lire coins, a 50 lire coin.

**MARIO-R:** You got it exactly right. Well done.

**ANNA-M:** You got it right, too.

**MARIO-R:** Ten 200 lire coins, three 100 lire coins, a 50 lire coin.

**ANNA-M:** Exactly right. A 1,000 lire banknote, thirteen 100 lire coins, a 50 lire coin.

**MARIO-R:** Exactly right.

**MARIO-R:** Eleven 200 lire coins, a 100 lire coin, and a 50 lire coin.

**ANNA-M:** That’s wrong.

**MARIO-R:** Why is it wrong?

**ANNA-M:** Because eleven 200 lire coins make 2,000 lire, a 100 lire coin makes 2,100, one-hundred, a 50 lire coin makes 2,150.

**MARIO-R:** No! I tricked you with eleven 200 lire coins; they make 250.

**MARIO-R:** I am wrong!! They make two thousand two-hundred.
The communication exposes the pupils to different ways of forming the given amount. To check a solution received from the partner, the student can reconstruct it in the coin microworld. In this way, the partner’s strategy can be acquired and elaborated to obtain a new solution. As the example shows, Mario started with the solution sent by Anna and applied single-step changes to produce alternative solutions. In this case, the dialogue structure (decomposition of the amount into its components in written language) also contributed to supporting this activity.

Anna and Mario approached the goal at different levels. Although Mario was able to elaborate on Anna’s solution, Anna initially reasoned in terms of the coins she wanted to use, drawing on the support offered by the coins microworld to obtain the required amount. Checking Mario’s solutions constituted a focus shift for Anna because, as the communication evolved, it supported her use of strategies that are new to her. By exposing pupils to interlocutors’ procedures, the game instructions mediated the possibility to perform an imitative approach to learning.

During the game, breakdowns emerged when errors are detected (“That’s wrong,” “Why is it wrong?” etc.). This induced students to perform strict control of their counting procedures to justify their actions (“I tricked you with eleven 200 lire coins; they make two hundred and fifty”) and negotiate a correct outcome.

**Example 3.** The following example shows how the system had been used to mediate the transition from the rules of the field of experience of purchase and sales (additive conception of numbers) to the rules of the arithmetic domain (positional conception of numbers).

The Problem is as follows: Ann opens her piggy bank and finds four 1,000 lire banknotes, three 500 lire coins, eleven 100 lire coins, and eleven 50 lire coins. Represent the coins that Ann possesses and count the total amount.

Working in the coins microworld, the students solved this problem by generating coins in the working space and organizing them in an appropriate way to count them.

Up to this point the pupils represented monetary value by means of the microworld coins and written verbal language; no didactical activity had been carried out to develop the transition to positional notation, even if the students had already come into contact with this notation both inside and outside school.

At the end of problem solving, some pupils spontaneously tried to express the result using positional notation. Some expressed the total amount as “700010050” or “7000150” instead of “7,150,” drawing on an additive rules system that was coherent with their past experience with coins and the written verbal notation of numbers.

A contradiction emerged from the subsequent interaction with the teacher on this matter, one connected to the different rule systems that the teacher and the pupils respectively employed to represent numbers. It brought about the objectification of a new need that consequently led to an expansion of the cycle of activity. The focus of the teaching–learning activity changed to become development of the capability to use the socially shared rules of positional notation.

ARI-LAB offered powerful support for this focus shift in the activity. Comparison with the coins, abacus, and number building microworlds was useful to understanding of how the system can mediate the transition from an additive conception of numbers to a positional one.

Operations with numbers performed in a currency system are conditioned by that system’s constrains. For example, there is no coin in Italian currency that corresponds to “seven thousand, one hundred and fifty”; this value can only be created by grouping coins together. The world of Italian currency and the world of numbers differ greatly, while at the same time presenting complex reciprocal interrelations. One example of such an interrelation is the relationship between the abstract number “7,150” and
the addition–number name “seven-thousand one-hundred and fifty” obtained by grouping together the coins indicated in the problem text.

The abacus and number building microworlds are the two microworlds of reference where such interrelations can be explored. Students can represent the results they obtain in the coin microworld within the representative structure of the other two microworlds. In the abacus microworld, pupils can generate little balls on the poles of the instrument and cancel or change them according to a specific procedure.

Referring back to the previous example, to represent “seven-thousand one-hundred and fifty” on the abacus, the pupil had to restructure the solution developed in the coins microworld according to the constraints of the abacus microworld. The representation in the abacus format corresponded to the decimal polynomial structure of the abstract number “7,150.”

In the number building microworld, the pupil interacts with a working space to generate digits of the decimal positional notation, move and cancel them, as well as group them together to link them up (by means of a specific command, the glue command) to form a number with more than one digit. The operation possibilities of this microworld allowed the students to see the relationship between the addition–number name “seven-thousand, one-hundred and fifty” and the abstract number “7,150” through the decimal positional notation “7,150” of the number.

The interrelations between the three different representations of the number made available in each microworld could be explored by means of the system’s voice synthesizer. This is a validation instrument that gives the oral pronunciation of each number representation performed in the three microworlds. The possibility of validation this offers is particularly useful, from a didactical point of view, for exploring the interrelations among the different number representations.

The voice synthesizer is a tool that enables coordination of the three representation registers available in the activities of the three microworlds. As Duval (1995) showed this coordination is crucial to development of the semiosis entailed in conversion from one representation register to another.

Following solution of the previously described problem, the teacher requested that the pupils represent the total amount obtained in the coins microworld in the other two microworlds. He also asked them to explain why some pupils’ decimal positional representation of the number “seven-thousand one-hundred and fifty” were considered incorrect.

Obviously, the passage from an additional conception of numbers to a decimal one is neither immediate nor straightforward. Some pupils performed this task quickly, whereas others experienced difficulties at this stage. The feedback opportunities offered by the system played a crucial role in the acquisition of these capacities. Breakdowns and shifts of focus occurred of the same nature as those described in Example 1. These mechanism supported pupils in the production of hypotheses on how to apply the rules of the different representation systems and provide opportunities to test those ideas out. The teacher’s task was to keep the dialogue with the pupils focused on the different rules used and on their interrelations. This was done through a game of anticipation, verification, and justification that is the source of the dialogue mediated by the system.

How ARI-LAB Contributes to Mediate the Evolution of the Didactical Contract

According to Cole and Engstrom’s (1991) model, belonging to a community implies a division of labor, that is, the repeated and renegotiated distribution of work tasks, power, and responsibilities among the participants. In practice, the division of labor defines a system of reciprocal obligations that mediate the strategy by which
community members, interpreting specific roles, interrelate for the social construction of the object of the activity.

Brousseau (1986) showed how the system of reciprocal obligations binding participants in mathematics teaching and learning is regulated by a kind of contract that is specific to the mathematics knowledge at play. He called this system of obligations the “didactical contract.”

The didactical contract defines the set of obligations which, either explicitly or implicitly, determine the area of responsibility to be managed by each participant (student or teacher) within the activity with respect to the knowledge in question and for which each will respond to the community.

Construction of a suitable didactical contract for the learning of given knowledge takes place through a dynamic process in which contradictions may emerge. These appear as breakdowns between what the teacher expects in terms of the students’ acceptance of obligation (referred to as the knowledge at play) and the load of responsibility that the student is able to bear when tackling tasks. Overcoming these contradictions can lead to adaptation phenomena that do not bring about effective knowledge growth within the class (these phenomena have been extensively studied in the literature; see, for example, the “Topaze” and “Jourdain” effect in Brousseau, 1986). On the other hand, the contradictions may be overcome through the search for a new contract based on the readjustment of norms and the distribution of responsibilities and powers in accordance with the needs felt by the participants.

In this section, we draw on examples to show how the ARI-LAB system mediated the contract construction and its evolution so as to help students to gradually take on obligations involved in additive and multiplicative problems solution. This had been done, at the beginning, using the microworlds tools, then the oral arithmetic, and, eventually, written arithmetic expressions (according to the analysis of arithmetic knowledge performed previously). In that section, we pointed out how the development of problem-solving abilities is based on student’s capacity of converting the meaning of a solution process performed in a real-world field of experience in languages increasingly general from a mathematical point of view.

The means offered by the ARI-LAB system gave us the opportunity to define a general didactical strategy suited for promoting in students this capacity. For example, the system was tested for 4 years with a primary school class in which the problem-solving activity was characterized by the following requirements:

- to solve the problems presented by working in microworlds and copying the meaningful graphic representations thus obtained into the solution sheet, then writing appropriate notes to explain them;
- to convert the solution produced into written verbal language using the positional notation; and
- to convert the verbal written solution into arithmetical relations and explain, in verbal language, the meaning of those relations, making reference to the concrete situation at hand.

When students were able to fulfill the previous requirements, the didactical contract underwent an evolution to allow students to explore aspects of arithmetic knowledge more general than those related to the specific real-world field of experience. The evolution of the didactical contract was strictly linked with the evolution of the activity from the field of experience of purchase and sales to the arithmetic one. In the following, we try to show how the ARI-LAB system mediated the didactical strategy previously underlined allowing students to gradually overcome contradictions and take on obligations involved in the mathematical knowledge at play.
**Solution with Microworld Tools.** The student’s acceptance of responsibility when engaging in problem-solving in a microworld is linked to the possibility of receiving feedback from the system during system-mediated activity. This feedback may lead to evolution in the adopted solution strategy, permitting the student to gradually take on obligations in relation to the knowledge incorporated in the microworld.

The feedback can be obtained in three different ways: from direct interaction with the system, from system-mediated social interaction arising when students play specific roles within the didactical situation proposed, and from a social interaction specifically designed to support student performance. An example of feedback obtained from the interaction with the system is provided in the explanation given previously of the validation provided by the speech synthesis tool embedded in the coins microworld and its importance for appropriation of rules involved in the use of Italian currency. In that case the speech synthesis tool enabled the teacher to modify the didactical contract to allow the student to gradually become aware of the fact that choosing a coin to use for solving the problem entails adoption of a precise counting strategy. Acceptance of this responsibility is mediated by use of the validation tool, which enabled the students to evolve their counting solutions strategies. At the beginning, these were often naive or undifferentiated because they were based on a single system (e.g., counting by hundreds), irrespective of the value of the coin used.

As to the feedback obtained from system-mediated social interaction, we note that the communication tool embedded in the system supports the assignment of problems in which students must take on specific roles to meet situation requirements. For example, during experimentation, tasks were set in which the students were to play complementary roles (i.e., seller and buyer) to reach a solution. In other cases, the didactical situation had the characteristics of a game and was solved through competitive role playing (see Example 2 in the previous section). In the buying and selling situation, each pair had to simulate a transaction using the network and the coins microworld features. This situation was guided by instructions and assignments that conferred precise, differentiated roles and constraints on either student.

We note that in this type of didactical situation, adhesion to specific roles in the field of experience (seller and buyer) and the enactment of related practices (giving the cost, paying, giving change, etc.) are mediated by the feedback that each participant receives from his or her partner about the actions carried out. Information of this kind can provide feedback both about the meaningfulness of the actions performed with respect to the constraints entailed in the situation and about the way the subject has interpreted his or her role. It can therefore bring to light possible contradictions.

When contradictions arise in the written dialogue between participants the activity focuses on the specific actions that have brought to the contradiction. During experiments, we noted that a peer partner assumed a crucial role in supporting his or her classmate in taking on the obligations involved with current practice.

Identification with the respective roles and acceptance of the specific responsibilities they entail, with respect to the reference knowledge, develops progressively through a chain of actions and feedback. ARI-LAB's communication, action, and representation possibilities enabled the design of didactical situations entailing social interaction that allowed students progressively to take on responsibilities with respect to the knowledge incorporated within the microworlds.

Lastly, let’s look at feedback obtained from system-mediated social interaction that is designed to support student performance. As many educational studies have shown (see, e.g., Tharp & Gallimore, 1989), this support is necessary for the overall evolution of the student’s knowledge and, in particular, for the development of appropriate problem-solving action schemes. Making reference to the notion of the Zone of Proximal Development elaborated by Vygotsky (1978), we can observe that when individual students are unable to solve a problem by themselves, a more experienced
person has to provide help. In the case of problem-solving activities mediated by the ARI-LAB system, the assistance provided should help each pupil in three respects: to exploit the action possibilities offered by the computer, to define goals pertinent to the task at hand, and to manage the system interface in a way pertinent for task solution.

Providing the student with support usually entails modification of the type of responsibility assigned to the student within the didactical contract. It also means the teacher must set suitable strategies for creating the conditions necessary to the understanding and the performance of the assigned task. As shown by Tharp and Gallimore, a number of different strategies may be adopted, for example, modeling, contingency management, feedback, instruction, questioning, and cognitive structuring.

The particular characteristics of the ARI-LAB system allow the support role to be widely shared among participants in the activity rather than being the exclusive charge of the teacher. For instance, the ARI-LAB communication feature used to share solutions (and corresponding monitoring) between students allows the teacher to orchestrate situations whereby those experiencing difficulty can be provided with models and strategies for imitating more proficient students, who, in this way, assume a cardinal role in steering classmates toward action schemes conducive to problem solution.

As the matter of fact, solutions received by a student from a classmate (or from the teacher) cannot be copied in the student’s own solution sheet but need to be reconstructed within the microworlds. To do so, the student interacts with the computational objects made available by the microworlds to represent all the solution steps accomplished by his or her partner. In this way he or she receives help for the definition of objectives necessary to pursue the task at hand.

**Solution Based on Oral Arithmetic.** To investigate students’ acceptance of responsibilities for constructing a solution based on oral arithmetic, let’s consider the third-grader’s solution to the proportionality problem reported at the beginning of the section addressing the theoretical framework underpinning ARI-LAB design (page 763).

As shown in Fig. 29.2, the girl expressed the solution through a discourse, using written language and numbers in positional notation. The skills involved in the reported solution are typical of oral arithmetic, as described previously. These skills developed over time through the conversion of solutions produced in the coins microworld into solutions based on natural language. The girl’s solution still bears explicit references to the actions performed in the microworld (“divide the banknotes making up 27,000 into three groups”). Nevertheless, it would appear that the microworld experience had been transformed by the use of natural language and positional notation.

Performing an activity in a microworld does not necessarily lead to its interiorization at the level of consciousness. The interiorization process is dialectically linked with the externalization that the student is able to accomplish. Many of the meanings related to practice undertaken in microworlds are brought to the student’s consciousness through conversion because this offers the possibility of carrying out interpretation of the microworld-based activities. Initially, conversion is characterized as a detailed description in written language of the actions carried out in the microworld.

The system provides a support for allowing the student progressively to take on the obligations required for developing the capacity to produce an oral arithmetic solution. This is the monitoring tool. It can be used as a tool for validating the conversion activity performed.

Through monitoring, the solution process performed in the microworld is transformed into an object that can be used to support conversion. Using monitoring, the student can reexamine the entire solution process performed in the microworld to
describe it, and, with the teacher’s assistance, to verify if the description given fits the solution process previously accomplished, that is, if there is semantic correspondence between the student’s description and the actions undertaken in the microworld. If this is not the case, a contradiction arises that can be explicated through comparison between the written description and the monitoring. This contradiction may lead the teacher to modify the type of responsibility assigned to the student (e.g., to describe only a few steps of the solution process) and to provide suitable forms of support (e.g., putting forth questions and giving instructions) that help the student accomplish conceptual extensions to the concrete meanings incorporated in the practices performed in the microworlds, in accordance with the new means offered by the different externalization registers of oral arithmetic. Thus, monitoring, through the teacher’s mediation, is the tool that coordinates the conversion activity between the different representation registers involved in the activity.

With exploitation of the means that these new representation registers offer, the contents of the converted solution tend over time to differ from the initial solution. This is because the new means lead to the selection and reorganization of actions and operations undertaken in the microworld. This is evident in the previously mentioned solution produced by the third-grader in which we can see that many of the actions and operations performed in the coins microworld have been condensed and reorganized on the basis of the new possibilities offered by written language and positional notation.

We note that in the reorganization of the solution strategy into the written language students implicitly use operations properties (such as associativity, distributivity, commutativity) that constitute an important reference point for the subsequent conversion activity into arithmetic expressions.

When, through the conversion process, the student begins to acquire skills for producing oral arithmetic solution strategies, a change takes place in the didactical contract. At this point, the student is capable of using verbal written language to take on the construction of a solution strategy directly, without having first to build the solution in the microworld. The microworld remains a reference tool for possible validation of the oral solution strategy.

Production of a Written Number Sentence as the Solution to a Problem. The ability to convert the oral solution strategy into a written number sentence was developed progressively. With the teacher’s help, students progressively learned to use arithmetic symbols to express the meaning of the strategies based on oral arithmetic. At the beginning, arithmetic symbols were introduced by the teacher as a means to synthetize the reasoning based on oral arithmetic. Then, by means of a didactical practice based on a comparison of the different solution strategies accomplished in the class, the teacher focused attention on the fact that different arithmetic expressions can represent different solutions to the same problem.

Through this kind of didactical practice mediated by the teacher, the students gradually learned to master the meaning that arithmetic symbols can have in problem solving, reorganizing their individual oral solutions in accordance with the potential the new symbols offer. For example, through the strategy of comparison, the students learned to establish a link between the meaning of additive completion and that of total–part–remainder in additive problems and learned how to use the sign “–” when constructing a solution expression, irrespective of the oral strategy developed.

The system offers the teacher a range of possibilities for classroom management of comparison-based practice and for getting the students gradually to take on responsibility in the use of written number sentences for solving additive and multiplicative problems. For example, during one of the experiments, the teacher provided students with already-developed problem solutions to expose them to different solution
strategies (expressed as written number sentences) for addressing the same problem. After the students had devised their strategies in the coins microworld and converted them into written verbal language, they were asked to identify which of the strategies proposed by the teacher most closely matched their own oral strategies and to justify their choice. In addition, they were asked to reconstruct in the coins microworld the solution strategies represented in the other written number sentences that the teacher had proposed. In this case, the monitor function was used with a twofold purpose: to verify whether semantic correspondence had been maintained during the conversion and to highlight any contradictions that emerged in students’ interpretation of the meaning embedded in the arithmetic expressions.

We note that this kind of practice can lead to situated abstraction processes (Hoyles, 1993), regarded as processes that enable the student to grasp the sense of arithmetic symbolic expressions. This sense is broader than that emerging from practice carried out in the microworld, but nonetheless it can be interpreted in virtue of the mathematical experience conducted in the microworld. This is crucial for the process of situated abstraction in the microworld, that is, for shedding light on the mathematical relations underlying the practice.

To understand better the mediation role played by ARI-LAB tools for developing situated abstraction processes, let’s examine the evolution of activity regarding the proportionality problem presented in Fig. 29.2. The evolution of this activity was designed to bring about a transition from an additive to a multiplicative conception through appropriation of a general solution scheme based on reduction to the unit. The acquisition of a general solution scheme in the context considered, implies an evolution of the activity and may come about as a result of didactic strategies of different kind: essentially by means of mathematical rules given by the teacher without explicit reference to the resolution process performed or by means of a reflection about the solution strategies actually carried out by the students. These two approaches are very different at the cognitive level. In the first case, the student develops the solution by means of a formal abstraction process without having had the opportunity to discern the mathematical efficacy of the general scheme proposed in comparison with the other possible strategies. In the second case, the pupil has the possibility of developing an abstraction in situation, that is to say, he or she has the possibility to grasp the mathematical necessity for this generalization.

Let us briefly describe the evolution of the activity that had brought to the development of this process of abstraction in situation.

After describing the solution strategy in natural language, the third-grader expressed it through the following number sentences:

\[
27,000 + 27,000 + 27,000 : 3 = 63,000 \\
27,000 + 27,000 + 27,000 : 3 + 9,000 : 2 = 76,500
\]

Note that the two expressions translate the girl’s oral strategy into arithmetic signs. Once all the pupils individually solved the problem regarding the bottles of fruit juice, the teacher asked them to use the communication tool to exchange with their classmates the different solutions expressed through written number sentences. After studying the interlocutor’s solution, each pupil sent back his or her own interpretation of the meaning it expressed in relation to the concrete situation of the problem. The interlocutor then verified if this interpretation is in accordance with his/her own.

After this negotiation phase, the teacher put together the results produced, gave her or his opinion of them, and helped those pupils who were unable to interpret the solution. The pupils were then asked to write down in their solution sheets all the various expressions produced by the class. After this work phase, the teacher asked...
the pupils to create in the simplified spreadsheet microworld of ARI-LAB a table like the one shown in Fig. 29.5.

This table was to be completed with all the solution expressions produced in the class. In addition, it contained new values for the number of bottles required, and the students were required to find new solution expressions for them.

Because the teacher provided the correct numerical result for these new values, validation of the correctness of the expressions produced could be done with the support of the automatic computations performed by the spreadsheet. This validation activity allowed the pupils to reconsider incorrect expressions and to modify them in a sort of anticipation, production, and validation game that is effective at the pedagogical and cognitive level.

The attention progressively shifted away from the meaning of an expression in relation to the concrete situation of the problem toward its structural aspects. The main role of the teacher in this activity was to assist students’ performance. In Table 29.2, we report some expressions that the pupils produced according to given values for the number of bottles.

At the end of the work, the teacher asked pupils to produce expressions that make explicit reference to the specific cells of the spreadsheet in which the cost of six bottles and the quantity indicating the number of bottles to buy are stored. This request constrained the building of the solution expression and aimed to force the students to use the reduction to the unit scheme. The pupils produced expressions using the

<table>
<thead>
<tr>
<th>Table 29.2</th>
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<tbody>
<tr>
<td>Some Expressions Produced by the Pupils</td>
</tr>
<tr>
<td>27,000 + 27,000 + 27,000 : 3</td>
</tr>
<tr>
<td>27,000*3 + 27,000 : 2</td>
</tr>
<tr>
<td>27,000*2 + 27,000 : 3</td>
</tr>
</tbody>
</table>
automatic computation offered by the spreadsheet to validate their work. In the end, most of the pupils produced expression such as the following:

\[ \frac{22,000}{6} \times B5, \quad \frac{22,000}{6} \times D5, \quad \frac{22,000}{6} \times F5, \quad \text{and} \quad \frac{22,000}{6} \times H5. \]

The teacher’s objective was then to make the pupils conscious of the mathematical meaning of the experience undergone, that is, students had to appreciate the common structure of the above expressions and its functionality for the purposes of generalization.

This reflection was performed during a classroom discussion based on comparison of all the solutions the students produced. In this way, they gradually realized that it is possible to apply the reduction to the unit structure to all numeric cases. Thus, they had the opportunity to understand the mathematical importance of generalization, at least at the computation level.

In the evolution of the activity, the spreadsheet revealed itself as a suitable tool to explore the structure of multiplication problems. This exploration had been carried out by exploiting its table structure, its computation opportunities, and the possibility to change values that the spreadsheet offers. These opportunities allowed students to explore the invariants at play in the solution of multiplication problems. In the mathematics discourse that developed, the experience carried out with ARI-LAB was a catalyst for discussion because it allowed the teacher to keep the focus on the learning objective of this phase of the activity.

**CONCLUSION**

In this chapter, we have analyzed the relationship between advanced technology and learning environments in the arithmetic problem-solving domain, making reference to an open learning system that integrates microworlds, communication, and monitoring tools: the ARI-LAB system.

Our work is grounded on the assumption that analysis of the cognitive processes, which develop in arithmetic problem solving, cannot be separated from the analysis of the specific cultural practices and symbolic tools used. Research in mathematics teaching and learning has highlighted the importance of developing problem-solving activities within cultural contexts that are of significance to students, that is, contexts that permit the linking up of mathematics with out-of-school motivations, experiences, and applications. During the ARI-LAB design phase, the field of experience notion enabled us to define significant, unitary, and homogeneous environments in which to develop medium- to long-term arithmetic problem-solving activities. This design work led to the creation of microworlds that model the resources and limitations of fields of experience both in the real world and in mathematics via computational objects. The student can interact with these microworlds by means of ordinary cognitive mechanisms (such as those used for basic spatial relations, like groupings, motion, distribution of things in space, etc.).

We have shown how activity with microworlds makes it possible to mediate new ways of accessing the concepts, procedures, rules and norms involved in problem solving within the field of experience concerned, fostering the emergence of objectives during task solution.

At the same time, we have highlighted the fact that construction of the mathematical experience needed to master the cultural practices involved in a specific field of experience is not just the result of the subject’s interaction with the microworld tools but also emerges from the social interaction developed in the classroom during activities mediated by the technological tools at hand. We have shown how the microworlds, integrated with the system’s communication, monitoring, and validation
tools, contributed to the building of didactical practice that is characterized by social interaction mechanisms that are particularly effective for learning. Within the activity theory framework, Cole and Engstrom’s (1991) model has allowed us both to perform detailed analysis of the changes brought about in the learning environment as a result of technology-mediated activity and to examine how these influence the students’ learning processes. In particular, the examples provided have illustrated how the technology has played a crucial role in

- developing a social practice that provides the students with assistance for overcoming the difficulties encountered while tackling the tasks set,
- favoring conversion of the solution into different representation registers and providing tools for coordinating them, and
- offering new possibilities for exploring the mathematical knowledge involved in solving additive and multiplicative problems, fostering the evolution of the activity from the real world to the mathematical field of experience.

The analysis performed within the framework of activity theory has enabled us to demonstrate how the process of developing skills in arithmetic problem solving is the result of the way in which the various components of the learning environment help to reveal and to overcome the contradictions arising out of use of the tools available. The traditional approach to the development of problem-solving skills has treated the solution to a problem mainly as something related to the individual’s mental capacity to adapt his/her preexisting knowledge to the new situation, for which he or she does not have procedures, techniques, and tools readily at hand. By contrast, the results of our study reveal how the knowledge involved in problem solving is above all distributed among the tools, relationships, and roles that characterize the social practices in which the solution activity develops. Accordingly, the acquisition of problem-solving capacities becomes more a matter of social interaction and mindful cultural engagement rather than a question of personal mental ability. In this light, the common and traditional separation of the individual’s attitudes and achievements from social–interpersonal variables fades, while a closer relationship between individuals’ learning and social interaction is assumed.

From this we can see how study of the relationship between advanced technologies and learning environments needs to bear in mind the whole teaching and learning activity that takes place in the given context. Technologies provide new tools that make it possible to alter didactical practices aimed at the acquisition of a certain knowledge; subsequently, they influence the learning processes brought into play. For this to happen, there must not be any disjoining of the technology design phase from the phase of planning and testing didactical practices focused on the object of learning. In other words, the design and evaluation of new didactical practices is to be considered an integral part of the design and implementation of an educational software application (Bodker, 1996).

The change in learning environments brought about by the introduction of an advanced technology needs to be considered as a two-way process: Not only do technological tools influence and transform the activities performed with their mediation, but the results of these activities also deeply influence the technology used. This influence can be seen at two levels. On the one hand, computer tools can change during use without being altered technically because use in context brings to light new possible uses of the features incorporated in the technology. On the other hand, use in context may contribute to the outlining of new practices and, as a consequence, may reveal new needs that in turn lead to the design of new tools. Our work does not seek to propose a generalization from the specific case considered but rather to subject the particular experience to a systematic and critical analysis within the
framework of activity theory, which may stimulate thinking about similar situations elsewhere.

REFERENCES


In this final chapter, we consider some of the many research issues that need attention in the advancement of our discipline. Specifically, we have identified the following questions as worthy of consideration:

1. What role can research play in illuminating the multidisciplinary debates on the powerful mathematical ideas required for the 21st century?
2. How can research support more equitable curriculum and learning access to powerful mathematical ideas?
3. How can research support the creation of learning environments that give learners better and more equitable access to powerful mathematical ideas?
4. How can research contribute to the kind of teacher education and teacher development programs that will be needed to facilitate student access to powerful mathematical ideas?

5. How can we assess the extent to which students have gained access to powerful mathematical ideas and their abilities to make effective use of these ideas? How can research inform such assessment?

6. How do we assess and improve research methodologies in mathematics education?

**ILLUMINATING THE MULTIDISCIPLINARY DEBATES ON POWERFUL MATHEMATICAL IDEAS FOR THE 21ST CENTURY**

In addressing this first question, we explore various perspectives on the nature of mathematics, its roles in society, and what count as important mathematical ideas for the new era. We consider some of the psychological, epistemological, philosophical, and sociocultural viewpoints on these issues.

How we conceive of mathematics has a major bearing on the mathematical ideas that we consider essential for the 21st century (cf. Hersch’s [1979] argument). No longer is mathematics seen as an arbitrary construction of the individual mathematician. Rather, mathematics and the direction of its growth are considered by many to be shaped by a complex system of cultural, social, and political forces (e.g., D’Ambrosio, 1999; Secada, 1995; Skovsmose & Valero, chapter 16 of this volume; Wilder, 1986). Others, such as Lakoff and Nunez (1997, 2000), see mathematics as structured by the human brain and limited by human mental capacities.

A focus on “mind-based mathematics” has emerged in recent years, with the work of Lakoff and Nunez (1997, 2000) featuring prominently. Their theory assumes that the way in which humans come to understand and know mathematics is through their sensory–motor experiences, which are largely metaphorical in nature. The means by which abstract ideas are understood in terms of the concrete is referred to as conceptual metaphor, which is a cognitive tool that allows one to use the inferential structure of one conceptual domain, such as geometry, to reason about another domain, such as arithmetic (e.g., when we conceptualize numbers as points on a line). The power of conceptual metaphors, argued Lakoff and Nunez, is that they allow us to apply our understanding of one branch of mathematics to reason about another branch. Although their theory does not specifically address mathematics education, Lakoff and Nunez claimed that mathematics can become far more accessible and comprehensible to learners if they are made aware of how these conceptual metaphors shape mathematical ideas. As Sfard (1997) pointed out, however, for students to develop metaphorical conceptualization, social mediation is required. That is, once a mathematical idea exists “on a social plane,” its reconstruction on an “individual plane” may be initiated and then fostered by others (Sfard, 1997, p. 363).

The role of social and political factors in shaping mathematical beliefs is illustrated nicely in Goldin’s work (chapter 9) in this volume. He shows how the different mathematical beliefs held by various educational, social, and political groups are fueling debates on what should constitute mathematics education for today’s students. Despite these conflicting ideological perspectives, it is generally recognized that mathematics is a major element of all human cultures, whether ethnic, urban, rural, or indigenous. Yet we still have a long way to go to bring this human element to the fore (D’Ambrosio, 1999). When mathematics is considered to be intertwined with human contexts and practices, it follows that social accountability must be applied to the discipline (Ernest, 1998). Issues of access and equity, which we address in the next
Ethnomathematics

One approach to the problems raised by the cultural, social, and political dimensions of mathematics and mathematics education has been offered by ethnomathematics (e.g., D’Ambrosio, 1999):

Ethnomathematics, as a program in history and epistemology with an intrinsic pedagogical action ... responds to a broader conception of mathematics, taking into account the cultural differences that have determined the cultural evolution of humankind and political dimension of mathematics. With the growing trend towards multiculturalism, ethnomathematics is recognized as a valid school practice, which enhances creativity, reinforces cultural self-respect, and offers a broad vision of humankind. (p. 150)

The implication of ethnomathematics for mathematics education is that the approach to mathematics should be rooted in the mathematics developed (in a more or less explicit way) in the culture of the learners. Ethnomathematics has emerged especially in developing countries to contrast the Eurocentric view. According to Vithal and Skovsmose (1997), four strands (partly overlapping) may be identified in related research: (a) the historical strand, in which the history of mathematics in different cultures is reconstructed; (b) the anthropological strand, in which the mathematical ideas found in traditional cultures are investigated; (c) the strand concerned with everyday settings, in which authentic practices are investigated, especially with respect to problem-solving strategies; and (d) the educational strand, in which the impact (if any) of ethnomathematics on mathematics curriculum development is studied.

This last strand has been questioned with respect to its impact on equity in mathematics learning. In simple terms, by linking mathematics education too closely to the culture of learners, there is the risk that, especially in developing countries, the learners will have not have access to the sophisticated ideas of mathematics on which our increasingly technological society is based. Indeed, in his plenary speech at the 22nd annual conference of the International Group for the Psychology of Mathematics Education, Ciril Julie (1998) strongly defended the right of South African students to study the powerful mathematical ideas that are made available to students in more affluent nations. In essence, as Skovsmose and Valero (chapter 16) pointed out, we are facing two paradoxes.

The paradox of inclusion refers to the fact that the current globalization model of social organization, which embraces universal access and inclusion as a stated principle, is also conducive to a deep exclusion of certain social sectors. The paradox of citizenship alludes to the fact that the learning society, claiming the need of relevant, meaningful education for current social challenges, at the same time reduces learning to a matter of necessity for adapting the individual to social demands. (p. 386)

Coping with these paradoxes seems to be one of the most demanding tasks for today’s mathematics education research.

Mathematics of Authentic Practices

Studies of the nature and role of mathematics used in the workplace and other everyday settings (e.g., nursing, carpentry, grocery shopping, dieting, architecture, street vending) are especially important in helping us identify some of the powerful mathematical ideas for the 21st century. Such studies (e.g., de Abreu, chapter 14; Lave, 1988, 1997; Nunes, Schliemann, & Carraher, 1993) have shown how the strategies and
decision making involved in these settings develop within, and thus become products of, the sociocultural communities of these practices. These studies have shown that the mathematics used in these practices is not simply an extension of school mathematics. Rather, practice-based mathematics tends to be situated in nature, shaped by the goals and the tools of the particular workplace or other sociocultural setting (Hoyles, Noss, & Pozzi, 2001).

Research that has attempted to characterize the mathematics used in working practices and in other life activities has not been without obstacles, however. Hoyles et al. (2001) pointed out that in such research, most employees don’t describe their duties in mathematical terms and then often state that they use very little mathematics in their work. The basic problem here seems to be that the “mathematics of work is hidden beneath the surface of cultures and practices, so that any superficial classification of it in terms of school-mathematical knowledge will inevitably result in its reduction to simple measurement and arithmetic” (Hoyles et al., 2001, p. 5). Stevens’s (1999) research has shown how mathematical practices can be considered as a part of another discipline, however, such as architectural design, rather than the discipline of mathematics itself. He argued that if we are to understand better how mathematics is a consequential part of the broader social world, we need to shift our attention from mathematics per se to mathematics within the disciplines.

Although a good deal more research is needed on the powerful mathematics required for 21st-century life settings, there is general agreement that economic, social, and technological change will have an increasing impact. As Clayton (1999) and others (e.g., Er-sheng, 1999; Pollack, 1997; Stevens, 1999; Roschelle, Kaput, & Stroup, 2000) have indicated, the availability of increasingly sophisticated technology has led to changes in the way mathematics is being used in workplace settings; these technological changes have led to both the addition of new mathematical competencies and the elimination of existing mathematical skills that were once part of the worker’s toolkit (Stevens, personal communication).

There seems to be general agreement that employers consider problem solving, including working collaboratively on complex problems, to be essential to productive outcomes, along with critical thinking, numerical reasoning, analysis of complex data sets, and appropriate applications of technology (Anderson, 1999; Seeley, 1999; Kaput & Roschelle, 1999; Lappan, 1999). With respect to the last component, a facility with various representational tools, such as spreadsheets and graphical software, is becoming increasingly important (Steen, 1997). Likewise, an understanding of ratio and proportion, probability, rate, change, accumulation, continuity, and limit is necessary, as is algebraic reasoning in general (Lappan, 1999; Jones, Langrall, Thornton, & Nisbet, chapter 6; Kaput & Roschelle, 1999). The applications of these key ideas and understandings to mathematical modeling in a variety of real-world situations have been cited as fundamental to success both in work environments and in life contexts in general (Clayton, 1999; Hall, 1999; Lesh & Doerr, 2002).

Despite the existing research on the mathematics needed for a productive society, there remain many issues in need of attention. These include what it means to analyze workplace settings from a mathematical perspective, the nature of the relationship between mathematical knowledge of the workplace and the mathematics taught formally in schools, and whether practical and formal mathematics are derived from different epistemologies (Hoyles et al., 2001). These issues are critical in addressing whether the mathematical ideas our students learn in formal settings are appropriate for their success beyond the classroom. Some say that the practices of many existing mathematics curricula are in opposition to those of the workplace, in which cooperation and teamwork are valued and a tolerance for ambiguity is the norm (Lott & Souhrada, 2000; Packer, 1997). At the same time, the teaching of narrow sets of skills and facts is unlikely to prepare students for life beyond the classroom.
walls; rather, Lappan (1999) suggested that we should be teaching students “how to learn mathematics and how to think with and invent with the mathematics that they know” (p. 157).

Such an approach to mathematics instruction, must, of course, reach all learners irrespective of gender, language, ethnicity, or disability. Every student has the right of access to powerful mathematical ideas, how to think effectively with these ideas, and how to apply their mathematical understandings beyond the walls of the classroom.

**SUPPORTING MORE EQUITABLE CURRICULUM ACCESS TO POWERFUL MATHEMATICAL IDEAS**

At the beginning of this new millennium, we face multiple challenges in our efforts to promote equity at all levels of education. Equity in mathematics education is a multidimensional issue, with many forces working against students’ democratic access to powerful mathematical ideas (see Tate & Rousseau’s discussion on this point in chapter 12). Given the complexity of the problem, it is not feasible to address the myriad of factors that require attention; rather, we will consider a selection of the ways in which researchers are attempting to promote more equitable curriculum access to powerful mathematics.

One of the problems that mathematics educators face is how to restructure the overcrowding in many existing curricula so that students from diverse backgrounds have access to the more sophisticated and complex mathematical ideas that society requires. Significant work is being done here by Kaput and his colleagues in achieving their goal of teaching “much more mathematics to many more people” (Kaput & Roschelle, 1999, p. 161). In highlighting change (economic, social, and technological) as a central phenomenon of this century, Roschelle et al. (2000) expressed concern that the mathematics of rate, change, and variation is “packed away in calculus courses,” with the result that only a small percentage of students are gaining access to these important mathematical ideas—the very concepts that students need to both participate in their physical and social lives and to make informed decisions in their personal and political lives (p. 47).

Using a combination of advanced technology and carefully constructed curricula, Roschelle et al. are democratizing students’ access to the mathematics of change. With a focus on visualization and simulation, their MathWorlds and SimCalc projects have aimed to provide ordinary students with the “opportunities, experiences, and resources” to develop “extraordinary understanding and skill” with the mathematics of change and variation (Roschelle et al., 2000, p. 48). The achievement of this goal is evident in the responses to Mathsworlds of inner-city students, the majority of whom were in the lowest quartile of academic achievement and socioeconomic standing. These students demonstrated an ability to construct viable mathematical concepts of change and variation. A caveat is in order, however. Roschelle et al. stressed that democratic access to the powerful mathematical ideas that innovations in computational media are providing is not simply a matter of choosing the right technology. Rather, it is imperative that conditions be created where students develop their abilities to understand and solve increasingly challenging and meaningful problems.

Although there have been numerous studies that have targeted inequity in schools (see Tate & Rousseau, chapter 12), there appear comparatively few that have addressed equity issues in mathematics teaching at the classroom level. Of these, the Cognitively Guided Instruction (CGI) program (e.g., Carey, Fennema, Carpenter, & Franke, 1995) and the Toward a Mathematics Equity Pedagogy (TEMP) Program (Fuson, De La Cruz, Smith, Lo Cicero, Hudson, Ron, & Steeby, 2000) have achieved substantial success. In aiming to provide a “diverse curriculum for diverse learners,” Carey et al. (p. 122)
claimed that CGI teachers are able to apply a knowledge of children’s learning to the development of a classroom that encourages all children to learn with understanding. However, as we note in the next section, CGI provides limited research-based knowledge of students’ thinking that will help us address broader issues of equitable access to powerful mathematics. Furthermore, as Carey et al. admitted, the success of CGI depends on teachers who have concern for all children and who are willing to confront their own teaching practices and decide how they can be improved. CGI researchers maintain that a critical feature of equitable classrooms is “empowering children to make decisions about what is appropriate for them in terms of context and content of mathematics” (Carey et al., 1995, p. 123).

Fuson’s TEMP program also draws on research on children’s thinking to help children from poor, urban backgrounds to learn high level mathematics with understanding. This is achieved by enabling all children to enter mathematical experiences at their own level. With a focus on problem solving, the program assists children to understand a meaningful mathematical situation through discussions involving rich and diverse language. This approach enables the children to mathematize the situation and to draw models to represent the mathematical features of the situation. Also of importance in the TEMP program is an emphasis on the affective, social, motivational, and self-concept aspects of participating effectively in mathematical activities: “children need to be helped to see themselves included in the world of mathematics” (Fuson et al., 2000, p. 201).

Numerous other authors have made related suggestions on how to achieve greater equity in mathematics learning, such as helping more students see the relevance and utility of mathematics by linking their school study more closely to the outside world including that of the workplace (Hoachlander, 1997) and by employing teaching strategies that engage children, challenge them mathematically, and generally show them that we value their mathematical ideas (Lappan, 1999). The reality is that a combination of many approaches is needed, however, and that broader issues beyond the immediate classroom must be addressed in our search for equity of access to powerful mathematics (Secada, 1995; Seeley, 1999).

**IMPROVING LEARNING ACCESS TO POWERFUL MATHEMATICAL IDEAS**

It is all very well to identify some of the powerful mathematical ideas we need to provide our students, but far more research is needed to improve students’ learning access to these ideas. Without such research, we will continue to see students turn away from mathematics before they leave the elementary grades.

Studies investigating the learning of mathematics have long been the focal point of mathematics education research (Adda, 1998; Mura, 1998). Learning access goes beyond learning, however, in that it involves issues of equity and willingness to learn. Consequently we need to consider what kind of research is needed to improve learning and how it should be designed to take cognizance of equity and willingness to learn.

Research from the 20th century reveals that students are not totally dependent on teaching for their mathematics learning. They bring a great deal of informal knowledge to any learning situation, and they are capable of constructing new knowledge by reorganizing their own existing conceptual schemas. Studies, largely based on clinical interviews, have already built a valuable cadre of students’ knowledge in number (Behr, Harel, Post, & Lesh, 1992; Carpenter & Moser, 1984; Fuson, 1992; Greer, 1992; Jones, Thornton, Putt, Hill, Mogill, Rich, & Van Zoest, 1996; Sowder, 1992; Verschaffel & De Corte, 1996), geometry and measurement (Clements & Battista, 1992; Lehrer & Chazan, 1998; Hershkowitz, Parzysz, & van Dormolen, 1996; van Hiele, 1959/1985),
algebra (Filloy & Sutherland, 1996; Kaput, 1989; Kieran, 1992) and probability and statistics (Jones, Langrall, Thornton, & Mogill, 1997; Jones, Thornton, Langrall, Mooney, Perry, & Putt, 2000; Shaughnessy, 1992; Shaughnessy; Garfield & Greer, 1996).

Regrettably few of these studies have provided data on the mathematical thinking of socially and economically disadvantaged students. Studies such as those of Ginsberg and Russell (1981), on the early number skills of preschoolers from disadvantaged backgrounds, however, suggest that these children do not suffer from massive mathematical deficits. Rather, as Silver, Smith, and Nelson (1995) claimed, “low levels of participation and performance in mathematics by females, ethnic minorities, and the poor were not primarily due to lack of ability or potential but rather to educational practices that deny access to meaningful high-quality experiences with mathematical learning” (p. 10).

Although there is a prima facie case for further clinical interview research into the mathematical learning of children from diverse cultures, it not clear how this research should proceed or to what extent it needs to be pursued to improve students’ access to powerful mathematical ideas. Certainly, research into students’ informal mathematical knowledge has come into sharper focus with the development of the CGI (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996). Although this model has shown promise for improving learning access, especially in areas such as whole numbers and their operations where the research is robust, serious issues remain. For example, in many domain specific areas of mathematics, it not clear what kind of research-based knowledge of students’ mathematical thinking is needed to create equitable learning access nor is it clear how this research needs to be presented to teachers especially on a large scale. In fact, the kind of research that has generally supported cognitively guided instruction, while longitudinal in nature, tends to represent learning access as a series of growth strategies or levels of thinking that characterize students at different stages of mathematical maturity. The research is often based on or is consistent with more general theories of development (e.g., Biggs & Collis, 1991; Case, Omoto, Griffin, McKeough, Bleiker, Henderson, Stephenson, Sigler, & Keating, 1996; Ginsburg, 1983; Piaget, 1952; Riley, Greeno, & Heller, 1982) and has a restricted perspective on willingness to learn, social interaction in learning and metacognitive processes of learning. For these reasons, much of the cognitive research on specific mathematical domains is seen to be limited by those who espouse situated cognition (e.g., Brown, Collins, & Duguid, 1989; Lave, 1988; Saxe, 1990) and more dynamic models of cognition in which learning is seen to develop in cycles rather than through linear growth (e.g., Pirie & Kieren, 1994). Moreover, it is also seen as narrow by those who adopt a social perspective to learning (Bauersfeld, 1980; Brousseau, 1984) or a socioconstructivist view of learning (Cobb, 1999) or indeed those who call for greater attention to special groups of learners (Secada, 1992).

On the other hand, teaching experiment methodologies that address psychological and sociological aspects of classroom learning (Cobb, 2000; Confrey & Lachance, 2000; Steffe, Thompson, & von Glasersfeld, 2000) offer promise with mathematical equity issues. In particular, Cobb and Yackel (1996) noted that if equity issues are to be addressed during a teaching experiment, learning activities need to be complemented by a strong sociocultural perspective. Cobb (1999) provided a glimpse of how this worked in a teaching experiment involving statistics. It embodied building a picture of the cultural background and social norms of the students at school and at home as well as analyzing classroom interactions for possible inconsistencies between the classroom microculture and the home cultures. As Cobb observed, “the purpose of these explorations was to delineate issues that will contribute to our understanding of equity as it related specifically to teaching and learning” (p. 37). Cobb also asserted that matters of instructional design, such as choice of activity, involve equity issues. For example, he claimed that the choice of statistics was fueled by its association with technology and the need for quantitative reasoning—both high stakes activities in our society.
Related methodologies that have both a sociological and a cognitive perspective have been conducted in other parts of the world: Carraher, Carraher, & Schliemann (1985) and Saxe (1990) in Brazil; Balacheff and Laborde (1988) in France; Bauersfeld (1988) and Wittman (1998) in Germany; Walkerdine (1988) in Great Britain; Bartolini-Bussi (1990) in Italy; and Gravemeijer (1998) in The Netherlands. The emphasis on learning access may vary among different researchers, but there is clearly the potential to look at learning access within the social conditions and interactions that prevail in school settings. Although in their embryonic stages, classroom teaching experiments and related methodologies do add a further dimension to research approaches that are already pervasive in relation to research on learning access and learning environments.

SUPPORTING LEARNING ENVIRONMENTS THAT PROVIDE LEARNERS WITH MORE EQUITABLE ACCESS TO POWERFUL MATHEMATICAL IDEAS

We now consider how research can support the establishment of learning environments that facilitate more equitable access to mathematics. Although this section concentrates on instructional environments, it does of necessity have a strong focus on learning and the reflexivity between teaching and learning. We direct our attention to issues related to learning environments: teaching and learning models, the role of technology in learning environments, and teacher development. We also look briefly at the kind of research that shows promise in clarifying and explicating these issues.

Much of the research on learning environments in the 20th century was predicated on the assumption that there existed an optimum model of teaching mathematics for all teachers and all students. When one revisits pedagogies such as drill and practice, incidental learning, meaningful learning, discovery learning, and direct instruction, the emphasis on learning was stronger in rhetoric than reality. These were essentially teaching models, and an often unstated assumption was that if the model were executed correctly, learning would follow. Moreover, the models were single minded; they were uncluttered by details about teachers different mathematical knowledge and beliefs and learners’ variability with respect to a range of factors including gender, race, and culture.

Given the assumption that there was a “right” teaching model, the role of research was to provide evidence authenticating the model or showing that it produced superior student performance to some earlier model or control model. Experimental design was ideally suited to this kind of research and hence the conclusions were strong on statistical comparison and light on theory building. Moreover, if the comparison was not favorable to the preferred model or, as was generally the case, produced insignificant differences, the model’s popularity waned, and a new model superseded it. In a real sense, the “math war” currently being waged in the United States is a historical extension of this preoccupation with the right teaching model. It not only embraces strong and conflicting beliefs about teaching models but also frustration that experimental research on the issue has often generated more heat than light.

What is the status on teaching models as we move into the 21st century? Thompson (1992) considered that the predominant models of mathematics teaching are those identified by Kuhs and Ball (1986). Following their extensive review of the literature in mathematics education, teacher education, the philosophy of education, and research on teaching and learning, Kuhs and Ball identified “at least four dominant and distinctive views of how mathematics should be taught:”

1. learner-focused: mathematics teaching that focuses on the learner’s personal construction of mathematics;
2. content-focus with an emphasis on conceptual understanding: mathematics teaching that is driven by the content itself but emphasizes conceptual understanding;
3. content-focused with an emphasis on performance: mathematics teaching that emphasizes student performance and mastery of mathematical rules and procedures;
4. classroom-focused: mathematics teaching based on knowledge of effective classrooms.

(p. 2)

Although these models of teaching and learning have been useful in providing broad characterizations of teachers’ beliefs and classroom practice, we don’t believe that they are indicative of research that has occurred during the last 20 years. The creation of learning environments is more complex than any of these models, and we believe that there is little value in pursuing further research on teaching models that reduce the complexity of teaching and learning to a single-minded perspective.

More promising research on teaching and learning environments is beginning to emerge, and this research is attempting to recognize the complexity of both teaching and learning. According to Koehler and Grouws (1992), the research not only involves multiple observations of classroom processes and products, it seeks to integrate research on teaching with research on learning, as well as research on teacher characteristics (knowledge, beliefs, enthusiasm) and learner characteristics (thinking, gender, race, confidence). Examples of research reflecting these perspectives include CGI (Fennema et al., 1996); classroom teaching experiments (Cobb, 2000; Lesh & Kelly, 2000); educational development and developmental research (Gravemeijer, 1998); “first-person” research or working from the inside (Ball, 2000; Lampert, 1998); mathematics knowledge as a collective enterprise (Bartolini-Bussi, 1990); psychosocial learning environments (Fraser & Walberg, 1981); recherches en didactique des mathématiques (Brousseau, 1992); the “open approach” method (Nohda, 2000; Shimada, 1977), and the teacher development experiment (Simon, 2000). These investigative programs are not only research processes or paradigms, they are philosophies of teaching and even theories of instruction. In fact, the interplay between the research process and classroom practice is mutually supportive, and conclusions drawn from such research have implications for instructional theory and practice, cognitive functioning of collective groups and individuals, and social and cultural perspectives in the classroom.

Within these research endeavors, theoretical knowledge is beginning to emerge on the special role of technology in learning environments (Balacheff & Laborde, 1988; Clements, Battista, & Sarama, 1998; Cobb, 1999; Tzur, 1999). Noss and Hoyles (1996) claimed that, in concert with teaching–learning research like we have outlined above, technology can be used to develop more effective learning models by providing a window for viewing children’s construction of meaning. Microworld spaces, for example, are especially germane for both individual and classroom research because they create self-contained worlds in which children “learn to transfer habits of exploration from their personal lives to the formal domain of scientific construction” (Pappert, 1980, p. 117). Microworld technology also seems to be capable of producing a language through which meanings can be externalized and emerging knowledge can be expressed, changed, and explored (Noss & Hoyles, 1996). The potential for insightful interactions between teachers, students, and researchers in a technology context where meaning is externalized seems boundless yet exceedingly fruitful for creating more accessible and more flexible learning environments.

In examining ways in which research can support the creation of more powerful learning environments, we have highlighted some of the more recent research paradigms in teaching and learning and have also focused on research into related areas such as technology learning environments. At the time of the writing of this chapter, evidence of the power of teaching–learning research with multiple perspectives was never more conspicuous than in the unfolding of both the methodology and
the results of the Third International Mathematics and Science Study (TIMMS) video study (Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999). This study involved 231 eighth-grade mathematics classrooms in three countries, Germany, Japan, and the United States, and its design and implementation involved a level of complexity that seemed unattainable 10 years ago. It has set new horizons for global research on learning environments that involved video observations, transcript development and translation, coding by international experts, reliability checks, and endless iterative processes in carrying out the analysis and reporting of results. However, the results of this video study, especially when combined with the TIMSS results on student achievement (National Center for Educational Statistics, 1996), have had a dramatic effect because they revealed significant differences in learning environments between cultures yet relatively little variation in learning environments within cultures (Stigler & Hiebert, 1999, p. 11). The far-reaching effects of such research not only have the potential to improve learning environments in mathematics across the world, they also have implications for teacher education and teacher development programs.

DEVELOPING TEACHER EDUCATION AND TEACHER DEVELOPMENT PROGRAMS THAT FACILITATE STUDENT ACCESS TO POWERFUL MATHEMATICAL IDEAS

For much of the 20th century, teacher development occurred through special projects with relatively small numbers of teachers. The expectation was that these projects would produce “lighthouse groups,” and knowledge would trickle down to other groups of educators and teachers. This approach has had limited success and in recent years teacher development programs that focus on large-scale teacher groups have begun to emerge. The QUASAR Project (Silver et al., 1995) and Project IMPACT (Campbell & Robles, 1997) both deal with large groups of teachers across schools with high minority student populations. QUASAR has satellite groups in six geographically dispersed sites that are centered around middle schools in economically disadvantaged areas. IMPACT works with all the elementary teachers in a large urban school district that is also located in an economically disadvantaged area. In addition to providing strong support for the teachers from colleagues and other qualified professionals, both projects create opportunities for teachers to work together to improve their practice as well as time to engage in personal reflection. This emphasis on teachers working together on “lesson research” (Yoshida, 1999) and using reflective practice has been endemic to the Japanese educational scene for many years, but the potential of it has only recently been realized (Lewis, 2000; Stigler et al., 1999). It seems that the valuable legacy from the Japanese experience and the two large-scale U.S. projects is a new perspective on large-scale teacher development and one that deserves further research in the present century.

Research on Teacher Change

A key component of research on teacher development is how we can more effectively promote teacher change. The mathematics education community has acknowledged the complexity of defining the professional knowledge base needed for teaching and has struggled with crucial issues concerning the relations among teachers’ knowledge of mathematics, their beliefs about mathematics teaching and learning, teachers’ classroom practices, and assessing students’ achievements. The growing body of research suggests that if teachers are to understand their students’ mathematical ways of thinking, make professional decisions regarding content and form of instruction,
and facilitate fruitful mathematical discourse in their classes, their knowledge and beliefs need to be challenged and changed (see, for instance, Ball, 1988; Jones, 1997; Ma, 1999; McDiarmid & Wilson, 1991; Thompson, 1992). This has led to the emergence of a new research trend that attempts to develop better understanding of the nature and processes of teacher change and the factors that affect these processes.

As for now, this rapidly growing trend has mainly yielded rich case studies describing processes of teacher change (see, for instance, Carpenter, Fennema, Peterson, & Carey, 1988; Clarke, 1997; Koch, 1997; Schifter & Fosnot, 1993; Cobb, Yackel, & Wood, 1992). Theories about the nature of teacher change are starting to emerge, however, (see, for instance, Cooney & Shealy, 1997), and new research designs and research methodologies for studying teacher change are constantly evolving (see, for instance, Kelly & Lesh, 2000a). The growing awareness of the crucial role of teachers’ knowledge and beliefs and the calls for substantial change in teachers’ practices resulted in the formation of many projects that develop new models of teacher education and professional development programs, and carefully monitor, evaluate, and study their impact on teacher change (e.g., Even, 1999; Jaworski, 1998).

Although research on teacher change is relatively new, the existing literature provides information about some characteristics of effective models of teaching in teacher education and in professional development programs. These are described in various documents calling for a reformed vision of the teaching and learning of mathematics (e.g., Australian Education Council, 1991; National Commission on Mathematics and Science Teaching for the 21st Century, 2000; National Council of Teachers of Mathematics [NCTM], 1989, 2000), in articles and books on mathematics education (e.g., Aichele & Coxford, 1994; Ferrini-Mundy & Schram, 1997; Fennema & Nelson, 1997; Jaworski, Wood, & Dawson, 1999; Lin & Cooney, in press; Loucks-Horsley, Hewson, Love, & Stiles, 1998) as well as in some of the chapters in this handbook. Here we shall briefly mention four characteristics of models of teacher change.

**Characteristics of Effective Models of Teacher Change**

Effective teacher education and professional development programs are those that

1. Provide teachers with opportunities to build their own knowledge and skills to promote their understanding of mathematics of how students learn mathematics and of ways of helping students learn. Such programs are driven by a well-defined image of the role of the teacher in the classroom and of effective classroom learning and teaching.

2. Provide opportunities for teachers to address issues of concern and interest, solicit teachers’ conscious commitment to the aims of the program, and encourage them to set goals for their professional growth. Professional development programs often attempt to create partnership and collaboration between researchers and teachers (e.g., see Ruthven, chapter 23).

3. Recognize that it often takes years for change to occur in teachers’ beliefs, acknowledge the different stages that individuals go through during this process and provide teachers with opportunities to experience the desired teaching approaches in real situations, with students. Such programs recognize that systems resist change and that successful changes on the individual level are more likely to occur when attention is given to the system within which the teacher works (e.g., see Middleton, Sawada, Judson, Bloom, & Turley, chapter 17).

4. Are continuously informed by the ongoing monitoring of the concerns, questions, and needs of the teachers and continuously assess themselves, adjusting their mechanisms to ensure positive impact on teacher change. Vivid descriptions of such ongoing processes can be found, for instance, in Middleton et al.’s chapter 17.
Teacher change is a relatively young area of study. The various chapters in this handbook suggest issues for further research; some relate to values, others to content, and still others to forms (see, e.g., Amit & Fried, chapter 15; Tirosh & Even, chapter 10). In the coming years, research on teacher change will undoubtedly develop and produce substantial knowledge. New models of teacher education and of professional development programs will be generated and explored, and new types of collaboration between researchers and teachers will be formed. This body of knowledge could significantly contribute to our understanding of fundamental, controversial issues related to mathematics teaching, including: (a) What do mathematics teachers need to know? (b) How do mathematics teachers come to know? and, (c) What are the relationships between teacher knowledge and classroom practices, including how teachers assess students’ complex achievements?

**EMERGING ISSUES IN ASSESSING STUDENTS’ (AND TEACHERS’) COMPLEX ACHIEVEMENTS**

Assessing students’ complex achievements is one of the most important and difficult tasks that teachers face. Likewise, the assessment of teachers’ professional growth and the assessment of instructional programs present great challenges to administrators and curriculum personnel. Undertaking such assessments raises many contentious issues that will continue to demand substantial research. In our own field, we must attend to the development of tools for documenting, assessing, and (in some cases) measuring the kind of complex achievements that we hope to elicit from students, teachers, and programs of instruction (Lesh & Clark, 2000). Yet even though, in some topic areas, mathematics educators have made enormous progress to clarify the nature of students’ developing mathematical knowledge and abilities, assessment instruments tend to be based on assumptions that are poorly aligned with modern views about the nature of mathematics, problem solving, learning, and teaching.

The test theory that dominates educational measurement today might be described as the application of twentieth century statistics to nineteenth century psychology. . . . The essential problem is that the view of human learning that underlies standard test theory is not compatible with the view rapidly emerging from cognitive and educational psychology. (Mislevy, 1990, p. 234)

These tests are based on different views of what knowing and learning mathematics means. . . . These tests are not appropriate instruments for assessing the content, process, and levels of thinking called for in the STANDARDS. (Romberg, Wilson, & Khaketa, 1991, p. 3)

Educators have tended to think about the mind (and about the nature of mathematical knowledge) as though it is similar to our most sophisticated technology (Kelly & Lesh, 2000b). Consequently, there has been a gradual transition:

- Away from analogies based on hardware, in which teachers are led to believe that the construction of mathematical knowledge in a child’s mind is similar to the process of assembling a robot, an automobile, or a string of condition-action rules in a computer program (that is, whole systems are considered to be no more than the sum of their parts).
- Beyond analogies based on software, in which silicone-based electronic circuits often involve layers of recursive interactions that may lead to emergent phenomena at higher levels that cannot be derived from characteristics of phenomena at lower levels.
- Toward analogies based on wetware, in which neurochemical interactions may involve “logics” that are fuzzy, partly redundant, partly inconsistent, and unstable.
As an age of biotechnologies gradually supersedes an age of electronic technologies, the systems that are priorities for mathematics educators to explain, create, modify, predict, or influence are no longer assumed to be inert. They are complex, dynamic, continually adapting, and self-regulating systems, and their behaviors and “ways of thinking” cannot be described in superficial ways using simple-minded, input–output rules (or simple algebraic or probabilistic equations) that ignore emergent phenomena and feedback loops in which second-order effects often outweigh first-order effects. Often, they are not simply given in nature; instead, their existence is partly the result of human constructions. Furthermore, it may not be possible to isolate them because their entire nature may change if they are separated from complex holistic systems in which they are embedded; they may not be observable directly but may be knowable only by their effects; Or when they are observed, changes may be induced in the “subjects” being investigated. So the researchers and their assessment instruments become an integral part of the system being measured.

Regardless of whether attention is focused on students, groups, teachers, or other learners or problem solvers, if we want to assess something more than low-level factual and procedural knowledge, attention generally needs to focus on models and conceptual systems that the “subjects” develop to interpret their experiences. But most modern theories of teaching and learning purport that the way learning and problem solving experiences are interpreted is influenced by both (internal) conceptual systems and (external) systems that are encountered, so the interpretations that learners and problem solvers construct involve interactions between (internal) structuring capabilities and (external) structured environments. Consequently, the following kinds of assessment issues arise: When different problem solvers are expected to interpret a single problem solving situation in fundamentally different ways, what does it mean to speak about “standardized” questions? When different learners are expected to interpret a given learning experience in fundamentally different ways, what does it mean to speak about “the same treatment” being given to two different participants? When assessments are integral parts of the systems being investigated, what does it mean to speak of “detached objectivity” for “outside” assessments?

Consider the case of high stakes standardized tests. It is widely recognized that such tests are powerful leverage points that influence (for better or worse) both what is taught and how it is taught. In particular, when such tests are used to clarify (or define) the goals of instruction, they go beyond being neutral indicators of learning outcomes; they become powerful components of the initiatives themselves. Far from being passive indicators of nonadapting systems, they have powerful positive or negative effects, depending on whether they support or subvert efforts to address desirable objectives. Therefore, when they are poorly aligned with the standards for instruction, they often create serious impediments to curriculum reform, teacher development, and student achievement.

**Impediments for Students**

If standardized tests focus on only narrow and shallow conceptions of the mathematical knowledge and abilities that are needed for success (beyond school) in a technology-based society, then such tests are likely to recognize and reward only students whose profiles of abilities fit this biased conception (Lesh, 2001). On the other hand, if new understandings and abilities are emphasized in simulations of “real-life” problem solving situations, than a broader range of students naturally emerge as having great potential (Lesh & Doerr, in press). Similarly, under the preceding circumstances, our research has shown that most students are able to invent better mathematics and science ideas than those associated with their prior failure
experiences with traditional textbooks, tests, and teaching (Lesh, Hoover, Hole, Kelly, & Post, 2000).

**Impediments for Teachers**

Even though gains in student achievement surely should be one factor to consider when documenting the accomplishments of teachers (or programs), it is foolish to assume that great teachers always produce larger student learning gains than their less great colleagues. What would happen if a great teacher chose to deal with only difficult students or difficult circumstances? What would happen if a great teacher chose to never deal with difficult students or difficult circumstances? No teacher can be expected to be "good" in "bad" situations (such as when students do not want to learn or when there is no support from parents and administrators). Not everything "experts" do is effective, and not everything "novices" do is ineffective. Furthermore, no teacher is equally effective across all grade levels (from kindergarten through calculus), with all types of students (from the gifted to those with physical, social, or mental handicaps), and in all types of settings (from those dominated by inner-city minorities to those dominated by the rural poor). In fact, characteristics that lead to success in one situation often turn out to be counterproductive in other situations. Consequently, it is naïve to make comparisons among teachers using only a single number on a "good–bad" scale (without identifying profiles of strengths and weaknesses and without giving any attention to the conditions under which these profiles have been achieved or the purposes for which the evaluation was made). Nonetheless, if tests include thought-revealing activities that teachers can use to learn about their students ways of thinking, then a variety of positive results may occur (Lesh et al., 2000).

**Impediments for Programs**

When evaluating large and complex program innovations, it is misleading to label them "successes" or "failures" (as though everything successful programs did was effective and everything unsuccessful programs did was not effective). All programs have profiles of strengths and weaknesses. Most programs "work" for achieving some types of results but "don’t work" for others, and most are effective for some students (or teachers or situations), but not for others. In other words, most programs "work" some of the time, for some purposes, and in some circumstances, but, none "work" all of the time, for all purposes, in all circumstances. For example,

- When the principal of a school doesn’t understand or support the objectives of a program, the program seldom succeeds. Therefore, when programs are evaluated, the characteristics and roles of key administrators also should be assessed, and these assessments should not take place in a neutral fashion. Attempts should be made to optimize understanding and support from administrators (as well as parents, school board members, and other leaders from business and the community). Then, during the process of optimization, documentation should be gathered to produce a simple yet high-fidelity trace of continuous progress.
- The success of a program depends on how much and how well it is implemented. For example, if only half of a program is implemented, or if it is only implemented in a halfhearted way, then 100% success can hardly be expected. Powerful innovations usually need to be introduced gradually over periods of several years, so when programs are evaluated, the quality and extent of the implementation should be assessed. This assessment should aim toward systemic validity. It should not pretend to be done in a neutral fashion. Optimization and documentation are not incompatible
processes. The goal is to improve performance, not just audit, so assessment should be longitudinal and recursive.

A systemically valid test (or item, or report) is one that induces in the education system curricular and instructional changes that foster the development of achievements that the test is designed to measure. (Fredericksen & Collins, 1989)

How can we assess the extent to which students, teachers, and programs achieve deeper and higher order goals of instruction? How can these assessments avoid biases related to cultural, ethnic, and gender differences? How can they provide information that is as useful as possible to a variety of different decision makers—ranging from students, to parents, to teachers, to policymakers? To begin to address such questions, it is important to emphasize that assessment is not the same as evaluation. Whereas the goal of evaluation is to assign a value to the subjects being inspected, the goal of assessment is to provide information for decision makers, perhaps by describing the subject’s location (and recent history of progress) in some landscape of ideas and abilities that it is desirable for them to learn.

To create assessment programs in which the preceding kinds of factors are taken into account, two distinct types of assessment designs are useful to sort out: pretest-posttest designs (referred to here as mechanistic) and continuous (systemic) documentation, monitoring, and feedback. Mechanistic pretest–posttest designs are intended to prove that “it” works, but they often promote conditions that minimize the chances of success by measuring progress in terms of deficiencies with respect to simplistic conceptions of success. Systemic documentation, monitoring, and feedback is intended to (simultaneously) document progress as well as encourage continuous development in directions that are increasingly “better” without using a simplistic definition of “best” to define the next steps. Table 30.1 displays some of the key features of these two types of assessment.

The important points to note in Table 30.1 include the following: (a) It is possible to examine students closely, without relying on nonproductive ordeals. (b) It is possible to document achievements and abilities without reducing relevant information to a

### TABLE 30.1

<table>
<thead>
<tr>
<th>Mechanistic Perspectives</th>
<th>Systemic Perspectives</th>
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<tbody>
<tr>
<td><strong>Testing</strong></td>
<td><strong>Examining</strong></td>
</tr>
<tr>
<td>Creating an ordeal (barrier, or filter) for accepting or rejecting (but not helping)</td>
<td>Inspecting closely and generating high-fidelity descriptions</td>
</tr>
<tr>
<td><strong>Measuring</strong></td>
<td><strong>Documenting</strong></td>
</tr>
<tr>
<td>Partitioning (fragmenting) into distinguishable pieces</td>
<td>Gathering tangible evidence which is credible to decision makers</td>
</tr>
<tr>
<td><strong>Evaluating</strong></td>
<td><strong>Assessing</strong></td>
</tr>
<tr>
<td>Assigning a value without specifying conditions or purposes</td>
<td>Taking stock, orienting with respect to known landmarks and goals</td>
</tr>
<tr>
<td><strong>Summative Information</strong></td>
<td><strong>Formative Information</strong></td>
</tr>
<tr>
<td>Focuses on decision-making issues of administrators</td>
<td>Focuses on decision-making issues of students and teachers</td>
</tr>
<tr>
<td><strong>Improving Tests</strong></td>
<td><strong>Improving Teachers’ Assessments</strong></td>
</tr>
<tr>
<td>Goal is to make credible evaluations better aligned with instructional goals</td>
<td>Goal is to make authentic assessments (such as those based on teachers’ judgments) more credible and reliable</td>
</tr>
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</table>
single-number “score” (or letter grade). (c) It is possible to assess “where students are” and “where they need to go” without comparing students with one another along a simplistic “good–bad” scale. (d) It is possible to address the decision-making needs of administrators without neglecting the decision-making needs of students, parents, and teachers. (e) It is foolish to say that a student (or teacher, or program) is “good” without saying “good for what?” or “good under what conditions?”

The preceding observations suggest that there should be close connections among assessments of achievement for students, teachers, programs, program administrators, and program implementations. Regardless of whether the entity being studied is a student, a teacher, or a program, the assessment should be expected to be wildly naive if (a) it takes information from only a single source, (b) it results in only simple-minded comparisons among individuals on a unidimensional “good–bad” scale, (c) it ignores conditions under which alternative profiles of achievement occur, and (d) achievement is assessed using tests that reduce expertise to simplistic lists of “condition–action rules” (e.g., Given . . ., the student will . . .).

Finally, assessment is about generating information that is useful to decision makers, and, in education, these decision makers range from students, to teachers, to parents or guardians, to program administrators, to policymakers. Decisions may range from high-stakes decisions that are irreversible decisions to low-stakes decisions where timeliness may be more important than high levels of precision or accuracy. In any case, in an age of technology, when most of these decision makers have access to a variety of tools that produce interactive, dynamic, and graphic displays of information, it is foolish for educators to adopt a “one-size-fits-all” policy that is static and that treats all “subjects” as if their abilities were accurately characterized by a single point on a number line extending from bad to good.

In an age when many applied sciences are using a variety of graphics-oriented and interactive global information systems to display complex information about complex systems, it is remarkable that educators continue to limit themselves to single-number descriptions of students, teaching effectiveness, and programs of instruction. For example, if the mathematics curriculum is visualized as a three-dimensional topographic map on which the mountains represent “big ideas” and surrounding valleys represent related lower level skills, a given student’s progress report might look similar to a map in a historical atlas that shows areas that have been conquered by an invading army. In the context of such a map, a simple arrow labeled “you are here” might provide a great deal of information about directions for future progress (Lesh & Lamon, 1993).

Mathematics educators need to design better tools for generating information about deeper and higher order achievements of students, teachers, and programs, but they also need to design new types of interactive, graphic, and dynamic displays of information.

ASSESSING AND IMPROVING RESEARCH DESIGNS IN MATHEMATICS EDUCATION

Educational innovations should be informed by the available scientific knowledge base and should be evaluated and analyzed with rigorous research methods; the advancement of education requires continued research efforts on a large scale. (Anderson et al., 2000, p. 13)

In this last main section, we review briefly some of the latest research designs in mathematics education and discuss criteria that might be applied to making these designs more rigorous.

The field of mathematics education research appears to be at a critical stage in its development. On the positive side, new research designs have been developed
that are based on new ways of thinking about the nature of students’ mathematical knowledge, problem solving, learning, and teaching, and that involve closer and more meaningful working relationships between many levels and types of both researchers and practitioners. Examples of these research designs include the following.

- **Action research** in which distinctions between researchers and practitioners often are blurred, as teachers participate as coresearchers or as researchers participate as teachers or as designers of instructional activities.
- **Carefully structured clinical interviews** in which relevant data include more than isolated pieces of information and patterns of behavior across iterative sequences of tasks.
- **Videotape analyses** in which decisions about whom to observe, when to observe, and what to observe may radically influence the nature of apparent results and in which many interpretation cycles may be needed to recognize relevant patterns of behavior.
- **Ethnographic observations**, which serve to identify the types of mathematical understandings needed for success in various out-of-school contexts; such observations usually need triangulation techniques to compensate for the fact that multiple theoretical perspectives often result in significantly different interpretations of people, places, and practices that are exhibited.
- **New approaches to assessment** that focus on deeper, higher order understandings and abilities and that focus on complex performances that must be classified in ways that are far more sophisticated than simply “correct” and “not correct.”
- **Teaching experiments** that go beyond investigating typical development in natural environments to focus on induced development in carefully controlled and **mathematically enriched** environments and that investigate the interacting development of students, teachers, and others (e.g., parents, policymakers).

Teaching experiment methodologies (Cobb, 2000; Confrey & Lachance, 2000; Lesh & Kelly, 2000; Steffe et al., 2000) offer considerable promise in providing more micro and dynamic evidence of students’ learning, in contrast to research that offers only a series of discrete snapshots of students’ mathematical thinking. In a teaching experiment, an ongoing series of teaching episodes is undertaken with individuals, groups, or complete classes of students in such a way that the planning of each exploratory teaching episode is based on the teacher’s and students’ thinking and actions in prior teaching episodes. Through these exploratory teaching episodes the teacher-researcher gains firsthand access to a continuous film of students’ learning and attempts to build dynamic models of “students’ mathematics” (Steffe et al., 2000, p. 268). These models, built through the researchers’ retrospective analysis of the teaching experiment, incorporate both a psychological and a sociological perspective (Cobb, 2000, p. 309). As such the models are expected to be able to justify the students’ mathematical language and social actions both individually and collectively. In essence, the researcher formulates an image of the students’ mental operations and an itinerary of what students might learn and how they might learn it.

- **The principled design experiment** (Hawkins, 1997; diSessa & Abelson, 1986) is a potent methodology that explores novel possibilities for learning in what are high-risk situations for software developers. In some sense, principled design experiments are the technological counterpart of the teaching experiment because they provide opportunities for observing students’ mathematical thinking in situations that take the students to the brink. In essence, they use software that has the potential not merely to amplify students’ cognition but to fundamentally change it (Döerfler, 1993). The Boxer design space (diSessa & Abelson, 1996) is a reconceptualization of Logo that is said to be at least a generation removed from incremental variations of Logo currently available commercially. Although this kind of research
is in its infancy, it is clear that we need to pursue it with vigor in the early part of this
century.

The negative side of the preceding research designs is that the development of
widely recognized standards for assessing (and improving) the quality of research
designs has not kept pace with the recognition of new levels and types of problems,
new theoretical perspectives, and new approaches to the collection, analysis, and
interpretation of data. Consequently, there is a growing concern that future progress
may be impeded unless criteria and procedures become more clear for optimizing
the usefulness, sharability, and cumulativeness of results that are produced by the
preceding kinds of innovative research designs.

Research is not just about learning a cluster of “accepted” techniques for gathering
data, analyzing data, and reporting results in some standard accepted form. Research
is about the development of knowledge; and, in particular, it is about the develop-
ment of shared constructs (models, prototypes, principles, and conceptual systems)
that provide useful ways to think about problems that are priorities for practition-
ers in the field to address (Lesh, Lovitts, & Kelly, 2000). Consequently, the design of
research involves the development of a coherent chain of reasoning that is power-
ful and auditable and that should be both meaningful and persuasive to practition-
ers, researchers, and skeptics. It cannot be reduced to a step-by-step formula-based
process.

Two of the most important factors that determine what kind of information to seek
and what kind of interpretative frameworks to use involve being clear about assump-
tions concerning the nature of the “subjects” being investigated and assumptions
concerning the “products” that are intended to be produced.

The Nature of the “Subjects” Being Investigated

Mathematics educators have come to recognize that students, teachers, classrooms,
courses, curricula, learning tools, and minds are complex systems taken singly, let
alone in combination. Dealing with complexity in a disciplined way is the essence of
research design (Kelly & Lesh, 2000b). In general, the systems we investigate are dy-
namic, interacting, self-regulating, and continually adapting. Their behaviors cannot
be described in superficial ways using simple input–output rules (or simple algebraic
or probabilistic functions), which ignore emergent phenomena and second-order ef-
fects that result from recursive feedback loops (e.g., when a factor X influences factor Y,
which in turn, influences factor Z, which in turn, influences factor X).

The Nature of the “Products” Being Investigated

If we recognize that research is about the development of sharable and reuseable
knowledge, then it is clear that not all knowledge consists of tested hypotheses and
answered quantitative questions. For example, in the history of more mature fields of
science, many of the most important products that research produces have consisted
of tools for creating, observing, classifying, or measuring “subjects” that are consid-
ered to be important. Or they are models or conceptual systems for constructing,
describing, or explaining complex systems. Or they are demonstrated possibilities in-
volving existence proofs in special circumstances. Similarly, in mathematics education,
using iterative research techniques of the type emphasized by Kelly and Lesh (2000b),
participants (which may include researchers as well as students and/or teachers)
are challenged to express their current ways of thinking in forms that must be tested
and revised repeatedly. Thus, after a series of such testing and revising cycles, au-
ditable trails of documentation are produced that enable retrospective analyses to
reveal the nature of significant developments that occur. Investigations that involve
these kinds of testing and revising cycles often are referred to as design studies because the products that participants generate consist of models or conceptual tools that must be developed and refined using iterative testing and revising cycles, similar to those commonly used by engineers and other scientists when they design complex systems ranging from spacecraft to artificial ecosystems.

On the other hand, research in mathematics education is not engineering. It is not physical or life science, it is not applied statistics, and it is not easy. In particular, no single means of understanding is likely to be sufficient, and no single style of inquiry is likely to take us very far, regardless of whether it emphasizes qualitative or quantitative methodologies. What we need is a shift from theory borrowing to theory building. Our issues, concerns, and assumptions are quite different from those of other disciplines.

One commonly suggested way of dealing with the complex issues of mathematics education research that we have been addressing is to have a clearly structured plan. In the whirring, buzzing confusion of real classrooms, students, tools, materials, and faculty, it is critical to have some sense of how your inquiry might proceed and where you are in the process. This is one reason why traditional descriptions of research processes often characterize research as a process involving the following list of steps (Romberg, 1992, pp. 51–53).

1. Identify phenomena of interest about which questions will be formulated and addressed.
2. Build a tentative model or description that helps sort out key aspects of the phenomena in question, especially distinguishing those that seem most relevant from those that seem less so.
3. Relate the phenomena and model to others’ ideas and results, both among those who share your worldview and those who do not.
4. Ask specific questions or make reasoned hypotheses or conjectures, trying to get at the essence of the phenomena in a way that supports a chain of inquiry that eventually affords some kinds of answers to the questions or specific tests of the conjectures.
5. Select a general research strategy for gathering evidence that fits all that has been decided to date to examine an existing situation in detail, to manipulate variables in a situation under your control, to compare situations, to look at the situation over an extended period in great detail, and so forth.
6. Select specific procedures and plan data collection. It is here that the usual research methods course content comes into play. Given the complexity of most research situations, however, a combination of procedures is usually required to address the critical questions.
7. Collect the information. At this point, the procedure(s) should be well specified. Most substantive research involves pilot stages of data gathering in which two or more cycles may be needed, however.
8. Interpret the information collected in the light of all the previous steps. Again, it may be the case that this step occurs as part of a sequence feeding back into many of the prior steps before a final sequence occurs.
9. Share the results. Even this step may involve cycles with the others, involving other scholars outside your immediate working community. This is especially the case in an environment where electronic communication supports rapid interchange and wide preliminary dissemination across a distributed community of working colleagues.
10. Anticipate the actions of others—other researchers, policymakers, practitioners, materials developers, and so forth. A given research activity virtually never occurs in isolation, and indeed a measure of its significance is the degree to which it spawns actions on the part of a wider community.
A great deal of experience and wisdom underlies the preceding guidelines. Nonetheless, even though most of these guidelines continue to be relevant to modern research in mathematics education, the list itself is a highly inadequate characterization of what really happens in a large share of the most productive research in our field. First, the development of useful knowledge is not restricted to products consisting of answered questions and tested hypotheses. Second, the design of models, conceptual tools, and other products of research often involves cyclic and iterative processes in which the design principles that must be emphasized do not conform to one-way, assembly-line characterizations of knowledge development. Third, the line between researchers and subjects is by no means as clear, as suggested by the preceding list of steps; many levels and types of researchers and practitioners may be involved, and communication is not simply in one direction. Fourth, because the systems being investigated are complex, dynamic, and continually adapting, researchers often must abandon notions of detached objectivity and naïve replicability. Similarly, naïve notions of reliability and validity may need to be reconceived to be consistent with modern theoretical perspectives. Yet when attempts are made to assess or improve the quality of specific research projects, issues such as usefulness, sharability, and cumulativeness continue to be relevant.

CONCLUDING POINTS

In this concluding chapter, we have addressed what we consider to be some of the key 21st-century issues in the advancement of mathematics education and mathematics education research. We have stressed the need for research to support more equitable mathematics curriculum and learning access for all students, with a focus on their exposure to the powerful mathematical ideas of the new millennium. We have considered ways in which research can facilitate the creation of learning environments that can increase this access, as well as ways in which teacher education and development can help achieve our goal of powerful mathematics for all. The need for research to inform the important and challenging issues of assessment—whether it be assessment of students, teachers, programs, or researchers—also has been emphasized here.

Our research endeavors in meeting these challenges need to be subjected to greater scrutiny, especially in terms of the way in which we deal with our “subjects” and the types of “products” we try to generate. Our subjects in mathematics education are complex systems—dynamic, interactive, self-monitoring, and continually adapting. We need to deal with such complex systems in a more disciplined way. At the same time, we need to broaden the nature of the products that we seek to generate from our research. More than ever before, our research needs to produce knowledge that can be shared and reused. Our products are many and varied, including tools for creating, observing, classifying, or measuring our subjects, and models or conceptual systems for constructing, describing, or explaining the complex systems with which we are dealing.

The proliferation of research designs and methodologies in the past decade has provided us with unprecedented opportunities for investigating (and ultimately improving) the mathematical learning of students and other members of our societies. With their varied emphases on cognitive, social, and cultural perspectives, these newer research designs place us in a unique position to look at mathematical learning from multiple perspectives. It is our hope that future studies will draw on the best aspects of these research designs to create rich, multidisciplinary approaches to mathematics education research. In so doing, we can generate products, including new theories, that provide more integrated and culturally sensitive frameworks for improving learners’ access to powerful mathematical ideas.
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